Evaluation of Howland-Type Integrals Involving $\tanh$ and $\coth$ Functions

By Chih-Bing Ling

Abstract. Four Howland-type integrals involving $\tanh$ and $\coth$ functions in their integrands are evaluated in this paper. The results are tabulated to 20D.

This paper presents tables of 20D values of the following four integrals of Howland type involving $\tanh$ and $\coth$ functions:

$$(1) \quad \begin{aligned} III_k &= \frac{2^k}{k!} \int_0^\infty \frac{x^k \tanh x \, dx}{\sinh 2x \pm 2x}, & (k \geq 0), \\ III_k^* &= \frac{2^k}{k!} \int_0^\infty \frac{x^k \coth x \, dx}{\sinh 2x \pm 2x}, & (k \geq 2), \\ IV_k &= \frac{2^k}{k!} \int_0^\infty \frac{x^k \coth x \, dx}{\sinh 2x \pm 2x}, & (k \geq 2), \\ IV_k^* &= \frac{2^k}{k!} \int_0^\infty \frac{x^k \coth x \, dx}{\sinh 2x \pm 2x}, & (k \geq 4), \end{aligned}$$

which are encountered in mathematical elasticity. These integrals can be evaluated in terms of the following integrals and series:

$$(2) \quad \begin{aligned} I_k &= \frac{2^k}{k!} \int_0^\infty \frac{x^k \, dx}{\sinh 2x \pm 2x}, & (k \geq 1), \\ I_k^* &= \frac{2^k}{k!} \int_0^\infty \frac{x^k \, dx}{\sinh 2x \pm 2x}, & (k \geq 3), \\ II_k &= \frac{2^k}{k!} \int_0^\infty \frac{x^k e^{-2x} \, dx}{\sinh 2x \pm 2x}, & (k \geq 1), \\ II_k^* &= \frac{2^k}{k!} \int_0^\infty \frac{x^k e^{-2x} \, dx}{\sinh 2x \pm 2x}, & (k \geq 3), \\ S_k &= \sum_{n=1}^{\infty} \frac{1}{n^k}, & (k \geq 2), \\ U_k &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^k}, & (k \geq 2), \\ s_k &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^k}, & (k \geq 1), \end{aligned}$$

where the four integrals are ordinary Howland integrals. Their relations are

$$(3) \quad \begin{aligned} III_k &= I_k + (I_{k-1} - II_{k-1} - s_k)/k, & (k \geq 2), \\ III_k^* &= I_k^* - (I_{k-1}^* - II_{k-1}^* - s_k)/k, & (k \geq 4), \\ IV_k &= I_k - (I_{k-1} + II_{k-1} - S_k)/k, & (k \geq 2), \\ IV_k^* &= I_k^* + (I_{k-1}^* + II_{k-1}^* - S_k)/k, & (k \geq 4). \end{aligned}$$

Received October 25, 1984.

1980 *Mathematics Subject Classification.* Primary 65A05, 65D20, 65D30.

Key words and phrases. Howland integrals, Howland-type integrals.

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The integrals III_0 , III_1 , III_2^* and III_3^* are not covered by (3), but they can be separately evaluated as

$$\begin{aligned}
 (4) \quad III_0 &= \sum_{n=1}^{\infty} \frac{1}{2n} (U_{2n} - I_{2n-1}), \\
 III_1 &= \sum_{n=1}^{\infty} (U_{2n+1} - I_{2n}), \\
 III_2^* &= \sum_{n=1}^{\infty} \frac{2n+1}{2} (I_{2n+1}^* - U_{2n+2}), \\
 III_3^* &= \sum_{n=1}^{\infty} \frac{(2n+1)(2n+2)}{6} (I_{2n+2}^* - U_{2n+3}).
 \end{aligned}$$

