Some Periodic Continued Fractions With Long Periods

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Abstract. Let $p(D)$ be the period length of the continued fraction for $\sqrt{D}$. Under the extended Riemann Hypothesis for $\mathcal{Q}(\sqrt{D})$ one would expect that $p(D) = O(D^{1/2} \log \log D)$. In order to test this it is necessary to find values of D for which $p(D)$ is large. This, in turn, requires that we be able to find solutions to large sets of simultaneous linear congruences. The University of Manitoba Sieve Unit (UMSU), a machine similar to D. H. Lehmer's DLS-127, was used to find such values of D . For example, if $D = 46257585588439$, then $p(D) = 25679652$. Some results are also obtained for the Voronoi continued fraction for $\sqrt[3]{D}$.

1. Introduction. Let D be any positive integer. In Williams [7] it was pointed out that if D is square-free, then $p(D)$, the period length of the continued fraction expansion of $\sqrt{D}$, should be bounded above by an expression of the form $cD^{1/2} \log \log D$. In fact, if

$$f(D) = \begin{cases} D^{1/2} \log \log D & \text{for } D \equiv 1 \pmod{8}, \\ D^{1/2} \log \log 4D & \text{otherwise,} \end{cases}$$

we should have

$$(1.1) \quad G(D) = p(D)/f(D) < k + o(1)$$

under the extended Riemann Hypothesis for $\zeta_{\mathcal{K}}$ when $\mathcal{K} = \mathcal{Q}(\sqrt{D})$. Here $k = 3.7012$, but we expect by Lévy's Law that the smaller value $12e^\gamma \log 2/\pi^2 \approx 1.50103$ could be used for k . In [7] values of D ($< 2 \times 10^9$) were examined in order to find large values of $G(D)$. The largest value found was that of $G(D) = 1.040452$ for $D = 1492180699$. In this paper we describe a further attempt to find values of D for which $G(D)$ is large. We also describe some analogous work done in the case of Voronoi's algorithm in $\mathcal{Q}(\sqrt[3]{D})$.

2. Numerical Results. A glance at the results and tables given in [7] reveals that, in order to find values of D for which $G(D)$ is likely to be large, one should examine integers of the form q or $2q$, where q is a prime and $q \equiv -1 \pmod{4}$. Further, if r_i is the i th odd prime, one should also attempt to have the Legendre symbols

$$(2.1) \quad (D/r_i) = 1 \quad (i = 1, 2, 3, \dots, n)$$

for as large a value of n as possible. Thus, for each such r_i we would want D to belong to one of the $(r_i - 1)/2$ congruence classes such that $(D/r_i) = 1$. To find such values of D requires that we find solutions of large numbers of simultaneous

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linear congruences, a problem best solved by using a number sieve (see Lehmer [3]). In Patterson and Williams [5] a very fast version of such a device, called the University of Manitoba Sieve Unit (UMSU), is described. This machine will solve such systems of congruences at the rate of 1.33×10^8 trials at a solution per second.

We searched for values of D of four different types:

- (i) $D \equiv 3 \pmod{8}$ D prime,
- (ii) $D \equiv 7 \pmod{8}$ D prime,
- (iii) $D \equiv 6 \pmod{8}$ $D/2$ prime,
- (iv) $D \equiv 1 \pmod{8}$ D prime.

We examined values of D of type (iv) to determine whether values of $G(D)$ would, as predicted by Shanks, tend to catch up to the larger values obtained for the other types of D . For each value of n (1, 2, 3, ...) UMSU was programmed to search for the first m (at least) values of D of a given type. For $n \leq 32$ ($r_{32} = 137$), we used 50 as our value of m . Because of the amount of time needed to go farther, we cut this value down to 10 for $33 \leq n \leq 36$ ($r_{36} = 157$). In addition, for $n = 37$ we used $m = 7$ and $m = 8$ for D of type (i) and type (iii), respectively. For D of type (iv) we used $m = 10$ for $n = 37$ and $m = 4$ with $n = 38$ for D of type (ii).

After these numbers had been found, a job requiring many weeks of continuous use of UMSU, we computed the corresponding continued fraction period lengths $p(D)$ and the values of $G(D)$. We summarize our results in the four tables given below. We give only those numbers $D > 2 \times 10^9$. Also, we print D and its corresponding p - and G -values only when $G(D)$ exceeds the value of $G(d)$ for all of our computed values of d of the same type with $d < D$.

In Table 5 we present the values of D , from among those found by UMSU, with the largest $p(D)$ values. We give five such numbers for each D -type.

