The Generalized Integro-Exponential Function

By M. S. Milgram

Abstract. The generalized integro-exponential function is defined in terms of the exponential integral (incomplete gamma function) and its derivatives with respect to order. A compendium of analytic results is given in one section. Rational minimax approximations sufficient to permit the computation of the first six first-order functions are reported in another section.

1. Introduction. The generalized integro-exponential function plays an important role in the theory of transport processes and fluid flow, yet no unified summary of its properties exists in the literature. First introduced by Van de Hulst [12] who gave a power series and numerical tabulation for some special cases, the function was abstracted by Chandrasekhar [8] and recently reviewed by Van de Hulst [13]. In the interim, tabulated values were presented by Stankiewicz [26] and Gussmann [11]; the latter as well as Abu-Shumays [2] and Kaplan [16] independently summarized some analytic properties. Recent work by this author [20]–[22] has reasserted the significant role played by this function in transport theory, and, by invoking the theory of Meijer's G -functions [18], many general properties of this function have been unveiled which were previously unknown. These were summarized in an Appendix to [20] and in a Table in a conference proceedings [22].

Since these summaries appeared, the problem of accurately evaluating this function numerically has arisen. For small values of the real independent variable x , power series evaluations work reasonably well with coefficients obtainable from the formulas given in [20] and [22]. For intermediate values of x , power series break down due to cancellation of significant digits when evaluating differences between large numbers, and for large values of x , asymptotic series do not do justice to the inherent accuracy available with modern computers.

The purpose of this paper is then twofold:

- (1) Reiterate, derive and present a unified summary of the analytic properties of these functions only some of which can be found elsewhere;
- (2) Proffer some approximations on which the numerical evaluation of a large number of these functions can be based.

2. Analysis. Let s and z be continuous (complex) variables and n and j represent nonnegative integers. Define the "generalized integro-exponential function" $E_s^j(z)$ by

$$(2.1) \quad E_s^j(z) = \frac{(-1)^j}{j!} \frac{\partial^j}{\partial s^j} E_s(z),$$

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where $E_s(z)$ is the usual* exponential integral generalized to $n = s$, or in terms of incomplete gamma functions

$$E_s(z) = z^{s-1} \Gamma(1-s, z).$$

We have the integral representation

$$(2.2) \quad E_s^0(z) \equiv E_s(z) = \int_1^\infty t^{-s} \exp(-zt) dt;$$

for the case $s = n$ this reduces to the exponential integral about which a considerable literature revolves [1], [3], [4], [6], [9], [14], [19], [23], [25], [27], [29]. Apply Eq. (2.1) to (2.2), interchange the order of integration and differentiation, and obtain the integral representation

$$(2.3) \quad E_s^j(z) = \frac{1}{\Gamma(j+1)} \int_1^\infty (\log t)^j t^{-s} \exp(-zt) dt$$

which, when $s = n$, reduces to Gussmann's [11] definition $E_{n,j}(z) = E_n^j(z)$, or that of van de Hulst [12] $E_1^{(j)}(z) = E_1^{j-1}(z)$. Equation (2.3) may be integrated by parts (integrate t^{-s} and differentiate the remaining factors) to obtain the recursion formula

$$(2.4) \quad E_s^j(z) = (zE_{s-1}^j(z) - E_s^{j-1}(z))/(1-s), \quad s \neq 1, j \geq 0,$$

defining $E_s^{-1}(z) = \exp(-z)$ for the case $j = 0$.

To obtain a unified theory of this function and its properties, start with the known identification of $\exp(-z)$ in terms of Meijer's G -function [18, No. 6.2.1(1) and 6.4(1)]

$$(2.5) \quad \exp(-z) = G_{0,1}^{1,0} \left(z \middle| \begin{matrix} ; \\ 0 \end{matrix} \right) = \frac{1}{2\pi i} \int_{L_0} \Gamma(-t) z^t dt,$$

substitute into Eq. (2.2), and use a known [18, No. 5.6.4(6)] integration formula to obtain

$$(2.6a) \quad E_s(z) = G_{1,2}^{2,0} \left(z \middle| \begin{matrix} ; s \\ 0, s-1 \end{matrix} \right) = \frac{1}{2\pi i} \int_{L_0} \frac{\Gamma(-t) z^t}{s-1-t} dt$$

$$(2.6b) \quad = \exp(-z) z^{s-1} \psi(s, s; z),$$

where L_0 is the negatively directed contour enclosing both the nonnegative t -axis, and the pole at $t = s - 1$. The third term in the equality (2.6a) is the contour integral representation [18, No. 5.2(1)] of the G -function according to prescription. In Eq. (2.6b), ψ is a confluent hypergeometric function of the second kind. Operating on Eq. (2.6a) as in Eq. (2.1), we obtain the contour integral representation

$$(2.7a) \quad E_s^j(z) = \frac{1}{2\pi i} \int_{L_0} \frac{\Gamma(-t) z^t}{(s-1-t)^{j+1}} dt$$

$$(2.7b) \quad = G_{j+1, j+2}^{j+2, 0} \left(z \middle| \begin{matrix} ; s, \dots, s \\ 0, s-1, \dots, s-1 \end{matrix} \right).$$

* "Exponential integral" (of order s) refers to the function $E_s(z)$. Some authors [13], [1] use this nomenclature to mean $E_n(z)$, while others [18] mean $E_1(z)$, or $\text{Ei}(z) = -E_1(-z)$ or both.

The result (2.7b) is obtained by identifying the contour integral (2.7a) according to prescription [18, No. 5.2(1)]; Eq. (2.7a) provides a means of generalizing the function to noninteger values of j . Set $j = -1$ in Eq. (2.7a), and with reference to Eq. (2.5) obtain

$$(2.8) \quad E_s^{-1}(z) = \exp(-z),$$

from which it is easy to see how Eq. (2.4) reduces to a well-known result [1, No. 5.1.14] when $j = 0$ and $s = n$.

The power series follows immediately from Eq. (2.7a) by evaluating the residues of the simple poles of $\Gamma(-t)$ and the multi-poles of order $j + 1$ when $s \neq n$ or $j + 2$ if $s = n$. The results are

$$(2.9) \quad E_s^j(z) = \sum_{l=0}^{\infty} \frac{(-z)^l}{(s-1-l)^{j+1}l!} + \frac{z^{s-1}(-1)^j}{j!} \sum_{l=0}^j (-1)^l \binom{j}{l} \log^{j-l} z \Gamma(1-s)^{(l)}, \quad s \neq n,$$

and

$$(2.10) \quad E_n^j(z) = \sum_{\substack{l=0 \\ l \neq n-1}}^{\infty} \frac{(-z)^l}{(n-1-l)^{j+1}l!} + \frac{z^{n-1}(-1)^{j+n}}{(j+1)!} \sum_{l=0}^{j+1} \binom{j+1}{l} \log^{j+1-l} z \Psi_{l,n},$$

where $\Gamma(1-s)^{(l)}$ means the l th derivative of $\Gamma(1-s)$ with respect to s , and

$$(2.11) \quad \Psi_{l,n} = \lim_{t \rightarrow n-1} \frac{\partial^l}{\partial t^l} \left(\frac{\Gamma(t-n+2)\Gamma(n-t)}{\Gamma(1+t)} \right).$$

Analytic expressions for $\Psi_{l,n}$ ($0 \leq l \leq 8$) have been obtained through the use of computer algebra [28]; they are given in Table 1. General, but complicated expressions can be found in [2, Eq. (4.12)]. The result (2.10) is also given in [11] and [2, Eq. (4.7)]. Similar coefficients may be obtained for the functions $E_{-n}^j(z)$; however an alternative method for the evaluation of the latter set of functions will be given shortly.