The following numerical tables are available: (i) 32D values of S_k , U_k and s_k [1], (ii) 25D values of I_k and I_k^* [3], [4], and (iii) 20D values of II_k and II_k^* [5]. With these tables of high precision, the integrals in (3) can be computed to 20D and those in (4) to 21D. The following formulas may be used for checking purposes:

$$\begin{aligned}
 (5) \quad & \sum_{k=1}^{\infty} \frac{1}{2} (III_{2k+1}^* - III_{2k+1}) = III_1, \\
 & \sum_{k=2}^{\infty} \frac{k}{2} (III_{2k}^* - III_{2k}) = III_2, \\
 & \sum_{k=2}^{\infty} \frac{k}{2} (IV_{2k}^* - IV_{2k}) = IV_2, \\
 & \sum_{k=2}^{\infty} \frac{k(2k+1)}{6} (IV_{2k+1}^* - IV_{2k+1}) = IV_3.
 \end{aligned}$$

Or

$$\begin{aligned}
 (6) \quad & 2 III_1 + \sum_{k=2}^{\infty} (-1)^k k (1 - III_k) = \frac{3}{4} + I_1 - II_1, \\
 & 2 III_3^* - \sum_{k=4}^{\infty} (-1)^k \binom{k}{3} (III_k^* - 1) = \frac{15}{16} + I_3^* - II_3^*, \\
 & \sum_{k=2}^{\infty} (-1)^k k (1 - IV_k) = \frac{3}{4} - I_1 - II_1, \\
 & \sum_{k=4}^{\infty} (-1)^k \binom{k}{3} (IV_k^* - 1) = I_3^* + II_3^* - \frac{15}{16}.
 \end{aligned}$$

It is noted that the evaluation of the four integrals in question was considered before by the author. The reader may consult the author's previous paper [2] for derivations of the relations in (3) and the first two relations in (4). The other two relations in (4) are new. Also, the second set of checking formulas as in (6) is new. The results are shown in the accompanying tables to 20D for k up to 60. By developing the integrands of the integrals into series of $\exp(-2x)$, we find

$$\begin{aligned}
 (7) \quad & III_k = 1 \mp \frac{k+3}{2^{k+1}} + \frac{4k^2 + 24k + 47}{3^{k+3}} \mp \dots, \\
 & IV_k = 1 \mp \frac{k-1}{2^{k+1}} + \frac{4k^2 + 23}{3^{k+3}} \mp \dots, \\
 & III_k^* = \dots
 \end{aligned}$$