On examining these tables, we see that the values of $G(D)$ are certainly growing sufficiently slowly for (1.1) to hold. Further, the values of $G(D)$ for D of type (iv) seem to be slowly catching up to those values for the other D -types.

TABLE 1. D – Type (i)

D	$p(D)$	$G(D)$
2186009851	151838	1.037297
2287905811	155710	1.039131
7528121899	288198	1.043420
30738225571	603178	1.061828
614886781051	2794390	1.063448
1260977393659	4081590	1.076694
55400066448211	28076486	1.078532

In Table 6, we extend part of Table I of Lehmer, Lehmer and Shanks [4]. That is, for various values of n we give the least prime $D \equiv 1 \pmod{8}$ such that (2.1) holds. We also mention here that D. H. Lehmer had already found previously (but not published) the first six lines of this table.

TABLE 2. D – Type (ii)

D	p(D)	G(D)
2763423391	170804	1.034456
4912298119	230048	1.036883
5097972751	234768	1.038196
12095524039	366384	1.040132
19672399231	471320	1.042810
24880707679	536964	1.053362
50151351559	772360	1.058250
62324011759	864408	1.059728
492210358039	2519212	1.074069
4944598510471	8181752	1.075383
22542868742839	17739532	1.076772
46257585588439	25679652	1.081244

TABLE 3. D – Type (iii)

D	p(D)	G(D)
2340752254	157036	1.035754
7636279366	288766	1.037853
8813799094	312690	1.044133
8932573654	316434	1.049406
31416841054	611088	1.063790
6730689687166	9585044	1.076654
13518648471574	13732410	1.081381

TABLE 4. D – Type (iv)

D	$p(D)$	$G(D)$
18901431649	433383	.996329
22945498489	479525	.997981
23258723401	483919	1.000142
28467424441	540685	1.007395
37312059409	625233	1.013966
40094470441	653345	1.021500
163965430561	1348681	1.024427
192052219969	1473213	1.032023
2570329924369	5552441	1.033038
2871842842801	5924695	1.041624
8103297298321	10135403	1.049695
457165855430761	79417945	1.055462

3. Some Analogous Results for $\sqrt[3]{D}$. It is well-known that the regular continued fraction expansion of $\sqrt[3]{D}$ is never periodic; however, Voronoi's [6] continued fraction is periodic for cubic irrationalities. Let $\mathcal{K} = \mathcal{Q}(\sqrt[3]{D})$ be the pure cubic field formed by adjoining $\sqrt[3]{D}$ to the rationals $\mathcal{Q}$, and let Δ be the discriminant of $\mathcal{K}$. Then, if D is cube-free and $D = ab^2$ with $(a, b) = 1$, we have

$$\Delta = \begin{cases} -3a^2b^2 & \text{when } a^2 \equiv b^2 \pmod{9}, \\ -27a^2b^2 & \text{otherwise.} \end{cases}$$

If ε_0 is the fundamental unit of $\mathcal{K}$, $R (= \log \varepsilon_0)$ the regulator of $\mathcal{K}$, and P the period of Voronoi's continued fraction, then by (8.3) of Williams [8], we get

$$(3.1) \quad R > [P/4] \log 2.$$

Unfortunately, we do not yet have a rule like Lévy's for this case, but it seems from empirical evidence that

$$(3.2) \quad R \approx vP,$$

TABLE 5

Type	D	p(D)	G(D)
(i)	152290419440611	46274886	1.062983
	165427035605659	48190146	1.061386
	206546921647291	54350198	1.069334
	215226414830491	54450146	1.049121
	300272328240091	65344634	1.063030
(ii)	133051755648751	42848636	1.054226
	142368153139039	44889152	1.067078
	146936775525439	45349180	1.060843
	166290530163319	48736480	1.070583
	174346066249111	49611996	1.063923
(iii)	246406633037854	57923528	1.041889
	256397742215806	60536004	1.067108
	285278695393246	64119584	1.070606
	301938138430366	64551980	1.047187
	350240722763374	70400728	1.059121
(iv)	229297977151681	54793321	1.034296
	259853252349289	58673599	1.039268
	273323976657169	60545353	1.045206
	366525636221761	69241975	1.029650
	457165855430761	79417945	1.055462

where $1.12 < v < 1.13$. Thus, if we can bound R , we can certainly get a result like (1.1).

