Noninterpolatory Integration Rules for Cauchy Principal Value Integrals

By P. Rabinowitz* and D. S. Lubinsky**

Dedicated to the memory of Peter Henrici

Abstract. Let $w(x)$ be an admissible weight on $[-1, 1]$ and let $\{p_n(x)\}_0^\infty$ be its associated sequence of orthonormal polynomials. We study the convergence of noninterpolatory integration rules for approximating Cauchy principal value integrals

$$I(f; \lambda) := \oint_{-1}^1 w(x) \frac{f(x)}{x - \lambda} dx, \quad \lambda \in (-1, 1).$$

This requires investigation of the convergence of the expansion

$$I(f; \lambda) \sim \sum_{k=0}^{\infty} (f, p_k) q_k(\lambda), \quad \lambda \in (-1, 1),$$

in terms of the functions of the second kind $\{q_k(\lambda)\}_0^\infty$ associated with w , where

$$(f, p_k) := \int_{-1}^1 w(x) f(x) p_k(x) dx \quad \text{and} \quad q_k(\lambda) := \oint_{-1}^1 w(x) \frac{p_k(x)}{x - \lambda} dx,$$

$$k = 0, 1, 2, \dots, \lambda \in (-1, 1).$$

1. Introduction. In the third volume of his monumental work, *Applied and Computational Complex Analysis*, Henrici [8, pp. 139-142] gave an algorithm for the numerical evaluation of *Cauchy principal value* (CPV) integrals. This algorithm was presented in a more explicit form in a recent paper, by one of the authors [15]. In neither case were convergence questions considered. In this paper, we shall analyze the convergence questions arising from the use of this algorithm.

Consider the CPV integral of the form

$$(1) \quad I(f; \lambda) := \oint_{-1}^1 w(x) \frac{f(x)}{x - \lambda} dx, \quad -1 < \lambda < 1,$$

where w is an admissible weight function, $w \in \mathcal{A}$, that is, $w(x)$ is nonnegative and integrable in $[-1, 1]$ and

$$(2) \quad m_0 := \int_{-1}^1 w(x) dx > 0.$$

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*Research completed during a visit to N.R.I.M.S.

**Part-time at the Department of Mathematics, University of the Witwatersrand, P.O. Wits 2050, Republic of South Africa.

For such w , there exist sequences of *orthonormal polynomials*

$$(3) \quad \{p_n(x) := p_n(w, x) := k_n x^n + \cdots, \quad k_n > 0\},$$

with respect to the inner product

$$(4) \quad (f, g) := \int_{-1}^1 w(x) f(x) g(x) dx,$$

satisfying a three-term recurrence relation

$$(5) \quad x p_n(x) = \alpha_{n+1} p_{n+1}(x) + \beta_{n+1} p_n(x) + \alpha_n p_{n-1}(x), \quad n = 0, 1, 2, \dots,$$

where

$$\alpha_n := k_{n-1}/k_n, \quad n \geq 1; \quad \beta_{n+1} := (x p_n, p_n), \quad n \geq 0, \\ p_{-1}(x) \equiv 0 \quad \text{and} \quad p_0(x) \equiv k_0 = m_0^{-1/2}.$$

If we define $q_n(\lambda)$, the *function of the second kind*, by

$$(6) \quad q_n(\lambda) := q_n(w, \lambda) := I(p_n; \lambda) := \int_{-1}^1 w(x) \frac{p_n(x)}{x - \lambda} dx, \quad -1 < \lambda < 1,$$

then the $q_n(\lambda)$ satisfy the same recurrence relation as the $\{p_n(x)\}$, namely

$$(7) \quad \lambda q_n(\lambda) = \alpha_{n+1} q_{n+1}(\lambda) + \beta_{n+1} q_n(\lambda) + \alpha_n q_{n-1}(\lambda), \quad n = 0, 1, 2, \dots,$$

with starting values $q_{-1}(\lambda) \equiv -1$, $q_0(\lambda) \equiv I(p_0; \lambda)$ and $\alpha_0 := m_0^{1/2}$. If we denote by

$$(8) \quad a_k := (f, p_k)$$

the Fourier coefficient of $p_k(x)$ in the formal expansion of $f(x)$,

$$(9) \quad f(x) \sim \sum_{k=0}^{\infty} a_k p_k(x),$$

then we can write a formal expansion for $I(f; \lambda)$ in terms of the $q_n(\lambda)$,