It is of some interest to obtain other integral representations. From Eq. (2.7b) we recognize the G -function as an Euler transform [18, No. 5.6.4(6)]:

$$(2.12) \quad E_s^j(z) = \int_1^{\infty} t^{-s} E_s^{j-1}(zt) dt, \quad j \geq 0,$$

from which the name derives. The case $j = 1, s = 1$ has been obtained elsewhere [17] in the form of a power series; (2.12) may also be found in [2]. An alternate expression for Eq. (2.12) is the repeated integral form:

$$(2.13) \quad E_s^j(z) = \int_1^{\infty} dy_1 y_1^{-s} \int_1^{\infty} dy_2 \cdots \int_1^{\infty} dy_{j+1} y_{j+1}^{-s} \exp\left(-z \prod_{i=1}^{j+1} y_i\right).$$

From (2.2), (2.3) and (2.12), we find

$$(2.14a) \quad E_s^j(z) = \frac{1}{\Gamma(j)} \int_1^{\infty} (\log t)^{j-1} t^{-s} E_s(zt) dt, \quad j > 0,$$

from which it may be shown that

$$(2.14b) \quad \int_1^\infty E_s^l(zt) t^{-s} \log^k t \, dt = \frac{\Gamma(k+1)}{\Gamma(l)} \int_1^\infty E_s^{k+1}(zt) t^{-s} \log^{l-1} t \, dt, \quad l > 0, k \geq 0.$$

Finally, using (2.14a) and (2.14b) we find the remarkable representation

$$(2.14c) \quad E_s^j(z) = \frac{1}{\Gamma(j-l)} \int_1^\infty E_s^l(zt) t^{-s} \log^{j-l-1} t \, dt \quad \forall \quad 0 \leq l \leq j-1$$

by a process of induction, employing (2.19). This generalizes both (2.12) and (2.14a).

Starting from Frullani's integral

$$\log t = \int_0^\infty dv (\exp(-v) - \exp(-tv)) / v$$

and Eq. (2.3), the order of integration may, with care, be interchanged, giving the Laplace transform

$$(2.15) \quad E_s^1(z) = \int_0^\infty dv \exp(-v) [E_s(z) - \exp(v) E_s(z+v)] / v.$$

After applying Eq. (2.1) to Eq. (2.15), a more general form is found:

$$(2.16) \quad E_s^{j+1}(z) = \frac{1}{(j+1)} \int_0^\infty dv \exp(-v) [E_s^j(z) - \exp(v) E_s^j(z+v)] / v, \quad j \geq 0.$$

A number of G -function representations may also be derived. From Eq. (2.2) and known integration formula [18, No. 5.6.2(18)], we have

$$(2.17) \quad E_s(z) = \exp(-z) G_{1,2}^{2,1} \left(z \left| \begin{matrix} 0; \\ 0, s-1; \end{matrix} \right. \right) / \Gamma(s).$$

Starting with Eq. (2.7a), substitute $t := 2t$ and use the duplication formula for the gamma function to find [18, No. 5.2(1)]

$$(2.18) \quad E_s^j(z) = \frac{2^{-j-1}}{\sqrt{\pi}} G_{j+1, j+3}^{j+3, 0} \left(\frac{z^2}{4} \left| \begin{matrix} (s+1)/2, \dots, (s+1)/2 \\ 0, \frac{1}{2}, (s-1)/2, \dots, (s-1)/2; \end{matrix} \right. \right).$$

From Eq. (2.3) the differentiation rules are easy to uncover

$$(2.19a) \quad \frac{\partial^k}{\partial z^k} E_s^j(z) = (-1)^k E_{s-k}^j(z),$$

$$(2.19b) \quad \frac{\partial^m}{\partial s^m} E_s^j(z) = \frac{(-1)^m \Gamma(m+j+1)}{\Gamma(j+1)} E_s^{m+j}(z),$$

and from Eqs. (2.19) and (2.4) we discover that $E_s^j(z)$ satisfies

$$(2.20a) \quad (s-1) \frac{\partial}{\partial s} E_s^j(z) - z \frac{\partial^2}{\partial s \partial z} E_s^j(z) + (j+1) E_s^j(z) = 0$$

as well as [18, No. 5.8(1)]

$$(2.20b) \quad \left(1 + \frac{\partial}{\partial z} \right) \left(z \frac{\partial}{\partial z} - s + 1 \right)^{j+1} E_s^j(z) = 0.$$

A short table of integrals is included as Appendix A.

In applications [27] we need to evaluate $E_{-n}^j(x)$, $n > 0$, $x = \operatorname{Re}(z) > 0$. Integrating by parts in Eq. (2.3) leads to the result

$$E_0^j(z) = E_1^{j-1}(z)/z, \quad j \geq 0.$$

Repeated application of Eq. (2.4) gives

$$(2.21) \quad E_{-n}^j(z) = \frac{\Gamma(n+1)}{z^n} \left(\sum_{k=0}^{n-1} \frac{z^k E_{-k}^{j-1}(z)}{\Gamma(k+2)} + E_0^j(z) \right).$$

Similarly, repeated application of Eq. (2.21) leads to the simpler form

$$(2.22) \quad E_{-n}^j(z) = \frac{\Gamma(n+1)}{z^{n+1}} \left[\exp(-z) \sum_{l=0}^{n-j} \frac{z^l}{l!} \xi_{l,n}^j + \sum_{m=1}^j \xi_{0,n}^{m-1} E_1^{j-m}(z) \right],$$

where the (precomputed) constants $\xi_{l,n}^j$ are given analytically by

$$(2.23) \quad \xi_{l,n}^j = \sum_{k_j=0}^{n-j-l} \sum_{k_{j-1}=0}^{k_j} \cdots \sum_{k_1=0}^{k_2} \left(\frac{1}{k_j + l + j} \right) \left(\frac{1}{k_{j-1} + l + j - 1} \right) \cdots \left(\frac{1}{k_1 + l + 1} \right),$$

$$\xi_{n-j,n}^j = \frac{(n-j)!}{n!}, \quad \xi_{l,n}^0 = 1, \quad \xi_{l,n}^j = 0 \quad \text{if } n < j + l.$$

The derivation of Eq. (2.22) comes by proving its truth for the cases $j = 1$ and $j = 2$ and proceeding by induction, using the identities

$$(2.24a) \quad \xi_{0,n}^j = \sum_{k=0}^{n-1} \left(\frac{1}{k+1} \right) \xi_{0,k}^{j-1},$$

$$(2.24b) \quad \xi_{l,n}^j = \sum_{k=0}^{n-l-j} \left(\frac{1}{k+l+j} \right) \xi_{l,k+l+j-1}^{j-1}.$$

From Eq. (2.22) all the $E_{-n}^j(z)$ can be obtained for $n \geq 0$ if the functions $E_1^m(z)$ are known for $0 \leq m \leq j-1$. Approximations for $E_1^m(z)$ are given in Section 3. A lengthy form for the evaluation of $E_n^j(x)$ for $n > 1$ in terms of $E_1^j(x)$ is given by Gussmann [11, Eq. (A.18) to (A.21')]. Note that Eq. (2.22) is numerically stable since all its terms are positive.

