An Estimate of Goodness of Cubatures for the Unit Circle in $\mathbf{R}^2$

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Abstract. The Sarma-Eberlein estimate s_E is an estimate of goodness of cubature formulae for n -cubes defined as the integral of the square of the formula truncation error, over a function space provided with a measure. In this paper, cubature formulae for the unit circle in $\mathbf{R}^2$ are considered and an estimate of the above type is constructed with the desirable property of being compatible with the symmetry group of the circle.

1. Isometries and Two-dimensional Cubature Formulae. Let

$$(1.1) \quad S_2 = \{(x, y) \in \mathbf{R}^2: x^2 + y^2 \leq 1\}$$

be the unit circle in the two-dimensional Euclidean space $\mathbf{R}^2$ and let $\mathcal{U}(S_2)$ denote the symmetry group of S_2 . This group consists of all linear bijective maps $u: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ which preserve the Euclidean distance (that is, isometries of $\mathbf{R}^2$ leaving the origin invariant). Each element of $\mathcal{U}(S_2)$ can be identified with a 2×2 real orthogonal matrix and therefore

$$(1.2) \quad \mathcal{U}(S_2) = \{u_\alpha, u_\alpha \circ v; \alpha \in [0, 2\pi)\},$$

where u_α denotes the rotation of α radians around the origin and v is the reflection about any fixed straight line passing through the origin; thus

$$(1.3) \quad \begin{aligned} u_\alpha(x, y) &= (x \cos \alpha - y \sin \alpha, x \sin \alpha + y \cos \alpha), \\ v(x, y) &= (x, -y). \end{aligned}$$

Let $w(x, y)$ be a normalized weight function compatible with $\mathcal{U}(S_2)$, that is, a real positive continuous function in the interior of S_2 such that

$$(1.4) \quad \iint_{S_2} w(x, y) dx dy = 1 \quad \text{and} \quad w \circ u = w \quad \text{for all } u \in \mathcal{U}(S_2).$$

A cubature formula for the w -weighted circle S_2 has the form

$$(1.5) \quad I(f) = Q_N(f) + E(f),$$

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where

$$(1.6) \quad \begin{aligned} I(f) &= \iint_{S_2} w(x, y) f(x, y) dx dy, \\ Q_N(f) &= \sum_{i=1}^N A_i f(x_i, y_i), \quad (x_i, y_i) \in S_2, \end{aligned}$$

and the constants A_i are independent of f .

Let us consider a symmetry $u \in \mathcal{U}(S_2)$ acting on (1.5). Since $I(f \circ u) = I(f)$, it leads to another cubature formula

$$(1.7) \quad I(f) = Q'_N(f) + E'(f),$$

where

$$(1.8) \quad \begin{aligned} Q'_N(f) &= Q_N(f \circ u) = \sum_{i=1}^N A_i f(u(x_i, y_i)), \\ E'(f) &= E(f \circ u). \end{aligned}$$

Definition 1. For every $u \in \mathcal{U}(S_2)$, the cubature formulae (1.5) and (1.7) are said to be $\mathcal{U}(S_2)$ -equivalent or equivalent with respect to the symmetry group of S_2 .

The integration of a function on the w -weighted circle S_2 is independent of the pair of orthogonal axis OX, OY whose origin O lies in the center of the circle. Therefore, all $\mathcal{U}(S_2)$ -equivalent formulae have identical characteristics when they are considered as approximations of the operator I .

Therefore, any estimate of goodness for cubature formulae (1.5) should be compatible with the $\mathcal{U}(S_2)$ -equivalence relation, that is, all $\mathcal{U}(S_2)$ -equivalent formulae should have the same estimate of goodness. For instance, the degree of precision of a cubature formula (1.5) is an estimate compatible with $\mathcal{U}(S_2)$, because the space of polynomials of degree at most n is invariant under all the symmetries in (1.2).

