A Method for Computing the Iwasawa λ -Invariant

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Abstract. We present a method for computing the minus-part of the Iwasawa λ -invariant of an Abelian field K . Applying this method, we have computed λ^- for several odd primes p when K runs through a large number of quadratic extensions of the p th cyclotomic field. A report on these computations and an analysis of the results is included.

1. Introduction. Let K be an Abelian field, i.e., a finite Abelian extension of $\mathbf{Q}$. For a prime $p > 2$, consider the cyclotomic $\mathbf{Z}_p$ -extension K_∞ of K . Let K_n ($n \geq 0$) denote the intermediate field of K_∞/K which is cyclic of degree p^n over K . The p -part of the class number of K_n equals $p^{\lambda_n + \nu}$, for all sufficiently large n , where $\lambda = \lambda(p)$ and $\nu = \nu(p)$ are integral constants, $\lambda \geq 0$. Call λ the Iwasawa λ -invariant of K and write $\lambda = \lambda^+ + \lambda^-$, where λ^+ is the corresponding invariant of the maximal real subfield of K . In this paper we present a method for computing λ^- , developed by the second author, and report on computer calculations by the first author, performed by this method.

If the conductor f_K of the field K is divisible by p^2 , then K has a subfield L such that $p^2 \nmid f_L$ and the cyclotomic $\mathbf{Z}_p$ -extension of L equals K_∞ . Hence we assume, without loss of generality, that $p^2 \nmid f_K$. Denote by $\text{Ch}(K)$ the character group of K . It is known that λ^- decomposes as

$$\lambda^- = \sum_{\chi \in X} \lambda_\chi$$

with

$$X = X(K) = \{ \chi \in \text{Ch}(K) : \chi(-1) = -1, \chi \neq \omega^{-1} \},$$

where ω denotes the Teichmüller character mod p and λ_χ is the λ -invariant of the Iwasawa power series representing the p -adic L -function $L_p(s, \chi\omega)$.

Thus, the computation of λ^- is reduced to the determination of the components λ_χ . This will be done in two steps: We first relate λ_χ to the p -orders of certain generalized Bernoulli numbers and then show how to determine these p -orders by means of a series of character sum congruences. As an application we consider the fields $\mathbf{Q}(\sqrt{m}, \zeta_p)$, where m is an integer prime to p and ζ_p denotes a primitive p th root of 1. In this case the congruences in question are simply rational congruences mod p .

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The computational part of our work consists of the determination of λ^- for quite a large collection of fields $\mathbf{Q}(\sqrt{m}, \zeta_p)$, chosen so that either p or $|m|$ is small. More precisely, we computed for these fields the components λ_χ with $\chi = \theta_m \omega'$, where θ_m is the quadratic character of the field $\mathbf{Q}(\sqrt{m})$; this is sufficient since the λ^- -invariants of the cyclotomic subfields $\mathbf{Q}(\zeta_p)$ are known. Our results also give the λ -invariant of $\mathbf{Q}(\sqrt{m})$ for the negative m in the range under consideration.

There are previous numerical results about λ^- in [1], [2], [3], [4], [6]. These concern mainly quadratic fields and the fields $\mathbf{Q}(\sqrt{-1}, \zeta_p)$ and $\mathbf{Q}(\sqrt{-3}, \zeta_p)$, and in all cases the decomposition of λ^- is simple in the sense that either there is but one positive component λ_χ , or all the positive components are equal to 1. In the present results this is no longer the case.

A detailed description of our computations appears in Sections 7–9.

2. On p -Adic L -Functions. For the theory of this section, the reader is referred to Washington's book [11], in particular to Sections 5.2 and 7.2.

We fix an embedding of the field of algebraic numbers in an algebraic closure Ω_p of $\mathbf{Q}_p$, the field of p -adic numbers. Denote by ord_p the p -adic valuation on Ω_p , normalized so that $\text{ord}_p(p) = 1$.

Let χ be a character in $X(K)$ (all characters are assumed primitive). Since the conductor f_χ of χ divides f_K , it is not divisible by p^2 ; we say that χ is of the “first kind”. Put

$$f_\chi = d \text{ or } dp \quad \text{with } (d, p) = 1.$$

As in the introduction, let K_n denote the n th layer of the $\mathbf{Z}_p$ -extension K_∞/K ($n \geq 0$). The character group of K_n is of the form $\text{Ch}(K) \times \langle \pi_n \rangle$, where π_n is a character with order p^n and conductor p^{n+1} (or 1, if $n = 0$); π_n is called a character of the “second kind”.

Now consider the p -adic L -function $L_p(s, \psi)$ for the (nonprincipal) character $\psi = \chi \omega \pi_n$. This function is defined in Ω_p in a neighborhood of 1 containing $\mathbf{Z}_p$, the p -adic integers, and it has the fundamental property that

$$(1) \quad L_p(1 - k, \psi) = -(1 - \psi_k(p) p^{k-1}) B^k(\psi_k) / k \quad (k \geq 1),$$

where $\psi_k = \psi \omega^{-k}$ and $B^k(\psi_k)$ stands for the k th generalized Bernoulli number attached to the character ψ_k .

Denote by $\mathbf{Q}_p(\chi)$ the extension of $\mathbf{Q}_p$ generated by the values of χ . Iwasawa's theory of p -adic L -functions asserts that there exists a power series

$$(2) \quad f(x, \chi \omega) = \sum_{j=0}^{\infty} a_j x^j$$

whose coefficients $a_j = a_j(\chi)$ are integers of $\mathbf{Q}_p(\chi)$, such that

$$(3) \quad L_p(s, \chi \omega \pi_n) = f\left(\frac{(1 + dp)^s}{\pi_n(1 + dp)} - 1, \chi \omega\right).$$

According to the Ferrero-Washington theorem, the power series $f(x, \chi \omega)$ has $\mu = 0$, in other words, there is an index j for which $\text{ord}_p(a_j) = 0$. The least such j is called the λ -invariant (or Weierstrass degree) of $f(x, \chi \omega)$. This is the number λ_χ introduced in Section 1.