For large values of real z , the first few terms in the asymptotic series of $E_{-n}^j(x)$ are given by the first finite sum in Eq. (2.22). From Eq. (2.7b) and the theory [18, Section 5.10] of G -functions, the asymptotic series is theoretically known. The first terms are

$$(2.25) \quad E_s^j(z) \sim \exp(-z) z^{-(j+1)} [1 - (j+1)(j+2s)/(2z) + \cdots].$$

A more general result can be obtained by integrating Eq. (2.12) repeatedly by parts:

$$(2.26) \quad E_n^{j+1}(z) = \frac{1}{z} \sum_{k=0}^{\infty} \frac{(-1)^k}{z^k} \frac{\Gamma(n+k)}{\Gamma(n)} E_{n+k+1}^j(z).$$

The claimed asymptotic formula is then ($\operatorname{Re}(z) \rightarrow +\infty$)

$$(2.27) \quad E_n^j(z) \sim \exp(-z) z^{-j-1} \sum_{l=0}^P \frac{(-1)^l}{z^l} \frac{\Gamma(n+l+j)}{\Gamma(n)} \xi_{n-1,n+l+j-1}^j,$$

where P is an integer. This can be proven by establishing its truth for the case $j = 0$ and $j = 1$, using in Eq. (2.26) the well-known [1, Eq. 5.1.51] result

$$E_{n+k+1}(z) \sim \exp(-z) z^{-1} \sum_{m=0}^P \frac{(-1)^m}{z^m} \frac{\Gamma(n+k+m+1)}{\Gamma(n+k+1)},$$

and proceeding by induction, employing the identity

$$(2.28) \quad \sum_{k=0}^l \frac{1}{(n+k)} \xi_{n+k, n+l+j}^j = \xi_{n-1, n+l+j}^{j+1}$$

in Eq. (2.27).

Finally, from Eq. (2.9) we have the special value

$$(2.29) \quad E_s^j(0) = \left(\frac{1}{s-1} \right)^{j+1}, \quad \operatorname{Re}(s) > 1.$$

3. Rational Minimax Approximations to $E_l^j(x)$. With the aid of the computer code REMES2 [15], it proved possible to obtain rational minimax approximations to $E_l^j(x)$ for x real, $1 \leq j \leq 6$, and hence $E_{-n}^j(x)$ with $1 \leq j \leq 7$, $n > 0$ because of Eq. (2.22). The region $0 \leq x$ was conveniently divided into three ranges for the purpose of both evaluating the master functions and obtaining efficient fits.

3a. *Small x .* In this range, we evaluate $E_l^j(x)$ according to the power series (2.10) in the form

$$(3.1) \quad E_l^j(x) = P_k(x)/Q_l(x) + P_{1j}(\log x),$$

where $P_k(x)/Q_l(x)$ is the (k, l) minimax fit to the infinite series plus the constant ($l = j + 1$) term of the second sum in Eq. (2.10). $P_{1j}(\log x)$ is a polynomial in $\log(e^\gamma x)$ of order j obtainable from Table 1 by means of computer algebra. Although no proof has been found, for $1 \leq j \leq 7$, this polynomial $P_{1j}(\log x)$ can be written

$$(3.2) \quad P_{1j} = \sum_{l=1}^{j+1} u_{j,l} v^l,$$

where

$$(3.3) \quad v = \log(e^\gamma x)$$

and

$$(3.4) \quad u_{j,l} = -u_{j-1, l-1}/l, \quad l \geq 1,$$

with the constants $u_{j,0}$ given in Table 2, and γ being Euler's constant. The upper end of this range (X_U) was chosen to coincide with a point smaller than the first positive zero of $E_l^j(x) - P_{1j}(\log x)$. All master function calculations were done in 120 bit double-precision arithmetic (~ 28 s.d.) and the approximation chosen was the lowest $(k + l)$ entry in the Walsh array with more than 14 s.d. accuracy. The coefficients of the (k, l) minimax fits are given in Table 3.

3b. *Intermediate x .* In this range, $X_U \leq x \leq 10$, the master functions were obtained from the power series (2.10), with an exponential weighting such that