The aim of this paper is to construct an $\mathcal{U}(S_2)$ -compatible estimate of goodness of cubature formulae for S_2 similar to that defined by V. L. N. Sarma in [3] for cubatures for the square.

The next section is devoted to recalling briefly some characteristics of the Sarma-Eberlein estimate that are useful for our purpose. A detailed exposition of the construction of this estimate can be found in [3], [4] and [5] and an excellent summary of these results in [6, pp. 188–192].

2. The Sarma-Eberlein Estimate of Goodness s_E . Let us consider the square

$$C_2 = \{(x, y) \in \mathbf{R}^2: |x| \leq 1, |y| \leq 1\}$$

and cubature formulae

$$(2.1) \quad I(f) = Q_N(f) + E(f),$$

where

$$(2.2) \quad \begin{aligned} I(f) &= \frac{1}{4} \iint_{C_2} f(x, y) dx dy, \\ Q_N(f) &= \sum_{i=1}^N A_i f(x_i, y_i), \quad (x_i, y_i) \in C_2. \end{aligned}$$

Sarma in [3], [4] defines the estimate of goodness of the cubature formula (2.1) as

$$(2.3) \quad s_E^2 = \int_{FS_\infty} E(f)^2 df,$$

where the integral is defined over the unit sphere of a normed space of functions provided with a measure defined as follows:

Let l_1 be the space of real sequences

$$(2.4) \quad f = \{f_{nk}; n = 0, 1, \dots; k = 0, 1, \dots, n\}$$

such that

$$(2.5) \quad \|f\|_1 = \sum_{n,k} |f_{nk}| < \infty; \quad n = 0, 1, \dots; k = 0, 1, \dots, n.$$

The unit sphere $S_\infty = \{f \in l_1: \|f\|_1 \leq 1\}$ is compact in the weak*-topology of l_1 , and an elementary integral defined for the weak*-continuous real functions on S_∞ can be extended by the Daniell process inducing a countably additive measure on S_∞ .

Among the properties of this measure, let us recall that

$$(2.6) \quad \int_{S_\infty} f_{nk} f_{ml} df = 0 \quad \text{if } (n, k) \neq (m, l),$$

$$(2.7) \quad \int_{S_\infty} f_{nk}^2 df = \frac{2^{n+2}}{(n+2)!(n+3)!} = q_n^2.$$

Real two-dimensional power series

$$(2.8) \quad f(x, y) = \sum_{n,k} f_{nk} x^{n-k} y^k; \quad n = 0, 1, \dots; k = 0, 1, \dots, n,$$

whose coefficients satisfy the condition

$$(2.9) \quad \|f\|_1 = \sum_{n,k} |f_{nk}| < \infty$$

converge uniformly and absolutely for all points $(x, y) \in C_2$.

The space Fl_1 of all functions defined by (2.8) and (2.9) can be identified with the sequence space l_1 and is dense in the space $\mathcal{C}(C_2)$ of all real continuous functions on C_2 with the uniform norm. This identification allows us to consider the above integral as an integral over the unit sphere FS_∞ of the function space Fl_1 .

The truncation error $E(f)$ of the cubature formula (2.1) is a continuous linear form over $\mathcal{C}(C_2)$ with the uniform norm and therefore also over Fl_1 with the $\|\cdot\|_1$ -norm. Using (2.6) and (2.7), it follows that the estimate s_E defined by (2.3) can be written as

$$(2.10) \quad s_E^2 = \sum_{n=0}^{\infty} q_n^2 e_n^2,$$

where q_n is defined in (2.7) and

$$(2.11) \quad e_n^2 = \sum_{k=0}^n E(x^{n-k} y^k)^2.$$

It should be noted that the identification of l_1 and Fl_1 is made through the monomials $x^{n-k} y^k$ and the use of these particular functions makes s_E compatible with $\mathcal{U}(C_2)$, the symmetry group of C_2 , in the sense described in the previous