$$(10) \quad I(f; \lambda) \sim \sum_{k=0}^{\infty} a_k q_k(\lambda).$$

Hence, an approximation to $I(f; \lambda)$ will be given by the truncated sum

$$(11) \quad S_N(f; \lambda) := \sum_{k=0}^N a_k q_k(\lambda).$$

If we now have a sequence of integration rules

$$(12) \quad Q_m(g) := \sum_{i=1}^m w_{im} g(x_{im}),$$

which converges to

$$(13) \quad I(g) := \int_{-1}^1 w(x) g(x) dx$$

for all $g \in C[-1, 1]$ or all $g \in R[-1, 1]$, the space of bounded Riemann integrable functions on $[-1, 1]$, and if we approximate the Fourier coefficients a_k by

$$(14) \quad a_{km} := Q_m(f p_k),$$

then, in general, we obtain a noninterpolatory integration rule for $I(f; \lambda)$, namely

$$(15) \quad Q_m^N(f; \lambda) := \sum_{k=0}^N a_{km} q_k(\lambda).$$

The approximations $Q_m^N(f; \lambda)$ can be evaluated in a stable manner using backward recursion by the algorithm given in [15], provided that we have the value of $q_0(\lambda)$. We can also express $Q_m^N(f; \lambda)$ in a Lagrangian form that is more useful in the numerical solution of integral equations:

$$(16) \quad Q_m^N(f; \lambda) = \sum_{i=1}^m w_{im}^N(\lambda) f(x_{im}),$$

where the weights

$$(17) \quad w_{im}^N(\lambda) := w_{im} \sum_{k=0}^N p_k(x_{im}) q_k(\lambda), \quad i = 1, 2, \dots, m,$$

can also be evaluated in a stable manner by the backward recursion algorithm [15].

As indicated above, this general approach to the numerical evaluation of CPV integrals appears in Henrici [8, pp. 139–142]. However, there is no discussion there of convergence or of the integration rules $Q_m(g)$. In fact, it is precisely the freedom in the choice of these rules, subject only to the condition that they converge to $I(g)$ for all $g \in C[-1, 1]$ or all $g \in R[-1, 1]$, that affords this method for evaluating CPV integrals considerable interest. Thus, if f is well behaved in most of the interval $[-1, 1]$, but is irregular over a small subinterval $[a, b] \subset [-1, 1]$, then we can concentrate most of our integration points x_{im} in $[a, b]$.

This was also done by Gerasoulis [7] using a different approach, and the results he achieved were a considerable improvement over those achieved using a conventional spacing of integration points. There have been many approaches to noninterpolatory integration of CPV integrals [4], [14], [17], but these two are the only ones that cater to the situation indicated above.

In Section 2, we state and prove Theorems 1 to 5, which deal with convergence of $S_N(f; \lambda)$ to $I(f; \lambda)$. In Section 3, we state and prove Theorems 6 to 8, which deal with the convergence of $Q_m^N(f; \lambda)$ to $I(f; \lambda)$ as m and $N \rightarrow \infty$. It turns out that in the general case we shall be able to prove convergence only for the iterated limit

$$(18) \quad \lim_{N \rightarrow \infty} \lim_{m \rightarrow \infty} Q_m^N(f; \lambda).$$

In fact, we shall show that we cannot in general expect convergence of the double limit. However, in certain cases where we can convert the double limit to a single limit in which m depends on N in some specific manner, we shall again be able to prove convergence. A similar approach was used by Dagnino [3] in studying the convergence of noninterpolatory product integration rules.

2. Convergence Results for $S_N(f; \lambda)$. Before we can study the convergence of $Q_m^N(f; \lambda)$ to $I(f; \lambda)$, we must establish the convergence of $S_N(f; \lambda)$ to $I(f; \lambda)$. To this end, we shall use the methods presented in Natanson [11] and Freud [5] for

proving convergence of orthonormal expansions. Since the proofs in [11] depend on the Christoffel-Darboux formula

$$(19) \quad \sum_{k=0}^N p_k(x)p_k(y) = \alpha_{N+1} \frac{p_{N+1}(x)p_N(y) - p_N(x)p_{N+1}(y)}{x-y},$$

we shall first establish an analogous formula for the sum

$$(20) \quad K_N(x, \lambda) := \sum_{k=0}^N p_k(x)q_k(\lambda).$$

Throughout, $C, C_1, C_2, \dots$, and $B, B_1, B_2, \dots$ denote positive constants independent of N, m, x and λ .