$$(3.5) \quad E_l^j(x) = \exp(-x) P_k(x)/Q_l(x).$$

TABLE 1
Analytic expressions for $\Psi_{l,n}$ in Eq. (2.10)

```

CPSI (1)=GAMIN      $      CPSI (2)=GAMIN*(-PSIN)

CPSI (3)=+GAMIN*(PSIN**2-PSIN+2.*ZETA2)

CPSI (4)=+GAMIN*(3.*PSIN*PSIN1-6.*PSIN*ZETA2-PSIN**3-PSIN2)

CPSI (5)=
1 +GAMIN*(4.*PSIN*PSIN2-6.*PSIN**2*PSIN1+12.*PSIN**2*ZETA2+PSIN**4
1 -12.*PSIN1*ZETA2+3.*PSIN1**2-PSIN3+12.*ZETA2**2+12.*ZETA4)

CPSI (6)=
1 +GAMIN*(60.*PSIN*PSIN1*ZETA2-15.*PSIN*PSIN1**2+5.*PSIN*PSIN3-60.
1 *PSIN*ZETA2**3-60.*PSIN*ZETA4-10.*PSIN**2*PSIN2+10.*PSIN**3*PSIN
1 1-20.*PSIN**3*ZETA2-PSIN**5+10.*PSIN1*PSIN2-20.*PSIN2*ZETA2-PSIN
1 4)

CPSI (7)=
1 +GAMIN*(-60.*PSIN*PSIN1*PSIN2+120.*PSIN*PSIN2*ZETA2+6.*PSIN*PSIN
1 1 4-180.*PSIN**2*PSIN1*ZETA2+45.*PSIN**2*PSIN1**2-15.*PSIN**2*PSIN
1 3+180.*PSIN**2*ZETA2**2+180.*PSIN**2*ZETA4+20.*PSIN**3*PSIN2-15.
1 *PSIN**4*PSIN1+30.*PSIN**4*ZETA2+PSIN**6+15.*PSIN1*PSIN3-180.*PS
1 IN1*ZETA2**2-180.*PSIN1*ZETA4+90.*PSIN1**2*ZETA2-15.*PSIN1**3+10
1 *.PSIN2**2-30.*PSIN3*ZETA2-PSIN5+360.*ZETA2*ZETA4+120.*ZETA2**3+
1 240.*ZETA6)

CPSI (8)=
1 +GAMIN*(-105.*PSIN*PSIN1*PSIN3+1260.*PSIN*PSIN1*ZETA2**2+1260.*P
1 SIN*PSIN1*ZETA4-630.*PSIN*PSIN1**2*ZETA2+105.*PSIN*PSIN1**3-70.*
1 PSIN*PSIN2**2+210.*PSIN*PSIN3*ZETA2+7.*PSIN*PSIN5-2520.*PSIN*ZET
1 A2*ZETA4-840.*PSIN*ZETA2**3-1680.*PSIN*ZETA6+210.*PSIN**2*PSIN1*
1 PSIN2-420.*PSIN**2*PSIN2*ZETA2-21.*PSIN**2*PSIN4+420.*PSIN**3*PS
1 IN1*ZETA2-105.*PSIN**3*PSIN1**2+35.*PSIN**3*PSIN3-420.*PSIN**3*Z
1 ETA2**2-420.*PSIN**3*ZETA4-35.*PSIN**4*PSIN2+21.*PSIN**5*PSIN1-4
1 2.*PSIN**5*ZETA2-PSIN**7+420.*PSIN1*PSIN2*ZETA2+21.*PSIN1*PSIN4-
1 105.*PSIN1**2*PSIN2+35.*PSIN2*PSIN3-420.*PSIN2*ZETA2**2-420.*PSI
1 N2*ZETA4-42.*PSIN4*ZETA2-PSIN6)

CPSI (9)=
1 +GAMIN*(-3360.*PSIN*PSIN1*PSIN2*ZETA2-168.*PSIN*PSIN1*PSIN4+840.
1 *PSIN*PSIN1**2*PSIN2-290.*PSIN*PSIN2*PSIN3+3360.*PSIN*PSIN2*ZETA
1 2**2+3360.*PSIN*PSIN2*ZETA4+336.*PSIN*PSIN4*ZETA2+8.*PSIN*PSIN6+
1 420.*PSIN**2*PSIN1*PSIN3-5040.*PSIN**2*PSIN1*ZETA2**2-5040.*PSIN
1 **2*PSIN1*ZETA4+2520.*PSIN**2*PSIN1**2*ZETA2-420.*PSIN**2*PSIN1*
1 *3+280.*PSIN**2*PSIN2**2-840.*PSIN**2*PSIN3*ZETA2-28.*PSIN**2*PS
1 IN5+10080.*PSIN**2*ZETA2*ZETA4+3360.*PSIN**2*ZETA2**3+6720.*PSIN
1 **2*ZETA6-560.*PSIN**3*PSIN1*PSIN2+1120.*PSIN**3*PSIN2*ZETA2+56.
1 *PSIN**3*PSIN4-840.*PSIN**4*PSIN1*ZETA2+210.*PSIN**4*PSIN1**2-70
1 *.PSIN**4*PSIN3+840.*PSIN**4*ZETA2**2+840.*PSIN**4*ZETA4+56.*PSI
1 N**5*PSIN2-28.*PSIN**6*PSIN1+56.*PSIN**6*ZETA2+PSIN**8-280.*PSIN
1 1*PSIN2**2+840.*PSIN1*PSIN3*ZETA2+28.*PSIN1*PSIN5-10080.*PSIN1*Z
1 ETA2*ZETA4-3360.*PSIN1*ZETA2**3-6720.*PSIN1*ZETA6-210.*PSIN1**2*
1 PSIN3+2520.*PSIN1**2*ZETA2**2+2520.*PSIN1**2*ZETA4-840.*PSIN1**3
1 *ZETA2+105.*PSIN1**4+56.*PSIN2*PSIN4+560.*PSIN2**2*ZETA2-840.*PS
1 IN3*ZETA2**2-840.*PSIN3*ZETA4+35.*PSIN3**2-56.*PSIN5*ZETA2-PSIN7
1 +13440.*ZETA2*ZETA6+10080.*ZETA2**2*ZETA4+1680.*ZETA2**4+5040.*Z
1 ETA4**2+10080.*ZETA8)

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Note: $\text{CPSI}(\ell+1) = \Psi_{\ell,n}$ $\text{ZETA}\ell = \zeta(\ell)$ $\text{PSIN} = \psi(n)$ $\text{GAMIN} = 1/\Gamma(n)$ $\text{PSIN}\ell = \psi^\ell(n)$

TABLE 2
Coefficients of the polynomials $P_{1j}(\log x)$

j	coefficient $u_{j,0}$	(numerical)
	(analytic)	
-1	1	1
0	0	0
1	$\pi^2/12$	.82246 70334 24113
2	$-\zeta(3)/3$	-.40068 56343 86531
3	$\pi^4/160$	.60880 68189 62515
4	$-\zeta(3)\pi^2/36 - \zeta(5)/5$	-.53693 62760 78223
5	$\zeta(3)^2/18 + 61\pi^6/120960$	.56510 20417 78833
6	$-\zeta(3)\pi^4/480 - \zeta(5)\pi^2/60 - \zeta(7)/7$	-.55855 78221 73266
7	$\zeta(3)^2\pi^2/216 + \zeta(3)\zeta(5)/15 + 1261\pi^8/29030400$	.56127 49829 66818

also $P_{1j} = \sum_{k=1}^{j+1} u_{j,k} v^k$ where $v = \log(e^{\gamma}x)$ and $u_{j,k} = -u_{j-1,k-1}/k$

example: $P_{13}(v) = \zeta(3)v/3 + \pi^2v^2/24 + v^4/24$

TABLE 3.1

$E^1_1(X) = P(X)/Q(X) * (X - X_{11}) + P_{11}(\log(X))$

FIT TO $E^1_1(X) = \exp(-X) * P(X)/Q(X)$

$X_{11} = .91446 \ 48659 \ 7277$

IN THE RANGE 1.000.LE.X.LE.10.000

IN THE RANGE 0.000.LE.X.LE. 1.000

RATIONAL APPROXIMATION IS R(6,8)
WITH PRECISION= 14.09 DIGITS

RATIONAL APPROXIMATION IS R(3,4)
WITH PRECISION= 14.46 DIGITS

P00 (6) -.376341 57664 06520
P01 (5) -.635070 75949 96412
P02 (4) -.538768 20580 92452
P03 (2) -.584198 45557 93210
Q00 (6) .418437 62176 08333
Q01 (6) .121793 37377 23431
Q02 (5) .132630 73906 04789
Q03 (3) .551375 17596 10943
Q04 (1) .100000 00000 00000

P00 (1) -.338130 76971 20518
P01 (0) .891028 00665 24613
P02 (3) .211811 97726 44376
P03 (3) .283328 58929 93606