section. In effect, $\mathcal{U}(C_2)$ consists of the eight symmetries

$$(2.12) \quad (x, y) \rightarrow (\pm x, \pm y); \quad (x, y) \rightarrow (\pm y, \pm x)$$

and the equalities

$$(2.13) \quad \begin{aligned} e_n^2 &= \sum_{k=0}^n E(x^{n-k}y^k)^2 = \sum_{k=0}^n E((\pm x)^{n-k}(\pm y)^k)^2 \\ &= \sum_{k=0}^n E((\pm y)^{n-k}(\pm x)^k)^2 \end{aligned}$$

imply that $\mathcal{U}(C_2)$ -equivalent cubature formulae have the same estimate of goodness s_E . Unfortunately, this estimate of goodness is not useful for cubature formulae (1.5), (1.6) for the unit circle S_2 , because it is not compatible with $\mathcal{U}(S_2)$, as can be computationally checked. For instance, taking $w(x, y) = 1/\pi$, the cubature formula (degree 3, 4 points) given by

$$(2.14) \quad Q_4(f) = \frac{1}{4} [f(\sqrt{2}/2, 0) + f(-\sqrt{2}/2, 0) + f(0, \sqrt{2}/2) + f(0, -\sqrt{2}/2)]$$

has an estimate of goodness $s_E = (-4)1.75032$, whereas the $\mathcal{U}(S_2)$ -equivalent formula (use a rotation of $\pi/4$ radians) given by

$$(2.15) \quad \begin{aligned} Q_4(f) &= \frac{1}{4} [f(1/2, 1/2) + f(-1/2, 1/2) \\ &\quad + f(1/2, -1/2) + f(-1/2, -1/2)] \end{aligned}$$

has an estimate of goodness $s_E = (-4)3.81547$.

3. An Estimate of Goodness of Cubatures for the Unit Circle. In the previous section, the sequence space l_1 was identified with the space of functions Fl_1 by using the family of monomials $\{x^{n-k}y^k; n = 0, 1, \dots; k = 0, 1, \dots, n\}$, but we can also identify l_1 with other subspaces of $\mathcal{C}(C_2)$ or $\mathcal{C}(S_2)$ by using other families of polynomials. For each n , let us denote

$$(3.1) \quad M_n = \{a_0x^n + a_1x^{n-1}y + \dots + a_ny^n; a_i \in \mathbf{R}\}$$

and let

$$(3.2) \quad \Phi_n = \{\varphi_{n0}, \dots, \varphi_{nn}\} \subset M_n$$

be a basis of M_n , i.e., $M_n = \text{span } \Phi_n$.

If the polynomials φ_{nk} satisfy

$$(3.3) \quad \|\varphi_{nk}\|_\infty = \max_{(x,y) \in S_2} |\varphi_{nk}(x, y)| \leq c; \quad n = 0, 1, \dots; k = 0, 1, \dots, n,$$

then the series

$$(3.4) \quad f(x, y) = \sum_{n,k} f_{nk} \varphi_{nk}(x, y)$$

whose coefficients satisfy (2.9) converge uniformly and absolutely for all points $(x, y) \in S_2$. If we denote $\Phi = \{\Phi_1, \Phi_2, \dots\}$, the space $Fl_1(\Phi)$ of all functions defined by (3.4) and (2.9) can be identified with the sequence space l_1 . Let us note that $Fl_1(\Phi)$ contains all real polynomials in two variables and therefore is dense in $\mathcal{C}(S_2)$ with the uniform norm.