3. The p -Orders of Generalized Bernoulli Numbers. Let us decompose χ as

$$\chi = \theta \omega^{t-1} \quad \text{with } f_\theta = d (\geq 1), \quad 1 \leq t \leq p-1.$$

In this section we obtain a relation between λ_χ and the p -order of $B'(\theta\pi_n)$.

For a fixed $n \geq 1$, put

$$\alpha_k = \frac{(1+dp)^{1-k}}{\pi_n(1+dp)} - 1 \quad (k \geq 1).$$

It follows from (3) and (1) that, for all $t = 1, \dots, p-1$,

$$\begin{aligned} f(\alpha_t, \theta\omega^t) &= L_p(1-t, \theta\omega^t\pi_n) \\ &= -(1 - (\theta\pi_n)(p)p^{t-1})B'(\theta\pi_n)/t = -B'(\theta\pi_n)/t. \end{aligned}$$

By using this result we prove the following proposition in which ϕ denotes Euler's totient function and e is the ramification index of $\mathbf{Q}_p(\theta)/\mathbf{Q}_p$.

PROPOSITION 1. *Let $n \geq 1$ and $1 \leq t \leq p-1$. We have*

$$\begin{aligned} \text{ord}_p(B'(\theta\pi_n)) &= \lambda_\chi/\phi(p^n) < 1/e & \text{if } \lambda_\chi < \phi(p^n)/e, \\ \text{ord}_p(B'(\theta\pi_n)) &\geq 1/e & \text{if } \lambda_\chi \geq \phi(p^n)/e. \end{aligned}$$

Proof. We evaluate the p -order of $f(\alpha_t, \theta\omega^t) = \sum_{j=0}^{\infty} a_j \alpha_t^j$.

By the definition of π_n , the number $\pi_n(1+dp) = \zeta$ is a primitive p^n th root of 1. Since

$$\alpha_t = \frac{1 - \zeta(1+dp)^{t-1}}{\zeta(1+dp)^{t-1}},$$

we have $\text{ord}_p(\alpha_t) = \text{ord}_p(1 - \zeta) = 1/\phi(p^n)$.

As to the p -orders of the coefficients a_j , observe that these are integers of $\mathbf{Q}_p(\chi) = \mathbf{Q}_p(\theta)$. Therefore, if $\text{ord}_p(a_j) > 0$ then $\text{ord}_p(a_j) \geq 1/e$.

Recalling the definition of λ_χ we now see that

$$\begin{aligned} \text{ord}_p(a_j \alpha_t^j) &\geq 1/e & \text{for } 0 \leq j \leq \lambda_\chi - 1, \\ &= j \text{ord}_p(\alpha_t) = \lambda_\chi/\phi(p^n) & \text{for } j = \lambda_\chi, \\ &\geq j \text{ord}_p(\alpha_t) > \lambda_\chi/\phi(p^n) & \text{for } j > \lambda_\chi. \end{aligned}$$

Consequently, if $\lambda_\chi < \phi(p^n)/e$, then

$$\text{ord}_p(f(\alpha_t, \theta\omega^t)) = \lambda_\chi/\phi(p^n),$$

while otherwise this p -order is at least $1/e$. Hence the result. $\square$

Proposition 1 gives us the value of λ_χ , once we know $\text{ord}_p(B'(\theta\pi_n))$ for a sufficiently large n . For later purposes it is convenient to reformulate this proposition, actually in a bit weaker form, as follows.

Note that the congruence $\alpha \equiv \beta \pmod{p^r}$ in Ω_p means that $\text{ord}_p(\alpha - \beta) \geq r$.

PROPOSITION 2. *Let $n \geq 1$ and $1 \leq t \leq p-1$. Assume that $h \in \mathbf{Z}$, $1 \leq h \leq \phi(p^n)/e$. Then*

$$\lambda_\chi \geq h \quad \text{if and only if} \quad B'(\theta\pi_n) \equiv 0 \pmod{p^{h/\phi(p^n)}}.$$

Proof. Suppose that the above congruence holds. If $\lambda_\chi < \phi(p^n)/e$, then a comparison of this congruence with the first part of Proposition 1 shows that $\lambda_\chi \geq h$. If $\lambda_\chi \geq \phi(p^n)/e$, then the assertion follows directly from the assumption made about h .

To verify the converse, apply both parts of Proposition 1 separately. $\square$

Remark. Proposition 2 is of the same kind as the main result in the second author's paper [8]. This relates λ_χ to certain Kummer type congruences of $B^k(\theta)$, provided $\lambda_\chi \leq p - 1$. Proposition 2 would enable one to replace the proof presented in [8] by a somewhat simpler proof.

4. Bernoulli Numbers and Character Sums. We now express the residue of $B'(\theta\pi_n)$ modulo p in terms of suitable character sums.

