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The F-E-M-Test for Convergence of Nonconforming Finite Elements

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Summary. A new convergence test, the F-E-M-Test, is established for the method of nonconforming finite elements. The F-E-M-Test is simple to apply, it checks only the local properties of shape functions along each interface or on each element. The test is valid for a wide class of nonconforming elements in practical applications.

1. Introduction. A simple and widely used procedure for checking convergence of nonconforming finite elements is the patch test, first presented by Irons in [1], [4]. As originally phrased in terms of mechanics, the basic idea of the patch test is that if the boundary displacements of an arbitrary patch of assembled elements are subject to a constant strain state, then the solution of the finite element equations on the patch should reproduce this presumed solution exactly. The mathematical explanation of the Irons patch test was given by Strang [16], [17]. Let $a_h(u, v)$ be the discrete bilinear form of the given variational problem, u^* the true solution of the problem and u_h the finite element approximation. Then the patch test has the following mathematical formulation:

$$(1) \qquad d_h(u^*, v_h) \equiv a_h(u^*, v_h) - a_h(u_h, v_h) = 0 \quad \forall u^* \in P_m,\ v_h \in V_h,$$

where V_h is the finite element space in which the approximate solution u_h is sought, P_m is the space of polynomials of degree m and m is the highest order of derivatives appearing in the variational problem.

However, it has been proved in [19], [7], [8] that Irons's patch test or its equivalent, the formula (1), is neither necessary nor sufficient for convergence.

In a recent paper [21], Taylor et al. gave a discussion concerning the validity of the patch test from an engineer's point of view. A new form of the patch test, Test C, is formulated which checks not only the satisfaction of the basic differential equation but also of its natural or 'traction' boundary conditions, as well as of the stability requirement of approximate problems. Paper [21] claims that Test C is a correct interpretation of the patch test, which should provide a necessary and sufficient condition for convergence. The first counterexample of Stummel [19] to the patch test was checked in [21] by Test C and failed to pass the test. However, it is shown in [12] that Stummel's second example still passes Test C but fails to converge to the true solution for natural boundary conditions. Hence, Test C cannot be a sufficient

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condition for convergence of nonconforming finite elements. A further analysis and interpretation of the patch test is apparently needed.

Because of the limitations of the patch test, Stummel [18] has proposed the generalized patch test, which together with the approximability condition provides a necessary and sufficient condition for convergence of nonconforming elements applied to general elliptic boundary value problems. Many nonconforming elements have been successfully tested by the generalized patch test in [18], [7], [8], [9], [10]. The generalized patch test is a very powerful tool for the study of convergence properties of nonconforming elements. However, its usage seems to be difficult for engineers in practical situations.

The aim of this paper is to present a simple and effective convergence test which may easily be checked along each interface, the F-Test, or on each element, the E-M-Test. The new test has been first mentioned in [13] and later in [11], [3]. In this paper we describe the test in detail.

2. Formulation of the F-E-M-Test. 2.1. We consider variational equations of the form

$$(2) \qquad u_0 \in V;\ \sum_{|\sigma|,\,|\tau| \leqslant m} \int_G a_{\sigma\tau} D^\sigma u_0 D^\tau v\,dx = \sum_{|\sigma| \leqslant m} \int_G f_\sigma D^\sigma v\,dx \quad \forall v \in V,$$

where G is a polyhedral domain in $\mathbf{R}^n$ and V is a closed subspace of the Sobolev space $H^m(G) = \{v\colon D^\sigma v \in L^2(G), \forall\sigma \text{ such that } |\sigma| \leqslant m\}$, equipped with the norm

$$\|v\|_{m,G} = \left(\sum_{|\sigma| \leqslant m} \int_G |D^\sigma v|^2\,dx \right)^{1/2}$$

and the seminorm

$$|v|_{m,G} = \left(\sum_{|\sigma| = m} \int_G |D^\sigma v|^2\,dx \right)^{1/2}.$$

The coefficients $a_{\sigma\tau}$ are bounded measurable functions on G and $f_\sigma \in L^2(G)$ for $|\sigma| \leqslant m$. The variational equation (2) may be written in the form

$$(3) \qquad u_0 \in V; \quad a(u_0, v) = l(v) \quad \forall v \in V.$$

Dividing the domain G into a regular family of finite elements K with diameters $h_K \leqslant h$ and defining appropriate piecewise polynomial spaces V_h, the finite element approximation of the problem (3) then is to find $u_h \in V_h$ such that

$$(4) \qquad a_h(u_h, v_h) = l_h(v_h) \quad \forall v_h \in V_h,$$

where

$$a_h(u, v) = \sum_K \sum_{|\sigma|,\,|\tau| \leqslant m} \int_K a_{\sigma\tau} D^\sigma u D^\tau v\,dx,$$

$$l_h(v) = \sum_K \sum_{|\sigma| \leqslant m} \int_K f_\sigma D^\sigma v\,dx.$$

We assume, as usual, that the bilinear form $a(u, v)$ is continuous on $H^m(G) \times H^m(G)$ and V-elliptic over the space V, and that the discrete bilinear form $a_h(u, v)$ is uniformly V_h-elliptic over the spaces V_h. Then the Lax-Milgram Theorem guarantees the unique solvability of the variational equations (3), (4). Note that certain weaker stability conditions concerning the bilinear forms $a(u, v)$, $a_h(u, v)$ have been stated in [20].