TABLE I
*Values of III_k and III_k^**

k	III_k				III_k^*			
0	0.29661	99339	44595	97112	-	-	-	-
1	0.47442	96568	44400	38418	-	-	-	-
2	0.63084	14007	56394	71239	2.13561	79285	77824	94912
3	0.75383	61380	88364	70967	1.41506	33609	72854	32434
4	0.84283	92075	97609	05654	1.19555	30856	62227	22928
5	0.90323	68041	38941	72530	1.10049	26035	57012	36691
6	0.94219	18007	41668	47549	1.05357	96662	57560	48355
7	0.96631	46744	35813	14361	1.02903	43071	42502	10827
8	0.98077	05523	77242	24405	1.01583	42008	66031	23633
9	0.98920	65773	86084	21391	1.00864	67795	84489	60628
10	0.99402	49922	06105	00479	1.00471	53780	14052	97031
11	0.99672	96737	31447	25230	1.00256	42740	49514	14013
12	0.99822	66146	53647	40993	1.00138	96013	41781	18900
13	0.99904	56852	39138	46194	1.00075	01799	98261	39463
14	0.99948	96912	34774	36906	1.00040	34288	02376	36775
15	0.99972	85510	46781	31800	1.00021	61385	29685	15785
16	0.99985	62445	06325	07098	1.00011	53807	26827	69980
17	0.99992	41536	98507	71646	1.00006	13846	38756	86013
18	0.99996	01111	69596	89982	1.00003	25543	06883	72782
19	0.99997	90799	94077	83888	1.00001	72138	31286	34127
20	0.99998	90549	89633	34142	1.00000	90774	16519	06784
21	0.99999	42860	51251	56941	1.00000	47747	76000	10268
22	0.99999	70227	03086	80599	1.00000	25057	29053	29209
23	0.99999	84513	39381	89658	1.00000	13121	49875	40531
24	0.99999	91957	18898	51663	1.00000	06857	60310	13140
25	0.99999	95829	04425	04331	1.00000	03577	38642	20147
26	0.99999	97839	82165	46273	1.00000	01863	04385	46698
27	0.99999	98882	58781	31913	1.00000	00968	71860	10514
28	0.99999	99422	64228	25313	1.00000	00502	96543	86347
29	0.99999	99701	99890	33980	1.00000	00260	78862	81209
30	0.99999	99846	33962	08112	1.00000	00135	04829	85857
31	0.99999	99920	84035	76641	1.00000	00069	85151	40249
32	0.99999	99959	25561	81941	1.00000	00036	08957	36474
33	0.99999	99979	04558	79829	1.00000	00018	62674	34056
34	0.99999	99989	23170	45323	1.00000	00009	60436	72970
35	0.99999	99994	47031	50379	1.00000	00004	94768	76335
36	0.99999	99997	16239	15663	1.00000	00002	54659	80169
37	0.99999	99998	54481	37493	1.00000	00001	30967	68945
38	0.99999	99999	25421	61908	1.00000	00000	67302	76695
39	0.99999	99999	61801	28710	1.00000	00000	34560	85449
40	0.99999	99999	80445	88647	1.00000	00000	17735	16625
41	0.99999	99999	89995	56615	1.00000	00000	09094	95387
42	0.99999	99999	94884	09505	1.00000	00000	04661	16274
43	0.99999	99999	97385	20369	1.00000	00000	02387	42443
44	0.99999	99999	98664	17999	1.00000	00000	01222	13380
45	0.99999	99999	99317	87909	1.00000	00000	00625	27771
46	0.99999	99999	99651	83410	1.00000	00000	00319	74427
47	0.99999	99999	99822	36433	1.00000	00000	00163	42484
48	0.99999	99999	99909	40581	1.00000	00000	00083	48878
49	0.99999	99999	99953	81472	1.00000	00000	00042	63257
50	0.99999	99999	99976	46327	1.00000	00000	00021	76037
51	0.99999	99999	99988	00959	1.00000	00000	00011	10223
52	0.99999	99999	99993	89377	1.00000	00000	00005	66214
53	0.99999	99999	99996	89138	1.00000	00000	00002	88658
54	0.99999	99999	99998	41793	1.00000	00000	00001	47105
55	0.99999	99999	99999	19509	1.00000	00000	00000	74940
56	0.99999	99999	99999	59061	1.00000	00000	00000	38164
57	0.99999	99999	99999	79183	1.00000	00000	00000	19429
58	0.99999	99999	99999	89418	1.00000	00000	00000	09888
59	0.99999	99999	99999	94622	1.00000	00000	00000	05031
60	0.99999	99999	99999	97268	1.00000	00000	00000	02559