For any character ψ with conductor f we have, in the usual symbolic notation,

$$B^k(\psi) = \frac{1}{f} \sum_{a=1}^f \psi(a)(fB + a - f)^k \quad (k \geq 0)$$

(e.g., [7, p. 134]), where the B^m denote ordinary Bernoulli numbers. On changing the summation variable a into $f - a$ we obtain

$$B^k(\psi) = \frac{(-1)^k \psi(-1)}{f} \sum_{a=1}^f \psi(a)(a - fB)^k.$$

Let $\psi = \theta\pi_n$ with $n \geq 1$. Then $\psi(-1) = (-1)^t$ since the character $\chi = \theta\omega^{t-1}$ is odd and π_n , being of p -power order, is even. Hence we find that

$$\begin{aligned} B^t(\theta\pi_n) &= \frac{1}{dp^{n+1}} \sum_{a=1}^{dp^{n+1}} (\theta\pi_n)(a)(a - dp^{n+1}B)^t \\ &\equiv \frac{1}{dp^{n+1}} \sum_{a=1}^{dp^{n+1}} (\theta\pi_n)(a)a^t - tB^1 \sum_{a=1}^{dp^{n+1}} (\theta\pi_n)(a)a^{t-1} \pmod{p}. \end{aligned}$$

The second sum of the last expression vanishes mod p , as can be verified again by the transformation $a \rightarrow dp^{n+1} - a$. Therefore,

$$(4) \quad B^t(\theta\pi_n) \equiv \frac{1}{dp^{n+1}} \sum_{a=1}^{dp^{n+1}} (\theta\pi_n)(a)a^t \pmod{p}.$$

From this result we derive the following congruence which is of the same type as the classical Voronoï congruence for ordinary Bernoulli numbers. We point out that the congruence (in a sharper form) has also been proved by Slavutskii [9, congr. (6)].

PROPOSITION 3. *Let b be a positive rational integer with $(b, dp) = 1$. Then*

$$(b^t - (\theta\pi_n)(b)^{-1})B^t(\theta\pi_n) \equiv tb^{t-1} \sum_{a=1}^{dp^{n+1}} (\theta\pi_n)(a)a^{t-1} \left[\frac{ba}{dp^{n+1}} \right] \pmod{p},$$

where, as in the above, $n \geq 1$ and $1 \leq t \leq p - 1$.

Proof. Put $\psi = \theta\pi_n$. Let a and b be positive rational integers prime to dp . Keeping b fixed, we write

$$ba = dp^{n+1} \left[\frac{ba}{dp^{n+1}} \right] + r_a, \quad 0 < r_a < dp^{n+1}.$$

On raising this equation to the t th power and multiplying by $\psi(a) = \psi(b)^{-1}\psi(r_a)$, we get

$$\psi(a)b^t a^t \equiv \psi(b)^{-1}\psi(r_a)r_a^t + \psi(a)tr_a^{t-1}dp^{n+1} \left[\frac{ba}{dp^{n+1}} \right] \pmod{p^{2n+2}}.$$

If a runs through $1, \dots, dp^{n+1}$, excepting those numbers for which $(a, dp) > 1$, then so does r_a . Summing over a we find that (observe that $\psi(a) = 0$ if $(a, dp) > 1$)

$$(b^t - \psi(b)^{-1}) \sum_{a=1}^{dp^{n+1}} \psi(a)a^t \equiv tdp^{n+1} \sum_{a=1}^{dp^{n+1}} \psi(a)r_a^{t-1} \left[\frac{ba}{dp^{n+1}} \right] \pmod{p^{2n+2}}.$$

Since $r_a \equiv ba \pmod{p^{n+1}}$, this result together with (4) yields the claimed congruence. $\square$

5. The Main Result. Every rational integer a prime to p has the following unique representation mod p^{n+1} :

$$(5) \quad a \equiv \omega(a)(1+p)^{v(a)} \pmod{p^{n+1}}, \quad 0 \leq v(a) < p^n.$$

For $b \in \mathbb{Z}$, $(b, dp) = 1$, put

$$(6) \quad S_{nk} = S_{nk}(b) = \sum_{v(a)=k} \theta(a)a^{t-1} \left[\frac{ba}{dp^{n+1}} \right] \quad (k = 0, \dots, p^n - 1),$$

where the sum is extended over those numbers a for which $1 \leq a < dp^{n+1}$, $(a, dp) = 1$ and $v(a) = k$. Moreover, set

$$(7) \quad T_u = T_u^{(n)} = \sum_{k=u}^{p^n-1} \binom{k}{u} S_{nk} \quad (u = 0, \dots, p^n - 1).$$

THEOREM. Let $\chi = \theta\omega^{t-1} \in X(K)$, where $f_\theta = d$ is prime to p and $1 \leq t \leq p-1$. Let b be a positive integer such that

$$(b, dp) = 1, \quad \theta(b)b^t \not\equiv 1 \pmod{\mathfrak{p}},$$

where $\mathfrak{p}$ is the maximal ideal of the ring of integers of $\mathbb{Q}_p(\theta)$. Denote by e the ramification index of $\mathbb{Q}_p(\theta)/\mathbb{Q}_p$. Let $n \geq 1$ and let $h \in \mathbb{Z}$, $1 \leq h \leq \phi(p^n)/e$. With the above notations,

$$\lambda_\chi \geq h \quad \text{if and only if} \quad T_0^{(n)} \equiv T_1^{(n)} \equiv \dots \equiv T_{h-1}^{(n)} \equiv 0 \pmod{\mathfrak{p}}.$$

Proof. Since the nonzero values of π_n are p^n th roots of 1, we have $\pi_n(b) \equiv 1 \pmod{\mathfrak{p}}$. Hence

$$b^t - (\theta\pi_n)(b)^{-1} \not\equiv 0 \pmod{\mathfrak{p}},$$

and it follows from Propositions 2 and 3 that

$$\lambda_\chi \geq h \quad \text{if and only if} \quad \sum_{a=1}^{dp^{n+1}} \theta(a)\pi_n(a)a^{t-1} \left[\frac{ba}{dp^{n+1}} \right] \equiv 0 \pmod{p^{h\kappa}},$$

where $\kappa = 1/\phi(p^n)$.