If h is the class number of $\mathcal{K}$, we have

$$(3.3) \quad hR = \frac{\sqrt{|\Delta|}}{2\pi} \Phi(1),$$

where

$$\Phi(1) = \lim_{s \rightarrow 1} \zeta_{\mathcal{K}}(s) / \zeta(s) = \prod_q f(q).$$

TABLE 6

r_n	Least D
83	8114538721
89	9176747449
97, 101, 103	23616331489
107, 109, 113, 127	196265095009
131, 137, 139	2871842842801
149	26437680473689
151	89436364375801
157, 163, 167	112434732901969
173, 179	178936222537081

Here the (Euler) product is taken over all the primes q , and $f(q)$ is given below:

$$\begin{aligned}
 f(3) &= \begin{cases} 3/2 & \text{when } a^2 \equiv b^2 \pmod{9}, \\ 1 & \text{otherwise;} \end{cases} \\
 f(q) &= 1 \quad \text{when } q \nmid ab; \\
 &\text{if } q \equiv -1 \pmod{3} \text{ and } q \nmid ab, \quad \text{then } f(q) = q^2/(q^2 - 1); \\
 &\text{if } q \equiv 1 \pmod{3} \text{ and } q \nmid ab, \quad \text{then} \\
 f(q) &= \begin{cases} q^2/(q-1)^2 & \text{when } (D/q)_3 = 1, \\ q^2/(q^2 + q + 1) & \text{otherwise.} \end{cases}
 \end{aligned}$$

If we use the symbol $\prod_f^Q$ to denote the product over all primes less than or equal to Q and $\equiv j \pmod{3}$, and if we denote by $T(Q, D)$ the infinite product

$$\prod_{\substack{q > Q \\ q \equiv 1 \pmod{3}}} f(q)$$

taken over all the primes exceeding Q and $\equiv 1 \pmod{3}$, then, since the infinite product

$$\prod_{q \equiv -1 \pmod{3}} f(q)$$

taken over all the primes $\equiv -1 \pmod{3}$ converges, we have

$$\Phi(1) = \left(f(3) \prod_{-1}^Q f(q) \prod_1^Q f(q) \right) T(Q, D)(1 + o(1)).$$

Now

$$\prod_1^Q f(q) \leq \prod_1^Q q^2/(q-1)^2;$$

hence

$$\prod_{-1}^Q f(q) \prod_1^Q f(q) \leq (2/3) \prod_{-1}^Q q/(q-1) \prod_1^Q q/(q-\chi(q)),$$

where each of the products on the right-hand side is evaluated over all the primes $\leq Q$ and $\chi(q) = (-3/q)$. By Mertens' theorem

$$\prod_{-1}^Q q/(q-1) = e^\gamma \log Q (1 + o(1)).$$

Also, since

$$L(1, \chi) = \prod_q q/(q - \chi(q)) = \frac{\pi}{3\sqrt{3}}$$

(the product taken over all the primes q), we get

$$(3.4) \quad \Phi(1) \leq \frac{2\pi e^\gamma f(3)}{9\sqrt{3}} (\log Q) T(Q, D) (1 + o(1)).$$

If $\mathcal{E}$ is the extension $\mathcal{K}(\omega)$ of $\mathcal{K}$, where $\omega^2 + \omega + 1 = 0$, then the discriminant d of $\mathcal{E}$ is $3\Delta^2$ (see Barrucand and Cohn [1]). If we put

$$U(D) = T((\log d)^2, D),$$

then $U(D) < 1 + o(1)$ under the extended Riemann Hypothesis for $\zeta_{\mathcal{E}}$ (see, for example, Williams, Dueck and Schmid [9, pp. 282–283]). Combining this result and (3.4) with $Q = (\log d)^2$, we get

$$\Phi(1) < \frac{4\pi e^\gamma f(3)}{9\sqrt{3}} \log \log(3\Delta^2) (1 + o(1)).$$

It follows from (3.3) that

$$(3.5) \quad hR < \frac{2e^\gamma f(3)\sqrt{|\Delta|/3}}{9} \log \log(3\Delta^2) (1 + o(1)).$$

When, for example, D is square-free, then

$$(3.6) \quad hR < \begin{cases} (1/3)e^\gamma D \log \log 3^3 D^4 (1 + o(1)) & \text{when } D \equiv \pm 1 \pmod{9}, \\ (2/3)e^\gamma D \log \log 3^7 D^4 (1 + o(1)) & \text{otherwise.} \end{cases}$$

4. Further Numerical Results. From (3.3) we see that in order to maximize R we must minimize h and get $\Phi(1)$ as large as possible. Of the possibilities for D square-free, $D \not\equiv \pm 1 \pmod{9}$ and $3 \nmid h$ (see Honda [2]) we elected to examine prime values of $D \equiv 2$ or $5 \pmod{9}$. If r_i is the i th prime of the form $1 + 3t$, then the prime D values which should give large $\Phi(1)$ values are those for which $(D/r_i)_3 = 1$ ($i = 1, 2, 3, \dots, n$) for as large a value of n as possible. We now encounter a difficulty, however. The determination of P is very expensive for rather modest values of D (say ≈ 200000); thus, we decided to look at the values of R instead. By

using the methods described in [9] we can calculate R much more rapidly than P ; but, it still becomes very expensive to find R when $D > 2 \times 10^9$. (It should, of course, be borne in mind that the discriminants for such values of D are very large, exceeding 10^{20} .)