LEMMA 1. Let $\{p_n\}_0^\infty$ be a sequence of orthonormal polynomials on $[-1, 1]$, with respect to $w \in \mathcal{A}$, and let $q_n(\lambda) := I(p_n; \lambda)$, $n = 1, 2, 3, \dots$, exist for a given $\lambda \in (-1, 1)$. Then, for $N = 1, 2, 3, \dots$,

$$(21) \quad K_N(x, \lambda) = \frac{\alpha_{N+1}\{p_{N+1}(x)q_N(\lambda) - p_N(x)q_{N+1}(\lambda)\} + 1}{x - \lambda}.$$

Proof. We have from (5) and (7) that for $k = 0, 1, 2, \dots$,

$$(22) \quad xp_k(x) = \alpha_{k+1}p_{k+1}(x) + \beta_{k+1}p_k(x) + \alpha_k p_{k-1}(x),$$

and

$$(23) \quad \lambda q_k(\lambda) = \alpha_{k+1}q_{k+1}(\lambda) + \beta_{k+1}q_k(\lambda) + \alpha_k q_{k-1}(\lambda).$$

Multiply (22) by $q_k(\lambda)$ and multiply (23) by $p_k(x)$; then subtract the two and sum from $k = 0$ to N . This yields

$$(24) \quad (x - \lambda)K_N(x, \lambda) = \alpha_{N+1}\{p_{N+1}(x)q_N(\lambda) - p_N(x)q_{N+1}(\lambda)\} \\ - \alpha_0\{p_0(x)q_{-1}(\lambda) - p_{-1}(x)q_0(\lambda)\}.$$

Since $p_{-1}(x) \equiv 0$, $q_{-1}(\lambda) \equiv -1$ and $\alpha_0 = m_0^{1/2} = 1/p_0$, (21) follows. $\square$

COROLLARY 1. The sum $K_N(x, \lambda)$ can also be written as

$$(25) \quad K_N(x, \lambda) \\ = \alpha_{N+1} \left\{ \frac{p_{N+1}(x)(q_N(\lambda) - q_N(x)) - p_N(x)(q_{N+1}(\lambda) - q_{N+1}(x))}{x - \lambda} \right\} \\ = \alpha_{N+1} \left\{ \frac{q_{N+1}(\lambda)(p_N(\lambda) - p_N(x)) - q_N(\lambda)(p_{N+1}(\lambda) - p_{N+1}(x))}{x - \lambda} \right\}.$$

Proof. If we set $x = \lambda$ in (24), we find that

$$\alpha_{N+1}\{p_{N+1}(x)q_N(x) - p_N(x)q_{N+1}(x)\} \\ = -1 = \alpha_{N+1}\{p_{N+1}(\lambda)q_N(\lambda) - p_N(\lambda)q_{N+1}(\lambda)\}.$$

Substituting into (21) yields (25). $\square$

Before proving some convergence theorems for $S_N(f; \lambda)$, we recall some definitions and results connected with the existence of $I(f; \lambda)$ [1]. We say that a function f is of Dini type on an interval I of length $l(I)$, and write $f \in DT(I)$, if

$$(26) \quad \int_0^{l(I)} \omega_I(f; t)t^{-1} dt < \infty,$$

where $\omega_I(f; t)$ is the ordinary modulus of continuity of f on I , defined by

$$(27) \quad \omega_I(f; t) := \sup_{\substack{|x-y| \leq t \\ x, y \in I}} |f(x) - f(y)|, \quad t > 0.$$

Obviously, if $f \in DT(I)$, then $f \in C(I)$. Furthermore, it can easily be shown that if $f \in DT(I)$, then f satisfies the *Dini-Lipschitz condition on I* , that is

$$(28) \quad \lim_{t \rightarrow 0+} \omega_I(f; t) \log t = 0.$$

Finally, it is well known that if $\lambda \in (-1, 1)$ and if for some small enough $\varepsilon > 0$, $f \in DT(\lambda - \varepsilon, \lambda + \varepsilon) \cap R[-1, 1]$ and $w \in DT(\lambda - \varepsilon, \lambda + \varepsilon) \cap \mathcal{A}$, then $I(f; \lambda)$ exists. Hence, to ensure the existence of $I(f; \lambda)$ for all $\lambda \in (-1, 1)$, it is sufficient to require that $f \in R[-1, 1]$ and $w \in \mathcal{A}$ belong to $DT(-1, 1)$.

We are now ready to prove some convergence results about $S_N(f; \lambda)$ corresponding to the convergence theorems for orthonormal expansions in [11]. As usual, for $w \in \mathcal{A}$ and $0 < p < \infty$, we let

$$(29) \quad L_{p,w} := \left\{ g: [-1, 1] \rightarrow \mathbf{R} \mid g \text{ is measurable and } \int_{-1}^1 w(x) |g(x)|^p dx < \infty \right\}.$$