For a fixed $n \geq 1$, write

$$\pi_n(1+p) = 1 + \eta.$$

Then we have $\text{ord}_p(\eta) = \kappa$ and, by (5),

$$\pi_n(a) = (1 + \eta)^{v(a)} \quad \text{for } p \nmid a.$$

Consequently,

$$\sum_{a=1}^{dp^{n+1}} \theta(a) \pi_n(a) a^{t-1} \left[\frac{ba}{dp^{n+1}} \right] = \sum_{k=0}^{p^n-1} (1+\eta)^k S_{nk} = \sum_{u=0}^{p^n-1} T_u \eta^u,$$

and we are done, once the congruence

$$(8) \quad \sum_{u=0}^{p^n-1} T_u \eta^u \equiv 0 \pmod{p^{h\kappa}}$$

is shown to be equivalent to

$$(9) \quad T_0 \equiv T_1 \equiv \cdots \equiv T_{h-1} \equiv 0 \pmod{p}.$$

Suppose that the congruences (9) hold true. Then these congruences are satisfied mod $p^{1/e}$ as well, and so mod $p^{h\kappa}$ since $1/e \geq h/\phi(p^n) = h\kappa$. Moreover, $\eta^u \equiv 0 \pmod{p^{h\kappa}}$ whenever $u \geq h$. This proves (8). The converse implication is established with similar arguments by induction on h . $\square$

The above theorem enables us to determine λ_χ , once the numbers $T_u^{(n)}$ modulo p are known for a sufficiently large n . We state this more explicitly as follows.

COROLLARY. Put $z_n = [\phi(p^n)/e]$. With the notations of the theorem,

- (i) if $T_0^{(n)} \equiv T_1^{(n)} \equiv \cdots \equiv T_{h-1}^{(n)} \equiv 0$ and $T_h^{(n)} \not\equiv 0 \pmod{p}$, where $0 \leq h \leq z_n - 1$, then $\lambda_\chi = h$;
- (ii) if $T_0^{(n)} \equiv T_1^{(n)} \equiv \cdots \equiv T_{z_n-1}^{(n)} \equiv 0 \pmod{p}$, then $\lambda_\chi \geq z_n$.

6. A Special Case. Suppose that $\theta = \theta_m$ is the nontrivial character of the quadratic field $\mathbf{Q}(\sqrt{m})$, where m is prime to p . Then the character $\chi = \theta \omega^{t-1}$ dealt with in the previous sections belongs to the character group of the field $\mathbf{Q}(\sqrt{m}, \zeta_p)$. Note that $f_\theta = d$ equals the absolute value of the discriminant of $\mathbf{Q}(\sqrt{m})$.

In this case, $\mathbf{Q}_p(\theta_m) = \mathbf{Q}_p$, so that $e = 1$ and $p = p\mathbf{Z}_p$. Hence we can determine λ_χ , provided it does not exceed $p - 2$, through the numbers $T_u = T_u^{(1)}$ as follows (see the corollary):

If $T_0 \equiv T_1 \equiv \cdots \equiv T_{h-1} \equiv 0$, $T_h \not\equiv 0 \pmod{p}$, where $0 \leq h \leq p - 2$, then $\lambda_\chi = h$.

If this criterion fails, then the computation of λ_χ requires passing to a higher level, i.e., computing $T_u^{(n)} \pmod{p}$ for a higher value of n .

Remark. As is seen from (5), working on a level n involves computations with integers mod p^{n+1} . We point out that, for $n = 1$, the congruence (5) can be written as

$$a \equiv a^p(1 + v(a)p) \pmod{p^2}.$$

Thus, $v(a) \equiv -q_a \pmod{p}$, where q_a denotes the Fermat quotient for a , defined by $q_a \equiv (a^{p-1} - 1)/p \pmod{p}$, $0 \leq q_a < p$.

7. Numerical Results. Consider, for a moment, the case of the cyclotomic field $\mathbf{Q}(\zeta_p)$. Then $X = \{\omega, \omega^3, \dots, \omega^{p-4}\}$ and it is known that

$$\lambda_\chi > 0 \quad \text{with } \chi = \omega^{t-1} \quad \text{if and only if} \quad B^t \equiv 0 \pmod{p}$$

($t = 2, 4, \dots, p - 3$). The values of λ_χ have been computed for $p < 125000$ [10]; it has turned out that in this range every positive value of λ_χ equals 1. So the λ^- -invariant of $\mathbf{Q}(\zeta_p)$, say λ_0^- , equals the *index of irregularity* of p , i.e., the number of *irregular pairs* (p, t) . Tables of irregular pairs can be found in many books, e.g., [11].

Now let us enlarge the field to $K = \mathbf{Q}(\sqrt{m}, \zeta_p)$ with $p \nmid m$. Then the character set X is enlarged by the characters $\theta_m \omega^{t-1}$ discussed in Section 6. To be precise, we have

$$\lambda^- = \lambda_0^- + \sum_x \lambda_x,$$

where the sum is extended over the characters

$$(10) \quad \chi = \theta_m \omega^{t-1} \quad \text{with} \quad \begin{cases} t = 2, 4, \dots, p-1 & \text{if } m > 0, \\ t = 1, 3, \dots, p-2 & \text{if } m < 0, \end{cases}$$

$\theta = \theta_m$ being the quadratic character of $\mathbf{Q}(\sqrt{m})$. If $m < 0$, the component λ_θ is just the λ -invariant of this quadratic field.

The actual computations associated with the present work comprised the determination of λ_x for the characters (10) when p and m range through the following values (m squarefree):

$$\begin{aligned} p = 3 & \quad \text{and} \quad -3235 \leq m \leq 3454, * \\ p = 5 & \quad \text{and} \quad -5000 < m \leq 3147, \\ p = 7 & \quad \text{and} \quad -3002 \leq m < 1000, \\ p = 11 & \quad \text{and} \quad -1000 < m < 500, \\ 11 < p < 200 & \quad \text{and} \quad m = -7, -3, -2, -1, 2, 5. \end{aligned}$$

The asterisk above indicates that for a few values of m the computation was stopped at the result $\lambda_x \geq 6$ (see below).