P04 (3) .120102 26074 96511
P05 (2) .192587 99072 06423
P06 (0) .999999 85034 28858
Q00 (0) -.400917 37819 68016
Q01 (2) -.106209 25463 66475
Q02 (2) .419023 64057 58286
Q03 (3) .569128 54287 46715
Q04 (3) .964803 08155 39357
Q05 (3) .616101 47030 82591
Q06 (3) .175879 22239 84149
Q07 (2) .222587 85540 75955
Q08 (1) .100000 00000 00000

$E^1_1(X) = \exp(-X)/X^{*2} * P(Z)/Q(Z)$; $Z = 1/X$

IN THE RANGE 0.000.LE.Z.LE. .100

RATIONAL APPROXIMATION IS R(4,6)
WITH PRECISION= 15.13 DIGITS

P00 (-2) .368896 16541 46809
P01 (0) .111317 82274 12989
P02 (1) .111573 42563 49138
P03 (1) .430645 72608 32509
P04 (1) .522913 13537 89389
Q00 (-2) .368896 16541 46811
Q01 (0) .122384 70770 37284
Q02 (1) .144230 98012 73127
Q03 (1) .747160 29601 50458
Q04 (2) .168869 92689 15484
Q05 (2) .135627 18800 19350
Q06 (1) .100000 00000 00000

TABLE 3.2

FIT TO $E_1^2(x) = P(x)/Q(x) + P_{12}(\log(x))$				FIT TO $E_1^2(x) = \exp(-x) * P(x)/Q(x)$			
IN THE RANGE 0.000.LE.X.LE. .400				IN THE RANGE .400.LE.X.LE. 2.000			
RATIONAL APPROXIMATION IS R(3,4) WITH PRECISION= 16.20 DIGITS				RATIONAL APPROXIMATION IS R(5,8) WITH PRECISION= 14.90 DIGITS			
P00	(6)	.216257	42293 79812	P00	(0)	.534274	07496 24268
P01	(6)	-.493516	16951 20742	P01	(2)	.172848	04128 51346
P02	(5)	-.784750	37376 89728	P02	(2)	.602765	67841 77444
P03	(4)	-.381379	38950 85943	P03	(2)	.582331	73035 43360
Q00	(6)	-.539718	43355 22324	P04	(2)	.147552	63525 88969
Q01	(6)	-.115308	01225 47753	P05	(0)	.999989	73572 67234
Q02	(4)	-.773816	80672 69915	Q00	(-1)	.170899	18403 89456
Q03	(3)	-.122836	12906 79186	Q01	(1)	.295254	33350 91985
Q04	(1)	.100000	00000 00000	Q02	(2)	.651380	89668 07230
				Q03	(3)	.315340	67127 43816
				Q04	(3)	.573546	38415 89567
				Q05	(3)	.444897	80147 56000
				Q06	(3)	.147788	79716 87537
				Q07	(2)	.207544	71393 35389
				Q08	(1)	.100000	00000 00000
FIT TO $E_1^2(x) = \exp(-x) * P(x)/Q(x)$				$E_1^2(x) = \exp(-x)/x^{**3} * P(Z)/Q(Z)$; $Z=1/x$			
IN THE RANGE 2.000.LE.X.LE.10.000				IN THE RANGE 0.000.LE.Z.LE. .100			
RATIONAL APPROXIMATION IS R(5,8) WITH PRECISION= 15.55 DIGITS				RATIONAL APPROXIMATION IS R(5,5) WITH PRECISION= 13.58 DIGITS			
P00	(3)	.340433	44179 16653	P00	(-3)	.325912	28045 69083
P01	(3)	.876064	53575 85969	P01	(-2)	.976173	27349 42300
P02	(3)	.676598	41845 71783	P02	(-1)	.894008	30387 59118
P03	(3)	.203399	00769 19242	P03	(0)	.249087	11994 57106
P04	(2)	.246893	21393 49231	P04	(-1)	.139742	26502 81342
P05	(0)	.999999	81273 60712	P05	(-2)	-.401189	07964 43870
Q00	(2)	.206456	40742 55514	Q00	(-3)	.325912	28045 69169
Q01	(4)	.120509	38588 74781	Q01	(-1)	.117172	06417 64891
Q02	(4)	.511338	37482 96754	Q02	(0)	.148297	13910 05923
Q03	(4)	.778642	56924 73543	Q03	(0)	.802097	98704 24226
Q04	(4)	.547543	10587 36117	Q04	(1)	.174325	30019 86266
Q05	(4)	.194264	79236 68901	Q05	(1)	.100000	00000 00000
Q06	(3)	.352535	91968 35532				
Q07	(2)	.306893	01010 78642				
Q08	(1)	.100000	00000 00000				

In the cases $j = 5$ and 6 , multiple-precision (48 digit) arithmetic [7] was employed within the master functions to overcome large losses in accuracy near the upper part of the range, and again, the lowest $(k + l)$ approximation with more than 14 digit accuracy was chosen for inclusion in Table 3. In some cases, it was beneficial to split this range into two to obtain more efficient approximations.

3c. *Large x*. It proved fruitless to approximate $E_j^l(x)$ with the asymptotic formula (2.27) in the range $10 \leq x$. Accordingly, (k, l) minimax fits $R_j(1/x)$ were procured such that

$$(3.6) \quad E_j^l(x) = \exp(-x) x^{-j-1} R_j(1/x),$$

TABLE 3.3

FIT TO $E_1^3(X) = P(X)/Q(X) + P_{13}(\log(X))$

IN THE RANGE 0.000.LE.X.LE. .600

RATIONAL APPROXIMATION IS R(3,4)
WITH PRECISION= 15.65 DIGITS

P00	(6)	-.839003	82605	47848
P01	(7)	.119755	65275	64264
P02	(6)	.242909	62547	68230
P03	(5)	.108775	12253	31120
Q00	(7)	-.137811	17423	82510
Q01	(6)	-.296572	26101	03168
Q02	(5)	-.174055	92800	54023
Q03	(3)	-.157386	94906	03826
Q04	(1)	.100000	00000	00000

FIT TO $E_1^3(X) = \exp(-X) * P(X)/Q(X)$

IN THE RANGE .600.LE.X.LE.10.000

RATIONAL APPROXIMATION IS R(5,9)
WITH PRECISION= 13.37 DIGITS

P00	(2)	.157208	97422	57519
P01	(3)	.128368	95938	74056
P02	(3)	.270646	30105	87470
P03	(3)	.148490	63549	05473
P04	(2)	.228734	95971	24294
P05	(0)	.999998	89216	94849
Q00	(0)	.320385	15630	95068
Q01	(2)	.675468	89806	76635
Q02	(3)	.861412	12489	99020
Q03	(4)	.372041	91499	30951
Q04	(4)	.677596	67313	99825
Q05	(4)	.552376	35994	49519
Q06	(4)	.213346	91178	79909