2.2. The F-E-M-Test is now described for the problem (4) with $m = 1, 2$ corresponding to general elliptic boundary value problems of second and fourth order. The test consists of tests of two different types. The first is a face test, the F-Test, which checks the conditions (F) along each interface, and the second is an element test, the E-M-Test, which verifies the conditions (E) + (M) on each element. In the following, the notation $D_k v_h$ is used for the partial derivative of v_h with respect to x_k.

F_1-*Test*. The finite element space V_h is said to pass the F_1-Test for problems of order $2m$, if for every function $v_h \in V_h$ the jump of v_h, denoted by $[v_h]$, across each interface F of two adjacent elements K_1, K_2 satisfies the condition

$$\text{(F1)} \qquad \left| \int_F [v_h]\, ds \right| \leqslant o\left(h_K^{n/2}\right) \|v_h\|_{m, K_1 \cup K_2}, \qquad h_K = \max\left(h_{K_1}, h_{K_2}\right).$$

For every outer boundary $F \subset \partial K \cap \partial G$ with Dirichlet boundary conditions, we define the jump $[v_h]_F \equiv v_h|_F$ and the condition (F1) is understood as

$$\left| \int_F v_h\, ds \right| \leqslant o\left(h_K^{n/2}\right) \|v_h\|_{m,K}.$$

F_2-*Test*. For fourth-order problems the F_2-Test requires that the jumps $[D_k v_h]$ across each interface F satisfy the condition

$$\text{(F2)} \qquad \left| \int_F [D_k v_h]\, ds \right| \leqslant o\left(h_K^{n/2}\right) \|v_h\|_{2, K_1 \cup K_2}, \qquad k = 1, 2, \ldots, n.$$

For every outer boundary $F \subset \partial K \cap \partial G$ with Dirichlet boundary conditions we define $[D_k v_h]_F \equiv D_k v_h|_F$ and the condition (F2) reads

$$\left| \int_F D_k v_h\, ds \right| \leqslant o\left(h_K^{n/2}\right) \|v_h\|_{2,K}, \qquad k = 1, 2, \ldots, n.$$

In particular, if in the condition (F1) or (F2) the equality

$$\int_F [v_h]\, ds = 0 \quad \text{or} \quad \int_F [D_k v_h]\, ds = 0, \qquad k = 1, 2, \ldots, n,$$

holds for all $F \subset \partial K$, respectively, the F-Tests are called the strong F_1-Test or F_2-Test, respectively.

E_1-M_1-*Test*. The finite element space V_h is said to pass the E_1-M_1-Test for problems of order $2m$, if every function $v_h \in V_h$ can be decomposed into two parts, a continuous part $C_1(v_h)$ and a discontinuous part $N_1(v_h)$:

$$\text{(5)} \qquad v_h = C_1(v_h) + N_1(v_h),$$

such that on each element K the discontinuous part $N_1(v_h)$ satisfies the two conditions

$$\text{(E1)} \qquad \left| \int_{\partial K} N_1(v_h) n_r\, ds \right| \leqslant o\left(h_K^{n/2}\right) \|v_h\|_{m,K}, \qquad r = 1, 2, \ldots, n,$$

$$\text{(M1)} \qquad \left| \int_{\partial K} N_1(v_h)^2\, ds \right| \leqslant o\left(h_K^{-1}\right) \|v_h\|_{m,K}^2.$$

where n_r are the components of the unit outward normal vector on the boundary ∂K.

E_2-M_2-*Test*. For fourth-order problems, the E_2-M_2-Test requires that the first derivatives $D_k v_h$ can be decomposed into two parts

$$(6) \qquad D_k v_h = C_2(D_k v_h) + N_2(D_k v_h), \qquad k = 1, 2, \ldots, n,$$

where $C_2(D_k v_h)$ are continuous functions over all elements and $N_2(D_k v_h)$ are the associated remainder terms such that on each element K the discontinuous parts $N_2(D_k v_h)$ satisfy the conditions

$$(\mathrm{E2}) \qquad \left| \int_{\partial K} N_2(D_k v_h) n_r \, ds \right| \leqslant o\left(h_K^{n/2}\right) \| v_h \|_{2,K}, \qquad k, r = 1, 2, \ldots, n,$$

$$(\mathrm{M2}) \qquad \int_{\partial K} N_2(D_k v_h)^2 \, ds \leqslant o\left(h_K^{-1}\right) \| v_h \|_{2,K}^2, \qquad k = 1, 2, \ldots, n.$$

Similar to the strong F-Tests, if the equalities

$$\int_{\partial K} N_1(v_h) n_r \, ds = 0, \qquad r = 1, 2, \ldots, n,$$

or

$$\int_{\partial K} N_2(D_k v_h) n_r \, ds = 0, \qquad k, r = 1, 2, \ldots, n,$$

hold for every element K, respectively, the tests are called the strong E_1-M_1-Test or E_2-M_2-Test, respectively.