UMSU was programmed to find the first 50 values of D for each n until a value of n was reached for which the least of these 50 numbers exceeded $2^{31} - 1$, the word size of the machine used to compute R —an AMDAHL 470-V8. We then computed R for each D and $C(D) = R/(D \log \log(3^7 D^4))$. In the cubic case it takes very little time to find the D values and a much larger amount of time to find the R values, the reverse of the situation in the quadratic case.

Our results are summarized in the following tables. In Tables 7 and 8 we give only those values of D for which $C(D)$ exceeds the C values for any of the other numbers that we found which were less than D . In Tables 9 and 10 we give values of D for which the corresponding regulator exceeds any of those previously found. Since $2e^\gamma/3 \approx 1.18738$, we have nothing here that comes near to violating the Riemann Hypothesis for ζ_g . Also the growth of $C(D)$ is slow and getting slower as D increases.

TABLE 7. $D \equiv 2 \pmod{9}$

D	R	C(D)
29	40.27082	.454983
1721	3669.37913	.588309
39521	92172.43814	.596085
92009	218706.73901	.597544
343433	895028.71553	.640002
6616667	18089884.90792	.642420
7202369	19994005.36092	.651564
202306187	586455162.98256	.653911
562788101	1689849729.97072	.670149

TABLE 8. $D \equiv 5 \pmod{9}$

D	R	C(D)
41	56.28937	.440672
239	431.94224	.533495
1301	2549.94344	.545373
4523	9440.96250	.560767
19391	42811.86808	.572868
67829	154494.32105	.575923
72617	168197.50896	.584893
143879	361610.34278	.626614
1145327	3021373.73848	.635515
8596463	23331608.01905	.635544
8666393	23925356.23751	.646390
48487811	139358465.15040	.658771
55570523	163251776.10755	.672292
60435383	179011355.42037	.677194

TABLE 9. $D \equiv 2 \pmod{9}$

D	R	C(D)
689816063	1888303399.286361	.609701902921
780923333	2040735586.012364	.581327045940
807748787	2264449384.076498	.623423186470
911130401	2663628567.917647	.649341195701
947294867	2666732555.238140	.625039976249
1039506833	2941248070.747570	.627656937643
1090062947	3194601736.597826	.649803055418
1250773679	3481668991.375506	.616374865552
1345747619	3810698517.456939	.626570574715
1411121837	3967734472.270628	.621882987760
1627729013	4492140541.726865	.609547346456
1695130949	5107010533.454052	.665168464608
2044171163	5464205375.038442	.589124377005

TABLE 10. $D \equiv 5 \pmod{9}$

D	R	C(D)
78446831	195588785.889993	.568325353629
85474661	229039192.818766	.610210176803
140795537	399133674.604591	.641989245305
172132241	499066717.859134	.655156175458
226496759	590664701.273444	.587554486086
230154107	639837059.815727	.626247127877
246667721	681500286.460078	.621912563185
258947807	705470723.780494	.612941643650
262192559	708275870.201634	.607683348911
267667889	739791831.127160	.621604001102
313154087	892931349.895030	.640243198576
613655951	1737626269.841794	.631422697999
641290649	1782982379.770936	.619710166881
671319221	1958397914.780726	.649933701273
736002077	2030844203.759158	.614180052846
784288049	2197173781.242724	.623178495983
789581183	2210040440.336767	.622584176723
792812201	2226806639.150783	.624725493228
860248787	2595846960.356864	.670626975448
914070821	2652552259.996093	.644540413185
948371243	2660317585.609716	.622821760536
957302429	2743025183.193182	.636135024907
1400879507	4155081949.704781	.656054383521
1617735209	4322176122.273822	.590142344485
1632061859	4549031363.516906	.615612610599
1827261311	4810329644.671832	.580807712394
1831479161	5515724098.441698	.664430018640
2108312123	5713478707.454342	.597085411322
2124689657	6127255313.478815	.635344945017

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