Q07	(3)	.392231	03684	45133
Q08	(2)	.328733	75166	04005
Q09	(1)	.100000	00000	00000

 $E_1^3(X) = \exp(-X)/X^{**4} * P(Z)/Q(Z)$; $Z = 1/X$

IN THE RANGE 0.000.LE.Z.LE. .100

RATIONAL APPROXIMATION IS R(3,8)
WITH PRECISION= 14.31 DIGITS

P00	(-2)	-.174019	20356	94571
P01	(-1)	-.709402	14529	05107
P02	(0)	-.905606	10531	27851
P03	(1)	-.359745	25082	90533
Q00	(-2)	-.174019	20356	94563
Q01	(-1)	-.883421	34886	03280
Q02	(1)	-.164111	11311	14874
Q03	(2)	-.137785	23506	45778
Q04	(2)	-.514428	97426	13443
Q04	(2)	-.514428	97426	13443
Q05	(2)	-.685703	73818	80260
Q06	(2)	-.162601	49416	93719
Q07	(0)	-.957694	76113	15230
Q08	(1)	.100000	00000	00000

where $0 \leq 1/x \leq 0.1$. To evaluate the master functions, Eq. (2.12) was transformed into the integral form

$$(3.7) \quad E_1^j(x) = \exp(-x) x^{-j-1} \int_0^\infty dt \exp(-t) (1 + t/x)^{j-1} R_{j-1} \left(\frac{1}{t+x} \right),$$

where R_{j-1} was a 19 digit rational minimax fit to $x^j \exp(x) E_1^{j-1}(x)$ according to Eq. (3.6) obtained especially for that purpose. The integration was a 400-point Gauss-Laguerre calculation in double-precision arithmetic [5]. To begin the bootstrap, high-precision fits to $E_1(x)$ due to Cody and Thacher [9] in this range were employed. The fit in the Walsh table with smallest $(k + l)$ and greatest accuracy over 14 s.d. was chosen, and is reported in Table 3.

TABLE 3.4

FIT TO $E^4_1(X)=P(X)/Q(X)+P14(\text{LOG}(X))$				FIT TO $E^4_1(X)=\text{EXP}(-X)*P(X)/Q(X)$			
IN THE RANGE 0.000.LE.X.LE. .500				IN THE RANGE .500.LE.X.LE. 2.000			
RATIONAL APPROXIMATION IS R(3,3) WITH PRECISION= 15.13 DIGITS				RATIONAL APPROXIMATION IS R(6,6) WITH PRECISION= 13.75 DIGITS			
P00	(6)	.921856	63967 05455	P00	(-1)	.190007	72856 75969
P01	(7)	-.164159	41458 60650	P01	(0)	.145255	03101 02014
P02	(6)	-.112522	80878 61094	P02	(-1)	.812092	21046 70916
P03	(3)	-.607131	03056 26315	P03	(-2)	-.725683	52768 34330
Q00	(7)	-.171688	27675 48834	P04	(-3)	.559368	64008 87053
Q01	(6)	-.140218	91431 54977	P05	(-4)	-.321053	49168 10291
Q02	(4)	-.161995	44008 14657	P06	(-6)	.996850	83917 42953
Q03	(1)	.100000	00000 00000	Q00	(-4)	.383398	00706 66617
				Q01	(-1)	.950926	03238 66574
				Q02	(1)	.175435	39262 79244
				Q03	(1)	.827856	65799 05180
				Q04	(2)	.117836	83566 14576
				Q05	(1)	.609668	97953 70546
				Q06	(1)	.100000	00000 00000
FIT TO $E^4_1(X)=\text{EXP}(-X)*P(X)/Q(X)$				$E^4_1(X)=\text{EXP}(-X)/X**5*P(Z)/Q(Z)$; Z=1/X			
IN THE RANGE 2.000.LE.X.LE.10.000				IN THE RANGE 0.000.LE.Z.LE. .100			
RATIONAL APPROXIMATION IS R(4,8) WITH PRECISION= 13.93 DIGITS				RATIONAL APPROXIMATION IS R(5,7) WITH PRECISION= 14.84 DIGITS			
P00	(2)	.353537	04817 63005	P00	(-5)	.395133	17174 71982
P01	(2)	.293820	63659 48671	P01	(-3)	.204661	41023 99156
P02	(2)	.145436	89922 68919	P02	(-2)	.346003	84617 74784
P03	(0)	.999939	71607 90340	P03	(-1)	.204274	58543 62413
P04	(-6)	.520751	56693 51248	P04	(-1)	.211759	92902 12236
Q00	(1)	-.223458	69506 60041	P05	(-2)	-.228430	28733 96219
Q01	(3)	.204802	30991 30036	Q00	(-5)	.395133	17174 71976
Q02	(4)	.183199	07073 94430	Q01	(-3)	.263931	38600 20275
Q03	(4)	.333915	80415 91970	Q02	(-2)	.672752	62012 18175
Q04	(4)	.276686	23525 75097	Q03	(-1)	.828969	69188 14699
Q05	(4)	.128585	73754 40614	Q04	(0)	.515915	51447 03679
Q06	(3)	.297624	44755 03964	Q05	(1)	.155462	70935 57096
Q07	(2)	.295401	74603 13003	Q06	(1)	.206789	80159 55259
Q08	(1)	.100000	00000 00000	Q07	(1)	.100000	00000 00000

In all three ranges, the function $E^j_l(x)$ computed by rational (k, l) fits in single precision was compared with the multiple-precision master routines for 20,000 pseudo-random values of x and $1/x$. The largest fractional error noted was 1.1×10^{-13} .

Finally, the functions $E^j_{-n}(x)$ were calculated for $1 \leq j \leq 7$, $0 \leq n \leq 7$ and pseudo-random values of x and $1/x$ according to Eq. (2.22). Comparisons were made with the same functions calculated via Eq. (2.3) using the single-precision routine DCADRE [10]. No significant discrepancies were found at the relative error level of 1×10^{-12} , and it was observed that rational minimax methods require ~ 150 times less computation time than does DCADRE. For the case $j = 0$, a pre-existent library function based on [9] was used, although better methods are known [3]. Agreement with Gussman's tabulated values [11] was total.

TABLE 3.5

FIT TO $E^5_1(X)=P(X)/Q(X)+P15(\text{LOG}(X))$

IN THE RANGE 0.000.LE.X.LE. .300

RATIONAL APPROXIMATION IS R(3,2)
WITH PRECISION= 15.30 DIGITS