The numerical material thus obtained contains about 22000 values of λ_x , some 6400 of them being positive. Samples from this material are exhibited in Tables 1 and 2 of the appendix. Table 1 presents the results for $p = 5$, $m > 0$, and Table 2 for $p < 200$, $m = -1, \pm 2, -3, 5, -7$. Note that every odd prime p below 200 really appears in Table 2, i.e., to every p there is at least one m and t such that $\lambda_x > 0$ for $\chi = \theta_m \omega^{t-1}$.

For $p > 3$, only few cases were found in which $\lambda_x > p - 2$. These cases, which had to be settled on the level $n = 2$, are listed here:

p	m	t	λ_x	p	m	t	λ_x
5	439	4	4	5	-3178	1	4
5	1427	4	4	5	-3471	1	4
5	-311	1	4	5	-3547	3	4
5	-761	1	4	5	-3923	3	4
5	-966	1	4	5	-4026	1	5
5	-2861	3	4	5	-4774	1	4
5	-3081	1	4	7	-1371	1	7

For $p = 7$ it in fact turned out that λ_x varies between 0 and 4 (assuming all values $0, \dots, 4$) except in the single case given above. For $p = 11$ we have the maximum $\lambda_x = 3$ for $m = -723$, $t = 1$.

If $p = 3$, then $\lambda_x > 1$ ($= p - 2$) in about a third of the cases. These could be settled on the level $n = 2$ (i.e., $\lambda_x \leq 5$), except in six cases. In the latter cases the continuation of the procedure was given up since the values of λ_x can be found in [6]; they are as follows:

$$\begin{aligned} \lambda_x = 6 & \quad \text{for } m = -239, -1022, -1427, -1777; \\ \lambda_x = 7 & \quad \text{for } m = -458, \\ \lambda_x = 8 & \quad \text{for } m = -2789. \end{aligned}$$

An examination of the results shows that the values of λ_χ seem to be distributed in the expected way. For example, if we keep p and t fixed, $t \neq 1$, and let m vary, then the number of cases with $\lambda_\chi \geq k$ (for $\chi = \theta_m \omega^{t-1}$ and $k \geq 0$) should be about a p^k th part of the number of all λ_χ ; this corresponds to the natural hypothesis that the coefficients of the power series $f(x, \chi\omega)$ are randomly distributed mod p . In the following table, N_k denotes the number of $\lambda_\chi \geq k$ in our range:

p	t	N_0	N_1	N_2	N_3	N_1/N_0	$1/p$	N_2/N_0	$1/p^2$	N_3/N_0	$1/p^3$
3	2	1577	553	172	50	0.35	0.33	0.11	0.11	0.032	0.037
5	2	1596	326	55	9	0.20	0.20	0.034	0.040	0.006	0.008
5	4	1596	329	68	15	0.21	0.20	0.043	0.040	0.009	0.008
5	3	2535	490	88	14	0.19	0.20	0.035	0.040	0.006	0.008
7	3	1599	221	29		0.14	0.14	0.018	0.020		
7	5	1599	256	39		0.16	0.14	0.024	0.020		

If $t = 1$, the situation is different. Indeed, by Eqs. (1)–(3) the constant term of $f(x, \chi\omega)$ equals

$$(11) \quad a_0 = (\chi(p) - 1)B^1(\chi);$$

hence, in the present case λ_χ is positive whenever $\chi(p) = \theta_m(p) = +1$. We must therefore modify the above hypothesis so as to concern those $f(x, \theta_m\omega)$ only for which $\theta_m(p) = -1$. We tested this hypothesis for $p = 5$, $m > 0$, obtaining the following (N'_k denotes the number of $\lambda_\chi \geq k$ when $\theta_m(5) = -1$):

$$N'_0 = 1268, \quad N'_1 = 241, \quad N'_2 = 36; \quad N'_1/N'_0 = 0.19, \quad N'_2/N'_0 = 0.028.$$

We may also ask how often λ^- is, say, positive as p is fixed and $|m|$ increases. If $p \leq 11$, then $\lambda_0^- = 0$, and so $\lambda^- > 0$ exactly when at least one of the $s = (p-1)/2$ numbers T_0 corresponding to the characters $\theta_m \omega^{t-1}$ vanishes mod p . To avoid the exceptional case $t = 1$, consider positive m only. Then it is again natural to assume that the values of T_0 be randomly distributed mod p , and this implies that the proportion of the number of fields with $\lambda^- > 0$ to the number of all fields should be about $\rho_p = 1 - (1 - p^{-1})^s$. Below is a comparison between the observed and expected values of this proportion:

p	observed proportion	ρ_p
5	$587/1596 = 0.37$	0.36
7	$204/530 = 0.38$	0.37
11	$100/279 = 0.36$	0.38

A table including all the results of our computations has been deposited in the UMT file; see Review 29 in this issue.

8. Comparison with Previous Results. We next describe the contents of the previously published tables about λ^- . These tables were used by us to check our results.

Gold [3], [4] has computed, for $p = 3, 5, 7, 11$, the λ -invariant of the quadratic field with discriminant $-d < 0$. His results in [4, Table 2] cover the range $0 < d \leq 264$. They agree completely with ours, and so do also the additional results presented in [3, Tables 2 and 5] after the following apparent errors are corrected: In Table 2, the value 1253 for d should be 1263 (corresponding to the given class number 20); in

Table 5, lines 5 and 6, instead of $\lambda = 3$ and $\lambda = 4$ one should read $\lambda = 2$. The latter correction is confirmed not only by [6] quoted below, but also by Corollary 5 in [3]. The expressions for e_n in Table 5 should be correspondingly corrected.