THEOREM 1. *For second-order problems* ($m = 1$), *the* F_1-*Test or the* E_1-M_1-*Test implies convergence.*

THEOREM 2. *For fourth-order problems* ($m = 2$), *the* F_1-*Test or the* E_1-M_1-*Test together with the* F_2-*Test or the* E_2-M_2-*Test imply convergence.*

The proof of these two theorems will be given in Section 4.

2.3. According to the above convergence theorems, we summarize the procedure of the F-E-M-Test as follows:

For second-order problems ($m = 1$) the test is carried out in two steps.

Step 1. Verify the F_1-Test for each interface and each outer boundary where Dirichlet boundary conditions are prescribed. If it is passed, convergence is guaranteed.

Step 2. If the F_1-Test fails, verify the E_1-M_1-Test for each element. We need a decomposition of the shape function v_h. When the vertices of the element are nodal points of v_h, the corresponding linear or bilinear Lagrangian interpolating polynomial for v_h at the vertices is a good choice of a continuous part $C_1(v_h)$ in (5). The discontinuous part $N_1(v_h)$ now is the remainder term of the interpolating polynomial. By interpolation theory (see [18, Inequality 2.1.(5)]) and the inverse property, we then have

$$(7) \qquad \int_{\partial K} N_1(v_h)^2 \, ds \leqslant C h_K^3 | v_h |_{2,K}^2 \leqslant C h_K | v_h |_{1,K}^2.$$

Here and later, C denotes a generic constant, independent of the mesh size h, which may have different values at different places. The inequality (7) obviously implies the condition (M1). Therefore, only the condition (E1) has to be verified. If it is passed, convergence follows.

For fourth-order problems ($m = 2$) we carry out three steps.

Step 1. Verify the F_1-Test. The condition (F1) with $m = 2$ holds if the shape function v_h has two nodal points on each side of the elements, since in that case interpolation theory gives

$$\int_F [v_h]^2\, ds \leqslant Ch_K^3 |v_h|_{2,K_1\cup K_2}^2 \tag{8}$$

for each interface $F = K_1 \cap K_2$, and

$$\int_F v_h^2\, ds \leqslant Ch_K^3 |v_h|_{2,K}^2 \tag{9}$$

for each outer boundary $F \subset \partial K \cap \partial G$ with Dirichlet boundary conditions.

We may also verify the E_1-M_1-Test. The conditions (E1) + (M1) with $m = 2$ hold if the shape function v_h is continuous at the vertices of the elements, because in that case the first inequality in (7), that is,

$$\int_{\partial K} N_1(v_h)^2\, ds \leqslant Ch_K^3 |v_h|_{2,K}^2,$$

implies both the condition (E1) and (M1) for $m = 2$.

In particular, if a plate element under consideration is a C^0-element, then both the F_1-Test and the E_1-M_1-Test are satisfied a priori.

It is a common practice that for fourth-order problems every element has two nodal points of function values on each side of the elements, usually at the vertices. Therefore, the F_1-Test or the E_1-M_1-Test for fourth-order problems is valid in practice.

Step 2. Verify the F_2-Test. If it is passed and Step 1 was successful, convergence is guaranteed.

Step 3. If the F_2-Test fails, verify the E_2-M_2-Test. We need certain decompositions of the first derivatives $D_k v_h$. If the vertices of the elements are nodal points of $D_k v_h$, the corresponding linear or bilinear interpolating polynomials of $D_k v_h$ at the vertices are usually chosen as the continuous parts of $D_k v_h$ in the decomposition form (6). Then the remainder terms $N_2(D_k v_h)$ satisfy the inequalities

$$\int_{\partial K} N_2(D_k v_h)^2\, ds \leqslant Ch_K |v_h|_{2,K}^2, \qquad k = 1, 2, \ldots, n, \tag{10}$$

which imply the condition (M2). In this case, only the condition (E2) has to be verified. If it is passed and Step 1 was successful, convergence follows.

Remarks. 1. The strong F-Tests or the strong E-M-Tests imply the satisfaction of the Irons patch test.

2. The F-E-M-Test can be applied for assessing the convergence of certain nonconforming elements that do not pass Irons's patch test in the sense of the formulation (1), as will be demonstrated in Section 3.