P00	(3)	.838743	38519	74414
P01	(4)	-.142760	40312	21125
P02	(2)	-.880503	85483	46258
P03	(0)	-.556431	96858	95740
Q00	(4)	.148423	35068	49911
Q01	(3)	.100211	06179	41411
Q02	(1)	.100000	00000	00000

FIT TO $E^5_1(X)=\text{EXP}(-X)*P(X)/Q(X)$

IN THE RANGE .300.LE.X.LE. 2.000

RATIONAL APPROXIMATION IS R(5,10)
WITH PRECISION= 15.44 DIGITS

P00	(-3)	.500522	58419	15039
P01	(0)	.179471	65122	08398
P02	(1)	.222234	03877	87114
P03	(1)	.379468	68767	06738
P04	(0)	.991769	17992	22935
P05	(-3)	.144274	11935	15928
Q00	(-4)	.322199	80730	27793
Q01	(-3)	.967191	28566	38765
Q02	(1)	.114091	98728	01536
Q03	(2)	.364550	65926	78242
Q04	(3)	.320690	08398	60978
Q05	(3)	.977517	80646	43698
Q06	(4)	.126115	41159	04314
Q07	(3)	.736669	39202	94918
Q08	(3)	.201906	35278	47405
Q09	(2)	.245286	16349	83681
Q10	(1)	.100000	00000	00000

FIT TO $E^5_1(X)=\text{EXP}(-X)*P(X)/Q(X)$

IN THE RANGE 2.000.LE.X.LE.10.000

RATIONAL APPROXIMATION IS R(4,9)
WITH PRECISION= 14.90 DIGITS

P00	(2)	.185231	13891	15313
P01	(2)	.169071	89408	09323
P02	(2)	.163822	45019	47477
P03	(0)	.999973	45809	73645
P04	(-6)	.178967	24703	86040
Q00	(0)	-.332104	62172	40376
Q01	(3)	.127144	45160	39896
Q02	(4)	.223717	64184	09195
Q03	(4)	.632007	63231	39911
Q04	(4)	.803518	43255	59348
Q05	(4)	.632932	10036	44392
Q06	(4)	.259400	04237	29686
Q07	(3)	.479992	93424	76704
Q08	(2)	.373802	66825	57554
Q09	(1)	.100000	00000	00000

 $E^5_1(X)=\text{EXP}(-X)/X**6*P(Z)/Q(Z) ; Z=1/X$

IN THE RANGE 0.000.LE.Z.LE. .100

RATIONAL APPROXIMATION IS R(5,7)
WITH PRECISION= 14.05 DIGITS

P00	(-6)	.854783	10464	47894
P01	(-4)	.624139	17717	77879
P02	(-2)	.138649	42989	29153
P03	(-2)	.948336	88932	79712
P04	(-3)	-.423100	66195	40255
P05	(-4)	-.370718	93388	82177
Q00	(-6)	.854783	10464	47817
Q01	(-4)	.803643	62915	37229
Q02	(-2)	.279890	57603	97328
Q03	(-1)	.462603	61191	05448
Q04	(0)	.380244	88250	95635
Q05	(1)	.147127	35379	71468
Q06	(1)	.229031	45317	90874
Q07	(1)	.100000	00000	00000

TABLE 3.6

FIT TO $E_1^6(X)=P(X)/Q(X)+P16(\text{LOG}(X))$

IN THE RANGE 0.000.LE.X.LE. .100

RATIONAL APPROXIMATION IS R(2,2)
WITH PRECISION= 15.16 DIGITS

P00	(4)	-.828583	83172	51790
P01	(5)	.145297	02796	48708
P02	(3)	.486899	73737	68510
Q00	(5)	.148343	42996	06228
Q01	(3)	.545404	94731	07867
Q02	(1)	.100000	00000	00000

FIT TO $E_1^6(X)=\text{EXP}(-X)*P(X)/Q(X)$

IN THE RANGE .100.LE.X.LE. 2.000

RATIONAL APPROXIMATION IS R(6,12)
WITH PRECISION= 14.72 DIGITS

P00	(-3)	.271478	83310	87788
P01	(-1)	.195102	35413	21396
P02	(0)	.321565	23084	02674
P03	(1)	.138382	85866	93074
P04	(1)	.205248	36427	90331
P05	(0)	.989908	76978	43590
P06	(-3)	.167302	21625	00549
Q00	(-7)	-.771598	59706	76959
Q01	(-2)	.168613	64843	91203
Q02	(0)	.219806	16317	34924
Q03	(1)	.810987	78546	02619
Q04	(3)	.114098	17140	23387
Q05	(3)	.688229	02344	19855
Q06	(4)	.205074	91268	54280
Q07	(4)	.325821	03449	06946
Q08	(4)	.282942	94513	61645
Q09	(4)	.130505	73635	81354
Q10	(3)	.296709	78069	04683
Q11	(2)	.296971	60052	15741
Q12	(1)	.100000	00000	00000

FIT TO $E_1^6(X)=\text{EXP}(-X)*P(X)/Q(X)$

IN THE RANGE 2.000.LE.X.LE.10.000

RATIONAL APPROXIMATION IS R(7,7)
WITH PRECISION= 14.29 DIGITS

P00	(1)	.109840	75266	90525
P01	(-2)	-.628986	68217	39343
P02	(-3)	.352279	00224	82096
P03	(-4)	-.166654	15521	17304
P04	(-6)	.631863	11472	37745
P05	(-7)	-.177898	75053	81043
P06	(-9)	.327229	37007	14598
P07	(-11)	-.292513	40388	70733
Q00	(0)	.174130	46755	89622
Q01	(1)	.789054	90506	24093
Q02	(3)	.294284	05239	39408
Q03	(3)	.980479	98971	36415
Q04	(3)	.851988	71787	21233
Q05	(3)	.259397	92046	62330
Q06	(2)	.293770	63084	51261
Q07	(1)	.100000	00000	00000

 $E_1^6(X)=\text{EXP}(-X)/X^{**7}*P(Z)/Q(Z)$; Z=1/X

IN THE RANGE 0.000.LE.Z.LE. .100

RATIONAL APPROXIMATION IS R(6,7)
WITH PRECISION= 14.98 DIGITS

P00	(-6)	.117928	10446	90701
P01	(-5)	.887532	43497	71136
P02	(-3)	.210604	43465	37132
P03	(-2)	.154902	34850	49091
P04	(-3)	-.627848	92801	53793
P05	(-3)	.202298	30636	59744
P06	(-4)	-.409048	68342	68896
Q00	(-6)	.117928	10446	90700
Q01	(-4)	.121773	11274	90613
Q02	(-3)	.487180	40530	95863
Q03	(-2)	.965568	34655	76481
Q04	(0)	.100200	50733	04699
Q05	(0)	.527459	47929	77240
Q06	(1)	.125439	02841	63380
Q07	(1)	.100000	00000	00000

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Appendix A

A Short Table of Integrals

From the G -function representation (2.7) and integrals listed in [21], it is possible to generate a table of very general integrals, which contain many useful results as special cases, all of which are believed to be new. A short list follows.