Kobayashi [6] investigates, for $p = 3$, the power series $f(x, \chi\omega)$ with $\chi = \theta_m$ and $\chi = \theta_m\omega$. He has determined the coefficients $a_0, \dots, a_8 \pmod 9$ of this power series for $-10^4 < m < 0$ and $0 < 3m < 10^4$. From his table one can read the value of λ_χ , since in all cases $\lambda_\chi \leq 8$. Note that for $\chi = \theta_m$ the table is far more extensive than ours, while for $\chi = \theta_m\omega$ our computations go a bit farther. The overlapping parts of both tables are in agreement, except that the table in [6] omits the first negative m with $\lambda_\chi > 0$, namely $m = -2$. The nonvanishing of λ_χ in this case follows, by (11), from the fact that $\chi(3) = \theta_{-2}(3) = +1$. Our computation indeed shows, in agreement with [4], that $\lambda_\chi = 1$.

The first author has determined, for $p < 10^4$, the components λ_χ with $\chi = \theta_m\omega^{t-1}$ for $m = -1$ and $m = -3$ (see [2] and [1], respectively). For $t = 3, 5, \dots, p-2$, one has in this range $\lambda_\chi = 1$ if $(p, t-1)$ is an E -irregular or D -irregular pair, respectively, and $\lambda_\chi = 0$ otherwise. A comparison of the tables in [1] and [2] with the present Table 2 shows no discrepancies.

The paper [5] by Hao and Parry tabulates the “ m -irregular” primes $p < 5025$ for the values of m that appear in our Table 2. For a fixed m , the prime p is m -irregular if and only if there is at least one $t > 1$ such that $\lambda_\chi > 0$ with $\chi = \theta_m\omega^{t-1}$. It is easily checked that, for $p < 200$, the lists given in [5] coincide with the corresponding lists extracted from Table 2. Our computations show the somewhat interesting fact that every positive value of λ_χ in this region in fact equals 1, except for a single value $\lambda_\chi = 2$ occurring for $p = 23$ and $\chi = \theta_{-2}\omega^{10}$.

Let us finally mention that if $m = -q$, with q a prime, and $\theta_m(p) = -1$, then it follows from (11) that $\lambda_{\theta_m} > 0$ exactly when the class number of the field $\mathbf{Q}(\sqrt{-q})$ is divisible by p . Thus a partial check of our results is also provided by the class number tables of imaginary quadratic fields.

9. The computations. The computations were run on the DEC-20 computer at the University of Turku. The programs, written in Fortran, used only integer arithmetic.

As is seen from Sections 5 and 6, the main task was the computing of the sums $S_{n,k}$ (mostly for $n = 1$). This was started by searching a primitive root mod p and constructing the index table. After decomposing m into prime factors, the character values $\theta_m(a)$ were calculated via the Legendre symbol, using the congruence

$$\left(\frac{a}{q}\right) \equiv a^{(q-1)/2} \pmod{q} \quad (q \text{ an odd prime factor of } m)$$

and then checking that $\theta_m(a)$ indeed equals ± 1 or 0. For a fixed t , we chose a minimal $b > 0$ such that $(b, dp) = 1$ and $\theta_m(b)b' \not\equiv 1 \pmod{p}$. To find the value of $v(a)$ for $n = 1$ (see (6) and (5)), we computed $a^{p-1} \pmod{p^2}$ by employing the 2-adic expansion of $p-1$ and the residues of $a^2, a^4, a^8, \dots \pmod{p^2}$.

After computing the numbers $S_{1,k} \pmod{p}$ we searched for the first nonvanishing number in the sequence $T_0^{(1)}, \dots, T_{p-2}^{(1)} \pmod{p}$. The cases in which such a number did not exist were afterwards picked out by hand and dealt with on the level $n = 2$. The procedure on this level was similar, except that this time the determination of $v(a)$ required computations mod p^3 .

APPENDIX

TABLE 1

The positive values of λ_x for $p = 5$, $\chi = \theta_m \omega^{t-1}$ ($t = 2$ or 4) and $0 < m \leq 3147$.