THEOREM 1. *Assume that for some $\lambda \in (-1, 1)$, $I(f; \lambda)$ exists, that*

$$(30) \quad \sup_k |q_k(\lambda)| \leq B < \infty,$$

and that

$$(31) \quad \varphi_\lambda(x) := (f(x) - f(\lambda))/(x - \lambda), \quad x \in [-1, 1],$$

belongs to $L_{2,w}$. Then

$$(32) \quad \lim_{N \rightarrow \infty} S_N(f; \lambda) = I(f; \lambda).$$

Proof. Multiply (21) by $w(x)(f(x) - f(\lambda))$ and integrate between -1 and 1 . We obtain

$$(33) \quad \begin{aligned} S_N(f; \lambda) - f(\lambda)q_0(\lambda)/p_0 \\ = \alpha_{N+1}\{c_{N+1}q_N(\lambda) - c_Nq_{N+1}(\lambda)\} + I(f; \lambda) - f(\lambda)q_0(\lambda)/p_0, \end{aligned}$$

where $c_k := (\varphi_\lambda, p_k)$ is the k th Fourier coefficient of φ_λ with respect to p_k . Since $\varphi_\lambda \in L_{2,w}$, $c_k \rightarrow 0$ as $k \rightarrow \infty$. Hence, since $\alpha_{N+1} \leq 1$ [5, p. 41], while (30) holds, we obtain (32). $\square$

An important special case of this theorem is that of the *generalized smooth Jacobi weight* (we write $w \in \text{GSJ}$), studied by Nevai [13, p. 673], among others. It is defined by

$$(34) \quad w(x) := \psi(x) \prod_{j=0}^{m+1} |x - t_j|^{\gamma_j}, \quad x \in [-1, 1],$$

where $m \geq 0$, $-1 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$, $\gamma_j > -1$, $j = 0, 1, 2, \dots, m+1$, $\psi \in DT(-1, 1)$ and $\psi(x) > 0$ in $[-1, 1]$. Clearly, if

$$(35) \quad \mathcal{D} := [-1, 1] \setminus \{t_0, t_1, \dots, t_{m+1}\},$$

then $w \in \mathcal{A} \cap DT(\mathcal{D})$, so that if $\lambda \in \mathcal{D}$ and $f \in DT(\lambda - \varepsilon, \lambda + \varepsilon) \cap R[-1, 1]$ for some $\varepsilon > 0$, then $I(f; \lambda)$ exists. Furthermore, Criscuolo and Mastroianni [2] have shown that if $w \in \text{GSJ}$, then (30) holds for $\lambda \in \mathcal{D}$, and uniformly in any closed

subset of $\mathcal{D}$. Hence, we have the following corollary:

COROLLARY 2. *If $w \in \text{GSJ}$ and $f \in DT(\lambda - \varepsilon, \lambda + \varepsilon) \cap R[-1, 1]$ for some $\lambda \in \mathcal{D}$ and some small enough $\varepsilon > 0$, while $\varphi_\lambda \in L_{2,w}$, then (32) holds.*

In the sequel, we use the norm $\|f\| := \max_{[-1,1]} |f(x)|$ for any $f \in C[-1, 1]$.

THEOREM 2. *If (30) holds for some $\lambda \in (-1, 1)$, if*

$$(36) \quad \sup_k \|p_k(x)\| < \infty,$$

and if $\varphi_\lambda \in L_{1,w}$, then (32) holds.

Proof. By Theorem 3 in [11, p. 69], the Fourier coefficients c_k of φ_λ (defined by (31)) converge to 0 as $k \rightarrow \infty$ under the hypotheses of the theorem. Furthermore, since $\varphi_\lambda \in L_{1,w}$, $I(f; \lambda)$ exists, as shown by the identity

$$(37) \quad I(f; \lambda) = \int_{-1}^1 w(x) \varphi_\lambda(x) dx + f(\lambda) q_0(\lambda) / p_0.$$

Hence (32) follows from (33). $\square$

COROLLARY 3. *Assume that $w \in \text{GSJ}$, where $\gamma_0, \gamma_{m+1} \leq -1/2$ and $\gamma_j \leq 0$, $j = 1, 2, \dots, m$. Further assume that $\lambda \in \mathcal{D}$, and that $\varphi_\lambda \in L_{1,w}$. Then (32) holds.*

Proof. By Nevai [13, p. 674, (16)], there exists $C > 0$ such that for $x \in [-1, 1]$ and $k = 1, 2, 3, \dots$,

$$(38) \quad |p_k(x)| \leq C\{[w(x)(1-x^2)^{1/2}]^{-1/2} + 1\}.$$

Hence, under the hypotheses of the corollary, (36) is true. Furthermore, as above, (30) is true for all $\lambda \in \mathcal{D}$. Hence, by Theorem 2, (32) holds. $\square$

Theorems 1 and 2 are of a local nature, since they depend on the behavior of the Fourier coefficients c_k of $\varphi_\lambda(x)$. The following is a global theorem, and its proof requires much more delicate analysis. The proof is modelled on the proof of Theorem 2 in [11, p. 95].