3. For fourth-order problems the conditions (F2), (E2), (M2) are simply obtained from the corresponding conditions (F1), (E1), (M1) by replacing the shape functions v_h by their first derivatives $D_k v_h$, $k = 1, 2, \ldots, n$.

4. As we have seen in the above procedure of carrying out the F-E-M-Test, the essential conditions which have to be verified are (F1) or (E1), and (F2) or (E2), for second-order and fourth-order problems, respectively. The other conditions may simply be proved in most practical cases by the continuity assumptions on the shape functions or their first derivatives at certain nodal points of the elements. The F-E-M-Test is simple to apply.

3. Applications. It will be proved in this section that many well-known nonconforming elements, as well as some newly introduced elements, pass the F-E-M-Test.

3.1. *The Crouzeix-Raviart Elements.* This is a class of triangular elements. The nodal parameters are the function values at rth order Gaussian points on each side F of the triangle K. The shape functions $v_h \in V_h$ are piecewise polynomials:

$$v_h^K \in P_r(K), \quad v_h^K|_F \in P_r(F),$$

for each triangle K and for each side F of K.

Since every function $v_h \in V_h$ is continuous at rth order Gaussian points on each interface $F = K_1 \cap K_2$ and the quadrature formula having these Gaussian points as nodal points is exact for all polynomials of degree $2r - 1$ in one variable on F, we obtain

$$\int_F [v_h]\, ds = 0, \quad \forall F = K_1 \cap K_2.$$

For $F \subset \partial K \cap \partial G$ with Dirichlet boundary conditions,

$$\int_F v_h\, ds = 0.$$

The Crouzeix-Raviart elements thus pass the strong F_1-Test.

3.2. *Wilson's Element.* This is a rectangular element. The nodal parameters are the function values at the vertices of the rectangle K and the mean values of the second derivatives $D_1 v_h$ and $D_2 v_h$ on K, respectively. The latter are two internal degrees of freedom which can be eliminated at the element level. The shape function v_h on each rectangle is a full quadratic polynomial.

Let $Q_1(v_h)$ be the piecewise bilinear interpolating polynomial of the shape function v_h at the vertices of all elements and $R_1(v_h)$ be the associated remainder term. Then $Q_1(v_h)$ is a continuous function over all elements. We have the decomposition

$$v_h = Q_1(v_h) + R_1(v_h).$$

It can easily be verified that for every rectangle K

$$\int_{\partial K} R_1(v_h) n_r\, ds = 0, \qquad r = 1, 2,$$

so that the strong (E1) condition holds. Moreover, it is known from [18, Section 2.2] that the remainder term $R_1(v_h)$ satisfies the inequality

$$\int_{\partial K} R_1(v_h)^2\, ds \leqslant C h_K |v_h|_{1,K}^2,$$

thus the condition (M1) is also satisfied. Therefore, the rectangular Wilson element passes the strong E_1-M_1-Test.

Now we apply the E_1-M_1-Test to the quadrilateral Wilson element which violates the patch test. The convergence has been previously proved in [8] under the condition that the distance d_K between the midpoints of the diagonals of each quadrilateral K is of order $o(h_K)$ uniformly for all elements as $h \to 0$.

For the E_1-M_1-Test we need a decomposition formula (5). Following Step 2 of Subsection 2.3, the 4-node isoparametric bilinear interpolating polynomial of the shape function v_h is chosen as the conforming part $C_1(v_h)$. Then it can be shown (see [8]) that for every quadrilateral K the remainder term $N_1(v_h) = v_h - C_1(v_h)$ satisfies the inequalities

$$\left| \int_{\partial K} N_1(v_h) n_r \, ds \right| \leqslant C d_K |v_h|_{1,K}, \qquad r = 1,2, \tag{11}$$

and

$$\int_{\partial K} N_1(v_h)^2 \, ds \leqslant C h_K |v_h|_{1,K}^2.$$

Comparing the inequality (11) with the condition (E1), we find that the condition $d_K = o(h_K)$ makes the quadrilateral Wilson element pass the E_1-M_1-Test.

Remark. Two 8-node quadrilateral elements of Sander and Beckers [5] that do not pass Irons's patch test have been analyzed in [7], where it was shown that these elements also converge under the condition $d_K = o(h_K)$. Like the quadrilateral Wilson element, by use of the 8-node isoparametric Lagrangian interpolating polynomial of the shape functions as their conforming parts, it can also be proved that the two elements of Sander and Beckers pass the E_1-M_1-Test under the condition $d_K = o(h_K)$. Thus we have seen that our new F-E-M-Test is able to prove the convergence of these elements that do not pass the patch test.