This identification allows us to define, in a natural way, an estimate of goodness for cubatures (1.5) by

$$(3.5) \quad s_E^2(\Phi) = \int_{FS_\infty(\Phi)} E(f)^2 df,$$

where

$$(3.6) \quad FS_{\infty}(\Phi) = \left\{ f \in Fl_1(\Phi) : \sum_{n,k} |f_{nk}| \leq 1 \right\}.$$

It is straightforward to deduce that this estimate can be expressed by

$$(3.7) \quad s_E^2(\Phi) = \sum_{n=0}^{\infty} q_n^2 e_n^2(\Phi_n),$$

where q_n^2 is given in (2.7) and

$$(3.8) \quad e_n^2(\Phi_n) = \sum_{k=0}^n E(\varphi_{nk})^2.$$

Our problem at this stage is to choose suitable families Φ_n satisfying (3.3), such that the estimate $s_E^2(\Phi)$ is compatible with the symmetry group $\mathcal{U}(S_2)$ in the sense described in Section 1.

As the matrix

$$(3.9) \quad \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$$

associated with the rotation $u_{\alpha} \in \mathcal{U}(S_2)$ has eigenvalues $e^{i\alpha}$, $e^{-i\alpha}$ and eigenvectors $(1, i)^T$, $(1, -i)^T$, the use of complex arithmetic will simplify the calculations. Let us denote

$$(3.10) \quad M_n^* = \{a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n; a_i \in \mathbb{C}\},$$

and let

$$(3.11) \quad \Phi_n^* = \{\varphi_{n0}^*, \dots, \varphi_{nn}^*\}$$

be a basis of M_n^* , i.e., $M_n^* = \text{span}^*(\Phi_n^*)$.

Considering the natural complexification of linear operators

$$(3.12) \quad E(f + ig) = E(f) + iE(g)$$

with the standard complex notation

$$(3.13) \quad |E(f + ig)|^2 = \overline{E(f + ig)} E(f + ig) = E(f)^2 + E(g)^2,$$

we can define

$$(3.14) \quad e_n^2(\Phi_n^*) = \sum_{k=0}^n |E(\varphi_{nk}^*)|^2.$$

THEOREM 1. *For every n , let $\Phi_n^* = \{\varphi_{n0}^*, \dots, \varphi_{nn}^*\}$ and $\Phi_n = \{\varphi_{n0}, \dots, \varphi_{nn}\}$ be bases of M_n^* and M_n , respectively, satisfying*

(i) $(\varphi_{n0}, \dots, \varphi_{nn})^T = A_n (\varphi_{n0}^*, \dots, \varphi_{nn}^*)^T$ where A_n is an $n \times n$ complex unitary matrix, i.e., $A_n^H = A_n^{-1}$;

(ii) $\sum_{k=0}^n |E(\varphi_{nk}^*)|^2 = \sum_{k=0}^n |E(\varphi_{nk}^* \circ u_{\alpha})|^2 = \sum_{k=0}^n |E(\varphi_{nk}^* \circ u_{\alpha} \circ v)|^2$ for all $\alpha \in [0, 2\pi)$;

(iii) *there exists a $c \in \mathbb{R}$ such that $\|\varphi_{nk}\|_{\infty} \leq c$ for all n, k .*

Then, the estimate $s_E(\Phi)$ associated with the family $\Phi = \{\Phi_0, \Phi_1, \dots\}$ is compatible with the symmetry group $\mathcal{U}(S_2)$.

Proof. Let us remark that the operators

$$\begin{aligned} f^* &\in M_n^* \rightarrow E(f^*) \in \mathbb{C}, \\ f^* &\in M_n^* \rightarrow f^* \circ u_\alpha \in M_n^*, \\ f^* &\in M_n^* \rightarrow f^* \circ u_\alpha \circ v \in M_n^* \end{aligned}$$

are linear and therefore “pass through the matrix A_n ”.