THEOREM 3. *If $f \in DT[-1, 1]$ and $w \in \text{GSJ}$, then (32) holds uniformly for λ in each compact subset of $\mathcal{D}$.*

Proof. We first remark that $I(f; \lambda)$ exists for all $\lambda \in \mathcal{D}$ and that f satisfies the Dini-Lipschitz condition (28) on $J := [-1, 1]$. We shall start by proving that

$$(39) \quad L_N(\lambda) := \int_{-1}^1 w(x) |K_N(x, \lambda)| dx,$$

is $O(\log N)$, uniformly in a given compact subset $\mathcal{K}$ of $\mathcal{D}$. We first establish this bound for the case $m = 0$ in (34), that is when $w(x)$ has no zeros or infinities in $(-1, 1)$. To this end, we write $L_N(\lambda)$ as the sum of five integrals

$$\begin{aligned} L_N(\lambda) &= \int_{-1}^{-1+h/2} + \int_{-1+h/2}^{\lambda-1/N} + \int_{\lambda-1/N}^{\lambda+1/N} + \int_{\lambda+1/N}^{1-h/2} + \int_{1-h/2}^1 \\ &=: I_1 + I_2 + I_3 + I_4 + I_5 \end{aligned}$$

and choose N sufficiently large so that $[\lambda - 1/N, \lambda + 1/N] \subset \mathcal{D}$ for all $\lambda \in \mathcal{X}$ and choose $h > 0$ so small that $\mathcal{X} \subset [-1 + h, 1 - h]$. We consider first I_1 and use (21) for $K_N(x, \lambda)$. Now, for $x \in [-1, -1 + h/2]$ and $\lambda \in \mathcal{X}$, $|x - \lambda| \geq h/2$. Further, since (30) holds uniformly for $\lambda \in \mathcal{X}$, since $\alpha_{N+1} \leq 1$, and since

$$\begin{aligned} \int_{-1}^1 w(x) |p_k(x)| dx &\leq \left\{ \int_{-1}^1 w(x) dx \right\}^{1/2} \left\{ \int_{-1}^1 w(x) p_k^2(x) dx \right\}^{1/2} \\ &= \left\{ \int_{-1}^1 w(x) dx \right\}^{1/2}, \end{aligned}$$

it follows that $I_1 = O(1)$. Similarly, $I_5 = O(1)$. For $x \in [\lambda + 1/N, 1 - h/2]$, it follows from (38) and the fact that (30) holds uniformly for $\lambda \in \mathcal{X}$, that

$$|K_N(x, \lambda)| \leq C/|x - \lambda|,$$

where C is independent of N , x and λ . Hence,

$$\begin{aligned} I_4 &\leq C^{-1} \int_{\lambda+1/N}^1 \frac{w(x)}{x - \lambda} dx \\ &\leq \int_{-1}^1 \left| \frac{w(x) - w(\lambda)}{x - \lambda} \right| dx + w(\lambda) \int_{\lambda+1/N}^1 \frac{dx}{x - \lambda} = O(\log N). \end{aligned}$$

Similarly, $I_2 = O(\log N)$. Finally, since

$$|K_N(x, \lambda)| \leq (N + 1) \sup_k |p_k(x)| \sup_k |q_k(\lambda)|,$$

we obtain $I_3 = O(1)$. Combining these estimates, we obtain

$$(40) \quad \sup_{\lambda \in \mathcal{X}} |L_N(\lambda)| \leq C_1 \log N,$$

for some C_1 independent of N . For the general case, we let h be the distance of $\mathcal{X}$ from the set $T := \{t_0, t_1, \dots, t_{m+1}\}$ and denote by U the subset of $[-1, 1]$ such that the distance of T to U is at most $h/2$. As before, we can show that

$$\int_U w(x) |K_N(x, \lambda)| dx = O(1),$$

and that

$$\int_{V_N} w(x) |K_N(x, \lambda)| dx = O(\log N),$$

where $V_N := [-1, 1] \setminus ([\lambda - 1/N, \lambda + 1/N] \cup U)$. If we choose N large enough so that $1/N < h$, we obtain (40).