$$(A.1) \quad \int_0^\infty dt \, t^\gamma E_\mu^j(\alpha t) E_\nu^k(\beta t) \\ = \beta^{-\gamma-1} G_{j+k+3, j+k+3}^{j+2, k+2} \left(\frac{\alpha}{\beta} \middle| -\gamma, 1-\nu-\gamma, \dots, 1-\nu-\gamma; \mu, \dots, \mu \right).$$

Special case ($j = -1$)

$$(A.1a) \quad \int_0^\infty t^\gamma e^{-\alpha t} E_\nu^k(\beta t) dt \\ = \beta^{-\gamma-1} \sum_{l=0}^\infty \frac{\Gamma(1+\gamma+l)}{\Gamma(1+l)} \left(\frac{1}{\nu+\gamma+l} \right)^{k+1} \left(\frac{-\alpha}{\beta} \right)^l \quad \text{if } |\alpha| \leq |\beta|.$$

Special case ($k = \gamma = 0, \nu = \alpha = \beta = 1, \mu = m$; [20], [24])

$$(A.1b) \quad \int_0^\infty E_1(t) E_m^j(t) dt \\ = \frac{(-1)^j j!}{m^{j+1}} \left\{ \frac{1}{2} \sum_{l=0}^j \frac{(-1)^l}{l!} \left(\frac{m}{2} \right)^l \left[\psi' \left(\frac{m+1}{2} \right) - \psi' \left(\frac{m}{2} \right) \right] + \log 2 \right\}.$$

$$(A.2) \quad \int_0^\infty t^\gamma E_\nu^k(\beta t) dt = \beta^{-\gamma-1} \Gamma(1+\gamma)(\nu+\gamma)^{-k-1}.$$

$$(A.3) \quad \int_0^\infty t^{\gamma-1} (t+\beta)^{-\sigma} E_\nu^j(z t) dt \\ = \frac{\beta^{\gamma-\sigma}}{\Gamma(\sigma)} G_{j+2, j+3}^{j+3, 1} \left(\beta z \middle| \begin{matrix} 1-\gamma; \nu, \dots, \nu \\ \sigma-\gamma, 0, \nu-1, \dots, \nu-1 \end{matrix} \right).$$

Special case ($\gamma = 1, \sigma = 1 + \nu, j = 0$)

$$(A.3a) \quad \int_0^\infty (t+\beta)^{-\nu-1} E_\nu(z t) dt = \beta^{-\nu} \exp(\beta z) E_\nu(\beta z) / \nu.$$

Special case ($\sigma = 2, \gamma = 2 - \nu, j = 0$)

$$(A.3b) \quad \int_0^\infty t^{1-\nu} (t+\beta)^{-2} E_\nu(z t) dt = \exp(\beta z) z^{\nu-1} \Gamma(2-\nu) E_{2-\nu}(\beta z),$$

$$(A.4) \quad \int_0^\infty {}_2F_1 \left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix} \middle| -wt \right) t^\delta E_\nu^j(z t) dt \\ = \frac{z^{-\delta}}{w} \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} G_{j+3, j+4}^{j+4, 1} \left(\frac{z}{w} \middle| \begin{matrix} 0; \gamma-1, \nu+\delta, \dots, \nu+\delta \\ \delta, \nu+\delta-1, \dots, \nu+\delta-1, \alpha-1, \beta-1 \end{matrix} \right).$$

Special case ($\alpha = \beta = \delta = 1, \gamma = 2$)

$$(A.4a) \quad \int_0^\infty \log(1 + wt) E_\nu^j(z) dt = \frac{1}{z} G_{j+2,j+3}^{j+3,1} \left(\frac{z}{w} \middle| 0; 1 + \nu, \dots, 1 + \nu \right).$$

Special case ($\beta = \alpha - \frac{1}{2}, \gamma = 2\alpha, \delta = \alpha - 1 - \nu$)

$$(A.4b) \quad \int_0^\infty [1 + \sqrt{1 + wt}]^{1-2\alpha} t^{\alpha-1-\nu} E_\nu^j(z) dt \\ = \frac{z^{\nu+1-\alpha} (\alpha - \frac{1}{2})}{w\sqrt{\pi}} G_{j+2,j+3}^{j+3,1} \left(\frac{z}{w} \middle| 0; 2\alpha - 1, \alpha - 1, \dots, \alpha - 1 \right. \\ \left. \alpha - 1 - \nu, \alpha - \frac{3}{2}, \alpha - 2, \dots, \alpha - 2; \right),$$

$$(A.5) \quad \int_0^\infty \cos(\alpha t) E_\nu^j(\beta t) dt \\ = \frac{2^{-j-1}}{\alpha} G_{j+2,j+2}^{j+2,1} \left(\frac{\beta^2}{\alpha^2} \middle| \frac{1}{2}; (\nu + 1)/2, \dots, (\nu + 1)/2 \right)$$

$$(A.5a) \quad = \frac{2^{-j-1}}{\beta} \sum_{l=0}^\infty \left(\frac{1}{\nu/2 + l} \right)^{j+1} \left(\frac{-\alpha^2}{\beta^2} \right)^l \quad \text{if } |\alpha| \leq |\beta|,$$

$$(A.6) \quad \int_0^\infty \sin(\alpha t) E_\nu^j(\beta t) dt = \frac{2^{-j-1}}{\alpha} G_{j+2,j+2}^{j+2,1} \left(\frac{\beta^2}{\alpha^2} \middle| 0; (\nu + 1)/2, \dots, (\nu + 1)/2 \right)$$

$$(A.6a) \quad = \frac{2^{-j-1}\alpha}{\beta^2} \sum_{l=0}^\infty \left(\frac{1}{(1 + \nu)/2 + l} \right)^{j+1} \left(\frac{-\alpha^2}{\beta^2} \right)^l \quad \text{if } |\alpha| \leq |\beta|.$$

$$(A.7) \quad \int_1^\infty t^{-\alpha} (t - 1)^{\alpha-\beta-1} E_s^j(z) dt \\ = \Gamma(\alpha - \beta) G_{j+2,j+3}^{j+3,0} \left(z \middle| s, \dots, s, \alpha \right. \\ \left. 0, \beta, s - 1, \dots, s - 1; \right).$$

Special case ($s = -n, \alpha = 1 - l, \beta = l$)

$$(A.7a) \quad \int_1^\infty t^{l-1} E_{-n}^j(z) dt \\ = \frac{\Gamma(n + 1)}{z^{n+1}} \left[\sum_{k=0}^{n-j} \frac{z^k}{k!} \xi_{k,n}^j E_{n+2-k-l}(z) + E_{n+1-l}(z) \sum_{m=1}^j \xi_{0,n}^{j-m} \left(\frac{1}{n-l} \right)^m \right. \\ \left. - \sum_{m=1}^j \sum_{i=0}^{m-1} (n-l)^{i-m} \xi_{0,n}^{j-m} E_1^i(z) \right] \\ \text{if } l \neq n, n + 1,$$

$$(A.7b) \quad \int_1^\infty t^{n-1} E_{-n}^j(z) dt \\ = \frac{\Gamma(n + 1)}{z^{n+1}} \left[\sum_{k=0}^{n-j} \frac{z^k}{k!} \xi_{k,n}^j E_{2-k}(z) + \sum_{m=1}^j \xi_{0,n}^{m-1} E_1^{j-m+1}(z) \right].$$

For the special case $l = n + 1$, see Eq. (2.12).

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