3.3. *A New 4-Node Quadrilateral Element.* Taylor et al. introduced a new element in [21]. On each quadrilateral K the conforming part $C_1(v_h)$ of the shape function v_h is the standard 4-node isoparametric bilinear polynomial, as in the quadrilateral Wilson element stated above. The nonconforming part $N_1(v_h)$ is constructed as a linear combination of four special cubic polynomials vanishing at the vertices of the reference square $\hat{K} = [-1,1] \times [-1,1]$:

$$\begin{aligned} N_1(v_h) = {} & (1-\xi^2)(1-\eta)a_1 + (1+\xi)(1-\eta^2)a_2 \\ & + (1-\xi^2)(1+\eta)a_3 + (1-\xi)(1-\eta^2)a_4. \end{aligned} \tag{12}$$

Substitution of $N_1(v_h)$ into the condition (E1) to satisfy the strong (E1) condition yields two linear equations for the unknown parameters a_i, $1 \leqslant i \leqslant 4$. Eliminating two of the a_i gives two cubic polynomials which form the nonconforming part $N_1(v_h)$ and are added to the conforming part $C_1(v_h)$. Obviously, the new element so constructed satisfies the strong E_1-M_1-Test and thus yields convergence.

3.4. *Modifications of Stummel's Examples.* It is known [6], [12], [19] that two examples of Stummel pass the patch test but do not imply convergence to the correct solution. A simple modification of Stummel's first example has been given in [14], [21], which replaces the nonconforming step function w_j on each subinterval I_j, scaled to the reference interval $[-1, 1]$, by the new quadratic polynomial

$$\varphi_j = \left[1 + \varepsilon(1 - s^2)\right] w_j, \qquad -1 \leqslant s \leqslant 1, \tag{13}$$

where ε is an arbitrary small constant but larger than round-off.

Actually, the modification (13) may be further extended by introducing on each element I_j a general nonconforming basis function

$$\varphi_j = f(s) w_j, \qquad -1 \leqslant s \leqslant 1, \tag{14}$$

where $f(s) \in H^1[-1, 1]$, $f(-1) = f(1) = 1$, $f(s) \not\equiv 1$. The formula (13) is a special case of (14). Convergence of the modification (14) can be checked by the E_1-M_1-Test. In fact, let us decompose the shape function v_h of the new modified element as follows:

$$v_h = Y_h + Z_h, \tag{15}$$

where the conforming part Y_h is the usual continuous piecewise linear polynomial and the nonconforming part Z_h consists of the basis functions φ_j of (14). Evidently, the nonconforming part Z_h satisfies the strong (E1) condition on each element $I_j = [x_{j-1}, x_j]$:

$$Z_h(x_j - 0) - Z_h(x_{j-1} + 0) = 0. \tag{16}$$

As for the condition (M1), it is not obvious whether or not the nonconforming part Z_h satisfies this condition, because now the conforming part Y_h in (15) is not the linear interpolating polynomial of v_h at the nodal points, and interpolation theory, therefore, is not available for Z_h. However, after a direct calculation it is found that Z_h satisfies the following inequality

$$Z_h(x_{j-1} + 0)^2 + Z_h(x_j - 0)^2 \leqslant \frac{h}{\int_{-1}^{1} f'(s)^2\, ds} |v_h|^2_{1, I_j}, \tag{17}$$

which shows that the condition (M1) is still valid, so that the modification (14) passes the strong E_1-M_1-Test. A similar modification can be made for Stummel's second example.

We note that Stummel's examples pass the strong (E1) condition as well, but do not satisfy the condition (M1).

3.5. *Adini's Element*. This is a well-known C^0 rectangular plate element. The nodal parameters are the function values and the two first derivatives at the vertices of the rectangle K. The shape function v_h on K has the form

$$v_h^K \in P_3(K) + \left[x_1^3 x_2, x_1 x_2^3\right].$$

We use the E-M-Test. Since it is a C^0-element, both the F_1-Test and the E_1-M_1-Test are automatically passed. Now we verify the E_2-M_2-Test. By definition, the derivatives $D_k v_h$ are continuous at the vertices of the elements, so that the bilinear interpolating polynomials $Q_1(D_k v_h)$ of $D_k v_h$ at the vertices are continuous over all elements, which gives the following decompositions:

$$D_k v_h = Q_1(D_k v_h) + R_1(D_k v_h), \qquad k = 1, 2.$$

It has been shown in [18] that for every rectangle K,

$$\int_{\partial K} R_1(D_k v_h) n_r\, ds = 0, \qquad k, r = 1, 2.$$

In addition, from interpolation theory we have

$$\int_{\partial K} R_1 (D_k v_h)^2 \, ds \leqslant C h_K |v_h|^2_{2,K}.$$

The strong condition (E2) and the condition (M2) are then satisfied. Therefore, Adini's element passes the F_1-Test, the E_1-M_1-Test, and the strong E_2-M_2-Test.

3.6. *Morley's Element*. This is a triangular plate element. The nodal parameters are the function values at the vertices of the triangle K and the first derivatives in normal direction at the midside nodes. The shape function v_h on K is a quadratic polynomial $v_h^K \in P_2(K)$.