TABLE 2
*Values of IV_k and IV_k^**

k	IV_k				IV_k^*			
2	1.09596	74127	81010	85878	-	-	-	-
3	0.94313	77055	08848	67791	-	-	-	-
4	0.93632	30754	57122	67709	1.70756	94031	43097	55236
5	0.95164	35866	53520	53556	1.24000	07445	07150	12412
6	0.96751	54936	93683	27020	1.10538	54294	02360	29909
7	0.97953	75278	14738	14097	1.05082	23685	77313	70040
8	0.98763	45988	74063	43243	1.02560	82341	55080	92253
9	0.99274	59473	93955	13798	1.01319	92806	69181	95449
10	0.99583	85646	54282	37673	1.00688	59990	48109	07772
11	0.99765	38966	76473	56783	1.00361	50333	53774	75744
12	0.99869	54997	30458	46138	1.00190	34159	95424	91800
13	0.99928	27114	09536	83414	1.00100	31566	44122	71579
14	0.99960	91749	46440	43910	1.00052	85675	09881	64186
15	0.99978	86529	35058	24499	1.00027	82413	68014	66299
16	0.99988	64275	65132	83220	1.00014	62701	91356	62952
17	0.99993	92932	65386	26683	1.00007	67729	99299	90470
18	0.99996	76982	83520	95590	1.00004	02288	37576	83404
19	0.99998	28797	61580	86957	1.00002	10442	43279	59493
20	0.99999	09570	86200	06940	1.00001	09902	32217	83127
21	0.99999	52378	84050	98006	1.00000	57303	50517	17140
22	0.99999	74988	96428	71618	1.00000	29832	25992	41811
23	0.99999	86895	33451	09555	1.00000	15507	97293	91173
24	0.99999	93148	50075	46958	1.00000	08050	48882	93785
25	0.99999	96424	81947	61774	1.00000	04173	70724	82992
26	0.99999	98137	75087	63794	1.00000	02161	16192	95492
27	0.99999	99031	56689	86685	1.00000	01117	76296	71740
28	0.99999	99497	13685	05045	1.00000	00577	48254	35475
29	0.99999	99739	24792	88887	1.00000	00298	04542	48757
30	0.99999	99864	96473	60984	1.00000	00153	67609	06208
31	0.99999	99930	15312	34737	1.00000	00079	16520	08614
32	0.99999	99963	91207	29193	1.00000	00040	74634	49746
33	0.99999	99981	37384	00941	1.00000	00020	95510	42487
34	0.99999	99990	39583	91064	1.00000	00010	76853	91810
35	0.99999	99995	05238	52539	1.00000	00005	52977	06414
36	0.99999	99997	45342	76804	1.00000	00002	83763	85135
37	0.99999	99998	69033	21516	1.00000	00001	45519	67971
38	0.99999	99999	32697	55104	1.00000	00000	74578	75023
39	0.99999	99999	65439	25714	1.00000	00000	38198	84207
40	0.99999	99999	82264	87288	1.00000	00000	19554	15865
41	0.99999	99999	90905	05983	1.00000	00000	10004	44959
42	0.99999	99999	95338	84205	1.00000	00000	05115	91044
43	0.99999	99999	97612	57724	1.00000	00000	02614	79822
44	0.99999	99999	98777	86679	1.00000	00000	01335	82068
45	0.99999	99999	99374	72249	1.00000	00000	00682	12114
46	0.99999	99999	99680	25580	1.00000	00000	00348	16598
47	0.99999	99999	99836	57518	1.00000	00000	00177	63570
48	0.99999	99999	99916	51123	1.00000	00000	00090	59420
49	0.99999	99999	99957	36744	1.00000	00000	00046	18528
50	0.99999	99999	99978	23963	1.00000	00000	00023	53673
51	0.99999	99999	99988	89777	1.00000	00000	00011	99041
52	0.99999	99999	99994	33786	1.00000	00000	00006	10623
53	0.99999	99999	99997	11342	1.00000	00000	00003	10862
54	0.99999	99999	99998	52895	1.00000	00000	00001	58207
55	0.99999	99999	99999	25060	1.00000	00000	00000	80491
56	0.99999	99999	99999	61836	1.00000	00000	00000	40939
57	0.99999	99999	99999	80571	1.00000	00000	00000	20817
58	0.99999	99999	99999	90112	1.00000	00000	00000	10582
59	0.99999	99999	99999	94969	1.00000	00000	00000	05378
60	0.99999	99999	99999	97441	1.00000	00000	00000	02732

These series are useful in evaluation when k is large. The first two terms give 20D values when $k \geq 50$, 15D values when $k \geq 39$, or 10D values when $k \geq 28$. If the first three terms are used, they give 20D, 15D, or 10D values when $k \geq 41$, 28, or 22, respectively.

Department of Mathematics
Virginia Polytechnic Institute and State University
Blacksburg, Virginia 24061

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