m	t	λ_x	m	t	λ_x	m	t	λ_x	m	t	λ_x
14	2	1	267	2	2	509	2	2	734	2	1
23	2	2	271	4	1	509	4	1	734	4	2
26	2	1	278	4	3	514	4	1	741	2	1
31	2	1	281	2	1	519	4	1	743	2	1
37	2	2	282	2	1	523	4	1	753	2	1
38	4	1	287	4	1	526	4	1	753	4	2
39	4	1	293	2	1	534	4	1	754	2	1
42	4	1	298	2	1	537	4	1	758	2	1
51	4	1	298	4	1	541	4	1	759	2	1
53	4	1	307	2	1	543	4	1	761	4	1
59	2	1	313	2	2	554	2	1	763	2	1
62	4	1	314	4	2	559	2	1	766	2	1
69	4	1	326	4	1	574	2	1	767	4	1
73	4	1	347	2	1	574	4	1	781	2	1
82	4	1	358	2	1	577	2	1	789	4	1
86	2	1	366	4	1	581	2	1	791	4	1
89	4	1	382	4	1	581	4	2	794	2	1
107	4	1	391	2	1	587	2	1	798	4	1
109	4	2	398	2	2	587	4	2	809	2	1
114	4	3	401	4	1	589	4	1	814	2	1
123	2	1	407	4	1	591	2	1	817	4	1
127	2	2	422	4	1	597	4	2	822	2	1
127	4	2	426	2	1	602	4	1	839	2	1
129	2	1	426	4	1	606	4	1	842	2	1
134	4	1	427	2	1	611	2	1	851	2	1
139	4	1	427	4	1	617	4	1	857	4	1
143	4	1	433	4	2	622	4	1	861	2	2
149	2	2	434	4	2	623	4	2	869	4	2
159	2	1	438	2	1	626	2	2	874	4	1
161	4	1	439	4	4	626	4	1	881	2	1
183	4	1	446	2	2	627	4	1	881	4	1
186	4	1	453	4	2	629	2	2	887	2	1
187	2	1	457	2	3	629	4	1	889	2	1
191	2	1	457	4	1	631	2	1	893	4	1
191	4	1	458	2	1	633	2	3	903	2	1
199	2	1	466	4	2	634	4	1	911	2	1
202	4	1	467	2	1	643	2	1	917	2	1
211	4	1	467	4	1	654	2	1	921	2	1
213	2	1	469	2	1	662	4	1	922	2	1
214	4	1	471	2	2	673	4	1	922	4	1
217	4	1	473	4	1	674	4	1	923	4	1
222	2	1	479	2	1	678	2	1	926	2	1
223	2	1	489	4	1	679	2	1	933	4	1
227	2	1	497	4	1	681	2	1	937	2	1
237	2	1	498	2	2	683	2	1	939	2	1
238	4	1	499	2	1	687	4	1	943	4	1
241	4	1	499	4	1	689	2	1	946	2	1
253	2	2	501	2	1	699	4	1	947	4	1
257	2	1	501	4	1	717	2	1	949	2	1
257	4	1	502	2	1	719	4	2	957	2	1
259	4	1	502	4	1	727	4	1	966	4	1

TABLE 1 (*continued*)

m	t	λ_X	m	t	λ_X	m	t	λ_X	m	t	λ_X
973	2	1	1214	4	1	1483	2	1	1797	4	2
978	4	1	1217	2	1	1486	4	1	1798	4	1
982	4	1	1226	4	1	1493	2	1	1799	2	1
983	4	1	1231	2	3	1493	4	1	1803	4	1
986	2	1	1231	4	1	1506	2	1	1817	4	2
997	2	1	1238	2	1	1509	2	1	1829	2	1
1003	2	1	1238	4	1	1511	2	1	1829	4	1
1006	2	2	1243	4	1	1514	2	1	1834	2	1
1007	2	1	1247	4	2	1518	2	1	1834	4	1
1018	2	1	1253	2	1	1518	4	3	1837	2	2
1031	4	1	1254	2	1	1529	4	2	1838	4	1
1034	2	1	1261	4	1	1531	2	1	1846	2	1
1037	2	1	1262	4	1	1531	4	1	1847	2	1
1041	4	1	1267	4	1	1533	2	1	1851	4	1
1042	4	1	1273	2	1	1541	2	1	1853	2	1
1051	4	1	1279	2	1	1546	2	1	1853	4	1
1059	2	1	1279	4	1	1571	2	1	1861	4	1
1063	2	1	1281	4	1	1577	4	1	1874	4	1
1069	2	1	1289	4	1	1579	2	1	1882	2	1
1073	2	3	1291	2	1	1582	2	1	1891	4	1
1074	4	1	1293	4	1	1586	4	1	1897	4	3
1077	2	1	1294	2	1	1597	2	1	1898	4	1
1079	4	2	1301	4	1	1597	4	1	1907	2	1
1086	4	2	1313	4	2	1621	4	1	1907	4	1
1087	4	1	1317	4	3	1631	2	1	1913	2	1
1093	4	1	1321	4	2	1631	4	1	1913	4	1
1097	2	1	1327	2	1	1633	4	2	1914	2	1
1106	2	1	1327	4	1	1637	4	1	1914	4	1
1111	4	2	1338	4	1	1641	4	1	1921	4	1
1113	2	1	1339	4	1	1654	2	1	1923	2	1
1113	4	1	1342	2	1	1658	4	1	1934	2	1
1114	2	1	1351	4	1	1662	4	2	1937	2	1
1118	4	1	1354	2	1	1663	4	1	1938	2	1
1119	2	1	1366	2	1	1686	4	1	1938	4	1
1121	2	1	1366	4	1	1699	4	1	1941	2	1
1122	2	1	1379	2	1	1702	4	2	1943	4	3
1123	2	1	1382	4	1	1713	4	1	1949	2	2
1126	2	1	1389	2	1	1717	4	1	1954	2	1
1126	4	1	1389	4	1	1721	2	1	1957	2	1
1129	2	1	1393	4	2	1723	4	1	1959	4	1
1131	2	2	1398	4	1	1731	4	1	1966	4	1
1133	4	1	1401	2	1	1738	2	1	1969	4	1
1137	2	1	1402	4	2	1738	4	2	1973	2	1
1137	4	1	1406	4	1	1739	2	1	1977	2	1
1142	2	1	1407	4	1	1741	4	1	1979	2	1
1149	2	1	1426	4	1	1754	4	1	1982	4	1
1157	4	1	1427	4	4	1757	2	1	1986	4	1
1169	2	1	1429	4	1	1758	2	1	1999	2	1
1173	4	1	1434	2	2	1758	4	1	1999	4	1
1182	4	2	1434	4	1	1761	4	2	2001	2	1
1187	4	1	1441	4	1	1762	4	1	2002	2	1
1191	4	1	1443	4	3	1766	4	1	2002	4	1
1193	2	2	1451	2	1	1769	2	1	2003	2	1
1194	4	1	1461	4	1	1777	4	1	2003	4	3
1198	2	1	1466	2	2	1779	2	1	2014	4	1
1203	2	1	1478	2	3	1783	2	1	2027	2	1
1213	4	1	1479	4	2	1786	4	1	2027	4	1