We use the F-Test. First, it is easily seen that the F_1-Test and the E_1-M_1-Test are passed because of the continuity of v_h at the vertices of the elements. Next, by the definition of the element, on each interface F the jump $[D_n v_h]$ is a linear polynomial in one variable vanishing at the midpoint of F. The midpoint rule gives

$$\int_F [D_n v_h] = 0, \qquad D_n v_h = \partial v_h / \partial n. \tag{18}$$

On the other hand,

$$\int_F D_s v_h = v_h(b) - v_h(a), \qquad D_s v_h = \partial v_h / \partial s,$$

a, b being the endpoints of F. Since the shape function v_h is continuous at the vertices of all elements, we have

$$\int_F [D_s v_h] \, ds = 0. \tag{19}$$

The equalities (18), (19) imply

$$\int_F [D_k v_h] \, ds = 0, \qquad k = 1, 2, \ F = K_1 \cap K_2.$$

For $F \subset \partial K \cap \partial G$ with Dirichlet boundary condition, we also have

$$\int_F D_k v_h \, ds = 0, \qquad k = 1, 2.$$

Hence the strong F_2-Test is passed. Morley's element thus passes the F_1-Test, the E_1-M_1-Test, and the strong F_2-Test.

3.7. *De Veubeke's Element* [2, Fig. 6(b)]. This is a triangular element. The nodal parameters are the function values at the vertices of the triangle K and at the center and the values of the first derivatives in normal direction at the second-order Gaussian points on each side of K. The shape function v_h on K is a full cubic polynomial $v_h^K \in P_3(K)$.

We use again the F-Test. Like Morley's element, the F_1-Test and the E_1-M_1-Test are obviously valid. As for the F_2-Test, we note that the jump $[D_n v_h]$ of de Veubeke's element across each interface F is a quadratic polynomial in one variable vanishing at the second-order Gaussian points. Application of the quadrature formula, having these two Gaussian points as nodal points, yields

$$\int_F [D_n v_h] \, ds = 0, \qquad F = K_1 \cap K_2,$$

from which, by the same argument used in Morley's element, we conclude

$$\int_F [D_k v_h]\, ds = 0, \qquad F = K_1 \cap K_2,$$

and

$$\int_F D_k v_h\, ds = 0$$

for outer boundaries F with Dirichlet conditions. Therefore, de Veubeke's element passes the F_1-Test, the E_1-M_1-Test, as well as the strong F_2-Test.

3.8. *Specht's Element* [15]. This is a new triangular plate element. The nodal parameters are the function values and the two first derivatives at the vertices of the triangle K. The shape function v_h on K has the form

$$\begin{aligned} v_h^K = [\lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_3\lambda_1, \lambda_1^2\lambda_2, \lambda_2^2\lambda_3, \lambda_3^2\lambda_1, \\ \lambda_1^2\lambda_2\lambda_3, \lambda_1\lambda_2^2\lambda_3, \lambda_1\lambda_2\lambda_3^2], \end{aligned} \tag{20}$$

using the area coordinates λ_i of the triangle K. Three additional constraints

$$\int_{\lambda_i=0} P_2(s) D_n v_h\, ds = 0, \qquad i = 1,2,3, \tag{21}$$

along the sides of K are introduced, which in conjunction with the nine nodal parameters uniquely define a polynomial v_h of the form (20) on the triangle K. In (21), P_2 denotes the Legendre polynomial of degree 2 on the sides $\lambda_i = 0$.

We still apply the F-Test. Since the shape function v_h is continuous at the vertices of the elements, the F_1-Test and the E_1-M_1-Test are valid. Secondly, by assumption, the jump $[D_n v_h]_F$ across each interface F is a polynomial of third degree in one variable and vanishes at the endpoints of F. Then, using the constraints (21), the expansion of $[D_n v_h]_F$ in Legendre polynomials takes the form

$$[D_n v_h]_F = a_1 P_1(s) + a_3 P_3(s), \tag{22}$$

where P_j, $j = 1,3$, are the Legendre polynomials of first and third degree. In the expansion (22) there is no constant term P_0 by virtue of the fact that the function $[D_n v_h]_F$ has two zeros at the endpoints of F and P_1, P_3 are odd functions. From (22) it follows immediately that

$$\int_F [D_n v_h]\, ds = 0. \tag{23}$$

Once we have the above equality (23), using the continuity of v_h at the vertices and Dirichlet boundary conditions, we may conclude as in Morley's and de Veubeke's elements that the strong F_2-Test is valid. Hence Specht's element passes the F_1-Test, the E_1-M_1-Test, and the strong F_2-Test.