Moreover, $E(\varphi_{nk})$ and $E(\varphi_{nk} \circ u)$ are real and therefore

$$\begin{aligned} \sum_{k=0}^n E(\varphi_{nk} \circ u_\alpha)^2 &= (\overline{E(\varphi_{n0} \circ u_\alpha)}, \dots, \overline{E(\varphi_{nn} \circ u_\alpha)})(E(\varphi_{n0} \circ u_\alpha), \dots, E(\varphi_{nn} \circ u_\alpha))^T \\ &= (\overline{E(\varphi_{n0}^* \circ u_\alpha)}, \dots, \overline{E(\varphi_{nn}^* \circ u_\alpha)})A_n^H A_n (E(\varphi_{n0}^* \circ u_\alpha), \dots, E(\varphi_{nn}^* \circ u_\alpha))^T \\ &= \sum_{k=0}^n |E(\varphi_{nk}^* \circ u_\alpha)|^2 = \sum_{k=0}^n |E(\varphi_{nk}^*)|^2 \\ &= (\overline{E(\varphi_{n0}^*)}, \dots, \overline{E(\varphi_{nn}^*)})(E(\varphi_{n0}^*), \dots, E(\varphi_{nn}^*))^T \\ &= (\overline{E(\varphi_{n0})}, \dots, \overline{E(\varphi_{nn})})A_n A_n^H (E(\varphi_{n0}), \dots, E(\varphi_{nn}))^T = \sum_{k=0}^n E(\varphi_{nk})^2, \end{aligned}$$

given that A_n is unitary. Similarly, it can be shown that

$$\sum_{k=0}^n E(\varphi_{nk} \circ u_\alpha \circ v)^2 = \sum_{k=0}^n E(\varphi_{nk})^2,$$

and therefore it follows in a straightforward way that $s_E(\Phi)$ is compatible with $\mathcal{U}(S_2)$. $\square$

Now let us consider the complex polynomials

$$(15) \quad \varphi_{nk}^* = (x + iy)^{n-k} (x - iy)^k \in M_n^*$$

obtained from the monomials $x^{n-k}y^k$ by a linear transformation with Jacobian

$$J = \begin{vmatrix} 1 & i \\ 1 & -i \end{vmatrix} = -2i,$$

so that $\varphi_{n0}^*, \dots, \varphi_{nn}^*$ are linearly independent in M_n^* .

Also,

$$\begin{aligned} (\varphi_{nk}^* \circ u_\alpha)(x, y) &= \varphi_{nk}^*(x \cos \alpha - y \sin \alpha, x \sin \alpha + y \cos \alpha) \\ &= e^{i(n-k)\alpha} (x + iy)^{n-k} e^{-ik\alpha} (x - iy)^k = e^{i(n-2k)\alpha} \varphi_{nk}^*(x, y), \end{aligned}$$

thus

$$(16) \quad \sum_{k=0}^n |E(\varphi_{nk}^* \circ u_\alpha)|^2 = \sum_{k=0}^n |E(\varphi_{nk}^*)|^2.$$

Similarly,

$$\begin{aligned} (\varphi_{nk}^* \circ u_\alpha \circ v)(x, y) &= (\varphi_{nk}^* \circ u_\alpha)(x, -y) = e^{i(n-2k)\alpha} (x - iy)^{n-k} (x + iy)^k \\ &= e^{i(n-2k)\alpha} \varphi_{n, n-k}^*(x, y), \end{aligned}$$

and then

$$(3.17) \quad \sum_{k=0}^n |E(\varphi_{nk}^* \circ u_\alpha \circ v)|^2 = \sum_{k=0}^n |E(\varphi_{nk}^*)|^2.$$

Therefore, for each n , the family $\Phi_n^* = \{\varphi_{n0}^*, \dots, \varphi_{nn}^*\}$ is a basis of M_n^* which satisfies the hypothesis (ii) of Theorem 1.