Next, let P_N^* be the polynomial of best approximation to f in the uniform norm, let $r_N := f - P_N^*$, and let $E_N(f) := \|r_N\|$. Since f satisfies (28) on J , it follows from Jackson's Theorems that

$$(41) \quad \lim_{N \rightarrow \infty} E_N(f) \log N = 0.$$

Now, for any $g \in C[-1, 1]$, we have

$$|S_N(g; \lambda)| = |(g, K_N(x, \lambda))| \leq \|g\| L_N(\lambda).$$

Hence, uniformly for $\lambda \in \mathcal{X}$, we have from (40) and (41),

$$\lim_{N \rightarrow \infty} |S_N(r_N; \lambda)| = 0.$$

Since

$$I(P_N^*; \lambda) = S_N(P_N^*; \lambda),$$

we have

$$I(f; \lambda) - S_N(f; \lambda) = I(r_N; \lambda) - S_N(r_N; \lambda),$$

and it thus remains to show that

$$\lim_{N \rightarrow \infty} |I(r_N; \lambda)| = 0.$$

Now

$$\begin{aligned} I(r_N; \lambda) &= \int_{-1}^1 w(x) \frac{r_N(x) - r_N(\lambda)}{x - \lambda} dx + r_N(\lambda) q_0(\lambda) / p_0 \\ &= \int_{-1}^1 w(x) \frac{r_N(x) - r_N(\lambda)}{x - \lambda} dx + o(1). \end{aligned}$$

Furthermore, as in [1],

$$\begin{aligned} \int_{-1}^1 w(x) \frac{r_N(x) - r_N(\lambda)}{x - \lambda} dx &= \int_{-1}^{\lambda-1/N} + \int_{\lambda-1/N}^{\lambda+1/N} + \int_{\lambda+1/N}^1 \\ &=: J_1 + J_2 + J_3. \end{aligned}$$

Here

$$|J_1| \leq 2E_N(f) \int_{-1}^{\lambda-1/N} \frac{w(x)}{|x - \lambda|} dx = E_N(f) O(\log N) = o(1) \quad \text{as } N \rightarrow \infty.$$

Similarly, $J_3 = o(1)$ as $N \rightarrow \infty$. Finally,

$$\begin{aligned} \int_{\lambda-1/N}^{\lambda+1/N} w(x) \frac{r_N(x) - r_N(\lambda)}{x - \lambda} dx \\ = \int_{\lambda-1/N}^{\lambda+1/N} w(x) \frac{f(x) - f(\lambda)}{x - \lambda} dx - \int_{\lambda-1/N}^{\lambda+1/N} w(x) \frac{P_N^*(x) - P_N^*(\lambda)}{x - \lambda} dx. \end{aligned}$$

Since $f \in DT[-1, 1]$, the first integral on the right-hand side is $o(1)$. As for the second integral, we have from [9] that

$$\begin{aligned} \left| \frac{P_N^*(x) - P_N^*(\lambda)}{x - \lambda} \right| &\leq \max\{|P_N^*(t)| : t \in [\lambda - 1/N, \lambda + 1/N]\} \\ &\leq CN\omega(f; 1/N). \end{aligned}$$

Hence

$$\begin{aligned} \int_{\lambda-1/N}^{\lambda+1/N} w(x) \left| \frac{P_N^*(x) - P_N^*(\lambda)}{x - \lambda} \right| dx \\ \leq 2C\omega(f; 1/N) \max\{w(x) : x \in [\lambda - 1/N, \lambda + 1/N]\} \\ \rightarrow 0 \quad \text{as } N \rightarrow \infty, \end{aligned}$$

since $w(x)$ is uniformly bounded above for $\lambda \in \mathcal{X}$ and N large enough. This completes our proof. $\square$

Remark. Theorem 3 is similar to Theorem 2.2 in [1]. By following the proof of Theorem 3, we can prove a result similar to Theorem 2.1 in [1], namely, that if f satisfies (28) on J , if $w \in \text{GSJ}$, and if for some $\lambda \in \mathcal{D}$, $I(f; \lambda)$ exists, then (32)

holds. The proof of Theorem 3 holds in this case too, except that we must show that

$$J_N := \int_{\lambda-1/N}^{\lambda+1/N} w(x) \frac{f(x) - f(\lambda)}{x - \lambda} dx = o(1), \quad N \rightarrow \infty.$$

Since

$$J_0 := \int_{-1}^1 w(x) \frac{f(x) - f(\lambda)}{x - \lambda} dx = I(f; \lambda) - f(\lambda)I(1; \lambda),$$

and both $I(f; \lambda)$ and $I(1; \lambda)$ exist, it follows that J_0 exists. Hence $J_N = o(1)$, and the proof is complete. $\square$

We now give some additional conditions for (32) to hold, which impose less restrictions on the weight function $w \in \mathcal{A}$, but require more smoothness of f . To this end, we first prove a lemma:

LEMMA 2. Let $w \in \mathcal{A}$, and assume that for some $\lambda \in (-1, 1)$,

$$(42) \quad \Gamma(\lambda) := \int_{-1}^1 \left| \frac{w(x) - w(\lambda)}{x - \lambda} \right| dx < \infty,$$

while for some positive ε , B_1 and B_2

$$(43) \quad B_1 \leq w(x) \leq B_2 \quad \text{for } |x - \lambda| \leq 2\varepsilon.$$

Then there exists a constant $B_3 > 0$ such that

$$(44) \quad T_{n-1}(\lambda) := \sum_{k=0}^{n-1} q_k^2(\lambda) \leq B_3 n, \quad n = 1, 2, 3, \dots$$

If $\Gamma(\lambda)$ is uniformly bounded and (43) holds uniformly for $\lambda \in [a - \varepsilon, b + \varepsilon] \subset [-1, 1]$, then (44) holds uniformly for $\lambda \in [a, b]$.

Proof. We first establish the following analogue of the Christoffel function extremum problem, noting that in essence, it is contained in [6]: Defining

$$(45) \quad \rho_n(w; \lambda) := \inf \left\{ \frac{I(P^2)}{(I(P; \lambda))^2} : P \in \mathcal{P}_{n-1}, I(P; \lambda) \neq 0 \right\},$$

where $\mathcal{P}_m$ denotes the set of all polynomials of degree $\leq m$, we have

$$(46) \quad \rho_n(w; \lambda) = 1/T_{n-1}(\lambda).$$

To see this, we note that for any $P \in \mathcal{P}_{n-1}$, we can write

$$P(x) = \sum_{k=0}^{n-1} a_k p_k(x), \quad \text{where } a_k := (P, p_k), \quad k = 0, 1, 2, \dots, n-1.$$

Hence

$$\begin{aligned} |I(P; \lambda)| &= \left| \sum_{k=0}^{n-1} a_k q_k(\lambda) \right| \leq \left\{ \sum_{k=0}^{n-1} a_k^2 \right\}^{1/2} \{T_{n-1}(\lambda)\}^{1/2} \\ &= \{I(P^2)\}^{1/2} \{T_{n-1}(\lambda)\}^{1/2}, \end{aligned}$$

so that

$$\rho_n(w, \lambda) \geq 1/T_{n-1}(\lambda).$$

On the other hand,

$$\hat{P}(x) := \sum_{k=0}^{n-1} p_k(x) q_k(\lambda) \in \mathcal{P}_{n-1},$$

and satisfies

$$I(\hat{P}; \lambda) = T_{n-1}(\lambda) = I(\hat{P}^2).$$

Then (46) follows.

We now use (46) to prove (44). Choose ε such that $[\lambda - 2\varepsilon, \lambda + 2\varepsilon] \subset [-1, 1]$. Now for any $P \in \mathcal{P}_{n-1}$,

$$\begin{aligned} |I(P; \lambda)| &= \left| \int_{|x-\lambda| \leq \varepsilon} P(x) \frac{w(x) - w(\lambda)}{x - \lambda} dx + w(\lambda) \int_{|x-\lambda| \leq \varepsilon} \frac{P(x)}{x - \lambda} dx \right. \\ &\quad \left. + \int_{|x-\lambda| \geq \varepsilon} \frac{w(x)P(x)}{x - \lambda} dx \right| \\ (47) \quad &\leq \Gamma(\lambda) \max_{|x-\lambda| \leq \varepsilon} |P(x)| + w(\lambda) \left| \int_{|x-\lambda| \leq \varepsilon} \frac{P(x)}{x - \lambda} dx \right| + \varepsilon^{-1} I(|P|). \end{aligned}$$

Next, let χ be the characteristic function of $[\lambda - \varepsilon, \lambda + \varepsilon]$, that is, $\chi(x) := 1$ in $[\lambda - \varepsilon, \lambda + \varepsilon]$ and $\chi(x) := 0$ elsewhere. We have from (45),

$$\begin{aligned} (48) \quad \left| \int_{|x-\lambda| \leq \varepsilon} \frac{P(x)}{x - \lambda} dx \right|^2 &\leq \rho_n(\chi; \lambda)^{-1} \int_{|x-\lambda| \leq \varepsilon} P^2(x) dx \\ &\leq B_1^{-1} \rho_n(\chi; \lambda)^{-1} \int_{|x-\lambda| \leq \varepsilon} P^2(x) w(x) dx \leq B_1^{-1} \rho_n(\chi; \lambda)^{-1} I(P^2). \end{aligned}$$

Furthermore, by standard estimates for Christoffel functions for the Legendre weight (cf. [12]),

$$(49) \quad \max_{|x-\lambda| \leq \varepsilon} (P(x))^2 \leq Cn \int_{\lambda-2\varepsilon}^{\lambda+2\varepsilon} P^2(t) dt \leq Cn B_1^{-1} I(P^2).$$