4. Proof of the Theorems. We apply the generalized patch test of Stummel and prove the theorems stated in Section 2 by a series of lemmas. According to [18], for a second-order problem ($m = 1$), the generalized patch test consists in verifying that as $h \to 0$, the relations

$$T_r(\psi, v_h) = \sum_K \int_{\partial K} \psi v_h n_r\, ds \to 0, \qquad r = 1,2,\ldots,n, \tag{24}$$

hold for every bounded sequence $v_h \in V_h$ and for all test functions $\psi \in C_0^\infty(G)$ ($\psi \in C_0^\infty(\mathbf{R}^n)$ in the case of Dirichlet boundary conditions), where n_r is defined as in (E1), (M1). For a fourth-order problem ($m = 2$) the test requires that, as $h \to 0$, the relations

$$T_r(\psi, v_h) = \sum_K \int_{\partial K} \psi v_h n_r \, ds \to 0, \qquad r = 1, 2, \ldots, n, \tag{25}$$

$$T_{k,r}(\psi, v_h) = \sum_K \int_{\partial K} \psi D_k v_h n_r \, ds \to 0, \qquad k, r = 1, 2, \ldots, n, \tag{26}$$

hold for every bounded sequence $v_h \in V_h$ and for the same class of test functions as used in a second-order problem.

LEMMA 1. *The condition* (F1) *gives*

$$|T_r(\psi, v_h)| \leqslant Ch|\psi|_1 |v_h|_{1,h} + o(1)\|\psi\|_1 \|v_h\|_{m,h}, \qquad r = 1, \ldots, n,$$

using the norm

$$\|u_h\|_{m,h} = \left(\sum_K \sum_{|\sigma| \leqslant m} \int_K (D^\sigma u_h)^2 \, dx \right)^{1/2}.$$

Proof. For every function $f \in L^2(F)$ let

$$P_0^F f = \frac{1}{|F|} \int_F f \, ds, \qquad |F| = \int_F 1 \, ds \tag{27}$$

be the mean value of f over the side F. The associated remainder term is

$$R_0^F f = f - P_0^F f. \tag{28}$$

Then we write

$$\begin{aligned} T_r(\psi, v_h) = {} & \sum_K \sum_{F \subset \partial K} \int_F P_0^F \psi R_0^F v_h n_r \, ds + \sum_K \sum_{F \subset \partial K} \int_F R_0^F \psi R_0^F v_h n_r \, ds \\ & + \sum_K \sum_{F \subset \partial K} \int_F \psi P_0^F v_h n_r \, ds. \end{aligned} \tag{29}$$

By the definition of the operators P_0^F and R_0^F, the first term on the right-hand side of (29) vanishes,

$$\sum_K \sum_{F \subset \partial K} \int_F P_0^F \psi R_0^F v_h n_r \, ds = 0. \tag{30}$$

By an application of Schwarz's inequality and interpolation theory [18, 2.1(5)] the integrals in the second sum are bounded by

$$\left| \int_F R_0^F \psi R_0^F v_h n_r \, ds \right| \leqslant \left(\int_F (R_0^F \psi)^2 \, ds \right)^{1/2} \left(\int_F (R_0^F v_h)^2 \, ds \right)^{1/2} \leqslant Ch_K |\psi|_{1,K} |v_h|_{1,K},$$

and so

$$\left| \sum_K \sum_{F \subset \partial K} \int_F R_0^F \psi R_0^F v_h n_r \, ds \right| \leqslant Ch |\psi|_1 |v_h|_{1,h}. \tag{31}$$

The third term is

$$\sum_K \sum_{F\subset \partial K} \int_F \psi P_0^F v_h n_r \, ds = \sum_F |F| P_0^F \psi P_0^F [v_h] n_r^F .$$

Using the condition (F1), the regularity assumption of element partitions and an inequality of the type in [18, 2.1(3)], we have

$$\begin{aligned} |F| |P_0^F \psi P_0^F [v_h] n_r^F| &\leqslant \frac{1}{|F|} \int_F |\psi| \, ds \left| \int_F [v_h] \, ds \right| \\ &\leqslant \frac{1}{\sqrt{|F| h_K}} o\left(h_K^{n/2}\right) \|\psi|_{1, K_1 \cup K_2} \| v_h \|_{m, K_1 \cup K_2} \\ &\leqslant o(1) \|\psi\|_{1, K_1 \cup K_2} \| v_h \|_{m, K_1 \cup K_2} \end{aligned}$$

for each interface $F = K_1 \cap K_2$, and

$$|F| |P_0^F \psi P_0^F [v_h] n_r^F| \leqslant o(1) \|\psi\|_{1,K} \|v_h\|_{m,K}$$

for $F \subset \partial K \cap \partial G$ with Dirichlet boundary conditions. Otherwise,

$$|F| |P_0^F \psi P_0^F [v_h] n_r^F| = 0$$

for $F \subset \partial K \cap \partial G$, since in that case $\psi \in C_0^\infty(G)$. Therefore,

$$\left| \sum_K \sum_{F\subset \partial K} \int_F \psi P_0^F v_h n_r \, ds \right| \leqslant o(1) \|\psi\|_1 \|v_h\|_{m,h} . \tag{32}$$