For $k < n/2$ let us define

$$(3.18) \quad \begin{aligned} \varphi_{nk} &= \frac{1}{\sqrt{2}} (\varphi_{nk}^* + \varphi_{n,n-k}^*) \\ &= \frac{1}{\sqrt{2}} (x^2 + y^2)^k [(x + iy)^{n-2k} + (x - iy)^{n-2k}], \end{aligned}$$

$$(3.19) \quad \begin{aligned} \varphi_{n,n-k} &= \frac{1}{\sqrt{2}i} (\varphi_{nk}^* - \varphi_{n,n-k}^*) \\ &= \frac{1}{\sqrt{2}i} (x^2 + y^2)^k [(x + iy)^{n-2k} - (x - iy)^{n-2k}], \end{aligned}$$

and if n is even,

$$(3.20) \quad \varphi_{n,n/2} = \varphi_{n,n/2}^* = (x^2 + y^2)^{n/2}.$$

Then, $\Phi_n = \{\varphi_{n0}, \dots, \varphi_{nn}\}$ is formed by polynomials with real coefficients and is a basis of M_n . Also the matrix A_n of Theorem 1 that relates the elements of Φ_n and Φ_n^* is a unitary matrix, because the matrices

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}i} & \frac{-1}{\sqrt{2}i} \end{pmatrix}$$

that relate the pairs $(\varphi_{nk}, \varphi_{n,n-k})$ and $(\varphi_{nk}^*, \varphi_{n,n-k}^*)$ are unitary. Moreover, it can easily be shown that

$$\|\varphi_{nk}\|_\infty = \|\varphi_{n,n-k}\|_\infty = \sqrt{2}, \quad k < n/2,$$

and $\|\varphi_{n,n/2}\|_\infty = 1$ for n even.

Using the results above, and applying Theorem 1, we deduce the following

THEOREM 2. *Let $\Phi = \{\Phi_0, \Phi_1, \dots\}$ where, for each n , $\Phi_n = \{\varphi_{n0}, \dots, \varphi_{nn}\}$ is the basis of M_n defined by (3.18), (3.19) and (3.20). Then the estimate $s_E(\Phi)$ defined by (3.5) is an estimate of goodness of cubature formulae for the unit circle that is compatible with the symmetry group $\mathcal{U}(S_2)$.*

Following the proof of Theorem 1, we can also deduce that

$$(3.21) \quad s_E^2(\Phi) = \sum_{n=0}^{\infty} q_n^2 \sum_{k=0}^n E(\varphi_{nk})^2 = \sum_{n=0}^{\infty} q_n^2 \sum_{k=0}^n |E(\varphi_{nk}^*)|^2,$$

and therefore the estimate $s_E(\Phi)$ can be calculated using any of these two expressions.

TABLE 1

Formula	D	N	$s_E(\Phi)$
Centroid	1	1	$(-2)3.72941$
$S_2: 3 - 1$	3	4	$(-3)1.52574$
$S_2: 5 - 1$	5	7	$(-5)4.17361$
$S_2: 5 - 2$	5	9	$(-5)1.56155$
$S_2: 7 - 1$	7	12	$(-7)7.32827$
$S_2: 7 - 2$	7	16	$(-7)7.31334$
$S_2: 9 - 1$	9	19	$(-10)2.86050$
$S_2: 9 - 3$	9	21	$(-9)8.64763$
$S_2: 9 - 5$	9	28	$(-10)5.70093$
$S_2: 11 - 1$	11	28	$(-11)7.64307$
$S_2: 11 - 2$	11	28	$(-12)2.00147$
$S_2: 11 - 3$	11	28	$(-11)4.55280$
$S_2: 11 - 4$	11	32	$(-11)7.64002$
$S_2: 13 - 1$	13	37	$(-14)3.05146$
$S_2: 13 - 2$	13	41	$(-14)1.03972$
$S_2: 15 - 1$	15	44	$(-15)2.92306$
$S_2: 15 - 2$	15	48	$(-15)2.88250$
$S_2: 17 - 1$	17	61	$(-20)4.97655$

Table 1 shows the values of $s_E(\Phi)$ for some cubature formulae (1.5) for the unit circle with $w(x, y) = 1/\pi$. The nomenclature of these formulae corresponds to the one in [6, pp. 277–289]. N stands for the number of nodes and D for the degree of precision.

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