TABLE 1 (*continued*)

m	t	λ_X	m	t	λ_X	m	t	λ_X	m	t	λ_X
2029	2	2	2307	2	1	2571	4	1	2878	4	1
2031	4	1	2307	4	1	2573	2	1	2882	2	2
2033	4	1	2314	2	1	2573	4	1	2886	4	1
2038	4	1	2317	2	1	2577	4	1	2887	2	1
2039	2	1	2323	4	3	2578	4	1	2911	4	1
2051	2	2	2326	4	1	2579	4	1	2914	2	2
2081	4	1	2329	2	1	2581	2	1	2923	4	1
2083	2	1	2333	2	1	2599	4	1	2927	2	1
2083	4	1	2334	4	1	2602	4	1	2931	4	1
2087	2	1	2341	2	1	2603	2	1	2946	2	1
2089	2	1	2342	4	1	2609	4	2	2949	2	1
2093	2	1	2347	2	1	2633	2	1	2959	2	1
2098	2	1	2353	2	2	2634	4	1	2963	2	2
2101	4	2	2354	2	1	2639	2	1	2966	4	1
2102	2	1	2359	4	1	2647	2	1	2967	2	2
2111	4	2	2362	4	1	2654	4	2	2971	2	1
2119	4	1	2373	2	2	2657	4	1	2973	4	1
2122	4	2	2381	2	1	2661	4	1	2974	2	2
2123	2	1	2386	2	1	2667	4	2	2983	4	2
2126	2	1	2386	4	1	2669	2	1	2986	4	2
2127	2	1	2391	2	1	2671	2	1	2991	2	1
2129	2	1	2391	4	2	2671	4	1	2991	4	1
2131	2	1	2397	4	1	2683	4	3	2993	4	1
2143	2	1	2399	4	1	2686	2	1	2994	4	1
2146	2	1	2406	4	1	2687	2	1	2998	2	1
2153	2	1	2411	4	1	2687	4	1	3013	2	1
2153	4	1	2433	2	2	2694	4	1	3014	2	2
2157	4	1	2438	2	2	2698	4	2	3023	2	1
2158	4	1	2438	4	1	2706	2	1	3039	4	1
2159	2	1	2446	2	1	2711	2	1	3041	4	1
2161	2	2	2449	2	1	2714	2	3	3053	2	1
2171	4	1	2459	4	2	2722	2	2	3054	4	1
2177	2	2	2462	2	2	2723	2	3	3059	4	2
2181	2	1	2471	4	2	2723	4	2	3077	4	1
2182	2	1	2481	2	2	2731	4	1	3082	4	1
2189	2	1	2482	2	1	2739	4	2	3098	2	1
2189	4	1	2486	2	1	2741	4	2	3101	4	2
2194	2	1	2487	2	1	2742	2	1	3102	2	1
2219	2	1	2489	2	3	2743	4	2	3103	2	1
2221	2	1	2498	4	3	2746	4	1	3106	2	2
2222	2	1	2501	4	2	2759	2	2	3107	2	1
2229	2	1	2503	2	1	2766	2	1	3111	4	1
2231	2	3	2503	4	1	2771	2	1	3113	2	1
2234	4	3	2509	2	1	2773	2	1	3121	2	1
2243	2	1	2513	4	1	2778	2	1	3121	4	1
2243	4	1	2519	2	1	2803	4	1	3126	2	1
2257	4	1	2522	4	2	2814	2	1	3127	2	1
2262	4	1	2526	2	1	2821	2	1	3129	2	1
2263	2	2	2531	2	1	2822	4	1	3129	4	1
2263	4	1	2533	4	1	2829	2	1	3134	4	1
2271	2	1	2534	2	2	2829	4	3	3138	4	1
2273	2	1	2546	4	1	2841	2	1	3147	2	1
2278	2	2	2551	2	1	2843	4	1			
2279	4	1	2557	4	1	2851	4	1			
2281	2	2	2558	4	1	2859	2	1			
2287	4	1	2563	2	1	2877	4	1			
2306	2	1	2566	4	1	2878	2	2			

TABLE 2

The values of t , $1 \leq t \leq p-1$, for which $\lambda_x > 0$ with $\chi = \theta_m \omega^{t-1}$, in the region $2 < p < 200$, $m = -7, -3, -2, -1, 2, 5$. The dagger (†) indicates that $\lambda_x = 2$; in all other cases $\lambda_x = 1$.

$p \backslash m$	-7	-3	-2	-1	2	5	$p \backslash m$	-7	-3	-2	-1	2	5
3			1				101				1		
5				1							63		
7		1					103		1				
11	1		1		4		107	1		1		64	22
13		1 [†]		1	12							86	100
17			1	1		14	109	1	1		1		
19		1	1	11	6	8	105						
			5				113	1	55	1	1		
23	1	17	11 [†]				65			13			
29	1			1						109			
19							127	1	1			40	62
												56	
31		1		23	30		131			1			32
37	1	1		1	34					51			64
41			1	1		18				57			
43	1	1	1	13			137	1		1	1		84
47	25	13		15				45		57	43		
53	1	29		1			139	21	1	1	129		44
	19	45							99	19			104
	43						149	1		79	1	146	22
59	33		1		34			39			147		
			19		36		103						
					50		151	1	1			14	66
61		1		1		42	157	101	1		1		
				7			163	1	1	1			144
67	1	1	1	27		6	167						66
		47					173	13		121	1	74	
71	1			29	68			97					
73	7	1	1	1		70	153						
	11		31				179	1		1		74	
79	1	1		19	16					119			
					30		181	35	1 [†]		1		
83	53		1				177						
	65		15				191	1					10
89			1	1	32			31					
			33				193	1	1	1	1		
97		1	1	1				59			75		
							197	1	179	191	1		
									183				
							199		1			186	
									161				

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