Combining (29)–(32), we have proved Lemma 1. □

LEMMA 2. *The condition* (E1) + (M1) *gives*

$$|T_r(\psi, v_h)| \leqslant o(1) \|\psi\|_1 \|v_h\|_{m,h}, \qquad r = 1, 2, \ldots, n .$$

Proof. By virtue of the decomposition (5) of the function v_h we have

$$\sum_K \int_{\partial K} \psi C_1(v_h) n_r \, ds = 0,$$

and so

$$T_r(\psi, v_h) = \sum_K \int_{\partial K} \psi N_1(v_h) n_r \, ds . \tag{33}$$

For every function $f \in L^2(K)$ let

$$P_0^K f = \frac{1}{|K|} \int_K f \, dx, \qquad |K| = \int_K 1 \, dx$$

be the mean value of f over the element K. The associated remainder term is

$$R_0^K f = f - P_0^K f .$$

Then we write

$$
\begin{aligned}
T_r(\psi, v_h) &= \sum_K \int_{\partial K} \psi N_1(v_h) n_r \, ds = \sum_K \int_{\partial K} P_0^K \psi N_1(v_h) n_r \, ds \\
&\quad + \sum_K \int_{\partial K} R_0^K \psi N_1(v_h) n_r \, ds.
\end{aligned} \tag{34}
$$

Applying the condition (E1) and the regularity assumption of element partitions to the first term on the right-hand side of (34) gives

$$
\begin{aligned}
\left| \sum_K \int_{\partial K} P_0^K \psi N_1(v_h) n_r \, ds \right| &\leqslant \sum_K |P_0^K \psi| \left| \int_{\partial K} N_1(v_h) n_r \, ds \right| \\
&\leqslant \sum_K \frac{1}{\sqrt{|K|}} o(h_K^{n/2}) \|\psi\|_{0,K} \|v_h\|_{m,K} \leqslant \sum_K o(1) \|\psi\|_{0,K} \|v_h\|_{m,K} \\
&\leqslant o(1) \|\psi\|_0 \|v_h\|_{m,h}.
\end{aligned} \tag{35}
$$

The second term can be estimated by use of the condition (M1) and interpolation theory as follows:

$$
\begin{aligned}
\left| \int_{\partial K} R_0^K \psi N_1(v_h) n_r \, ds \right| &\leqslant \left(\int_{\partial K} (R_0^K \psi)^2 \, ds \right)^{1/2} \left(\int_{\partial K} N_1(v_h)^2 \, ds \right)^{1/2} \\
&\leqslant o(1) |\psi|_{1,K} \|v_h\|_{m,K}.
\end{aligned}
$$

Therefore,

$$
\left| \sum_K \int_{\partial K} R_0^K \psi N_1(v_h) n_r \, ds \right| \leqslant o(1) |\psi|_1 \|v_h\|_{m,h}. \tag{36}
$$

Lemma 2 now follows from (34)–(36). □

LEMMA 3. *The condition* (F2) *yields*

$$
|T_{k,r}(\psi, v_h)| \leqslant Ch |\psi|_1 |v|_{2,h} + o(1) \|\psi\|_1 \|v_h\|_{2,h}, \qquad k, r = 1, 2, \ldots, n. \tag{37}
$$

Proof. We recall that the condition (F2) is obtained from the condition (F1) by replacing the functions v_h by their derivatives $D_k v_h$. Therefore, Lemma 3 follows as in the proof of Lemma 1 by replacing v_h by $D_k v_h$. □

LEMMA 4. *The condition* (E2) + (M2) *yields*

$$
|T_{k,r}(\psi, v_h)| \leqslant o(1) \|\psi\|_1 \|v_h\|_{2,h}, \qquad k, r = 1, 2, \ldots, n. \tag{38}
$$

Proof. Using the same argument as in the proof of Lemma 2, Lemma 4 is obtained by replacing v_h by $D_k v_h$, and C_1, N_1 by C_2, N_2. □

From the above lemmas it follows that the F-E-M-Test, described in Section 2, provides a simple sufficient condition for the validity of the generalized patch test for second-order and fourth-order problems. Therefore, the F-E-M-Test can be used for assessing convergence of nonconforming elements applied to general elliptic boundary value problems of order two or four.

We remark in passing that the F-E-M-Test is not necessary for convergence. It is only a sufficient condition.

Finally, we would like to emphasize the fact that the F-E-M Test checks only local properties of the shape functions and/or their first derivatives, namely along each interface or on each element. In practice, most of the nonconforming elements, invented by engineers, are constructed by mechanical considerations and intuitions, based upon a local analysis of shape functions and their derivatives on an individual element, which leads exactly to the condition (F) or the condition (E) + (M), e.g., Morley's, de Veubeke's, Sander-Beckers', and Taylor's elements. Therefore, the F-E-M-Test should be able to deal with a sufficiently wide class of nonconforming elements in practical applications.

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