Combining (47), (48) and (49), and using the Cauchy-Schwarz inequality, we obtain

$$|I(P; \lambda)| \leq B_4 \{n^{1/2} \Gamma(\lambda) + \rho_n^{-1/2}(\chi; \lambda) + 1\} I(P^2)^{1/2}.$$

But

$$\rho_n(\chi; \lambda)^{-1} = \sum_{k=0}^{n-1} q_k^2(\chi; \lambda) \leq B_5 n,$$

since $q_k(\chi; \lambda)$ is the function of the second kind associated with the Legendre weight shifted to $[\lambda - \varepsilon, \lambda + \varepsilon]$, so that $q_k(\chi; \lambda) = O(1)$. Hence

$$|I(P; \lambda)| \leq B_6 n^{1/2} I(P^2)^{1/2},$$

so that

$$1/T_{n-1}(\lambda) = \rho_n(w; \lambda) \geq B_7/n.$$

If the assumptions on λ hold uniformly in $[a - \varepsilon, b + \varepsilon]$, it is not difficult to modify the proof to hold uniformly in $[a, b]$. $\square$

We now prove the analogue of Theorem IV.1.2 in Freud [5, p. 139].

THEOREM 4. Let $w \in \mathcal{A}$ and assume that for some $\lambda \in (-1, 1)$, (42) holds, while (43) holds for some positive ε, B_1 and B_2 . Define for $n = 1, 2, 3, \dots$,

$$(50) \quad E_n^{(2)}(f; w) := \inf_{P \in \mathcal{P}_n} (f - P, f - P)^{1/2}.$$

Then, if

$$(51) \quad \sum_{n=1}^{\infty} E_n^{(2)}(f; w) n^{-1/2} < \infty,$$

(32) holds. If $\Gamma(\lambda)$ is uniformly bounded for $\lambda \in [a - \varepsilon, b + \varepsilon] \subset [-1, 1]$, while (43) holds uniformly for $\lambda \in [a - \varepsilon, b + \varepsilon]$, then (32) holds uniformly in $[a, b]$.

Proof. First recall the notation (8). For any positive integer m ,

$$\begin{aligned} \sum_{k=2^{m+1}}^{2^{m+1}} |a_k q_k(\lambda)| &\leq \left\{ \sum_{k=2^{m+1}}^{2^{m+1}} a_k^2 \right\}^{1/2} \left\{ \sum_{k=2^{m+1}}^{2^{m+1}} q_k^2(\lambda) \right\}^{1/2} \\ &\leq \left\{ \sum_{k=2^{m+1}}^{\infty} a_k^2 \right\}^{1/2} \left\{ \sum_{k=0}^{2^{m+1}} q_k^2(\lambda) \right\}^{1/2} \\ &= E_{2^m}^{(2)}(f; w) \left\{ \sum_{k=0}^{2^{m+1}} q_k^2(\lambda) \right\}^{1/2} \\ &\leq C E_{2^m}^{(2)}(f; w) 2^{m/2}, \end{aligned}$$

where the last inequality follows from (44). Since $E_k^{(2)}(f; w)$ is nonincreasing with k ,

$$\begin{aligned} 2^{m/2} E_{2^m}^{(2)}(f; w) &\leq 2^{m/2} \left\{ 2^{-m+1} \sum_{k=2^{m-1}+1}^{2^m} E_k^{(2)}(f; w) \right\} \\ &= 2^{1-m/2} \sum_{k=2^{m-1}+1}^{2^m} E_k^{(2)}(f; w) \leq 2 \sum_{k=2^{m-1}+1}^{2^m} E_k^{(2)}(f; w) k^{-1/2}. \end{aligned}$$

Hence,

$$\sum_{k=2}^{\infty} |a_k q_k(\lambda)| = \sum_{m=1}^{\infty} \sum_{k=2^{m-1}+1}^{2^m} |a_k q_k(\lambda)| \leq B \sum_{k=1}^{\infty} E_k^{(2)}(f; w) k^{-1/2}. \quad \square$$

The next theorem is the analogue of Theorem IV.1.3 in Freud [5, p. 140].

THEOREM 5. Let w and λ be as in Theorem 4. Let $f \in C[-1, 1]$ and for $J := [-1, 1]$, suppose that $w_J(f; \delta)$ satisfies for some $\eta > 0$,

$$(52) \quad \lim_{\delta \rightarrow 0+} w_J(f; \delta) \delta^{-1/2} |\log \delta|^{1+\eta} = 0.$$

Then (32) holds. If $\Gamma(\lambda)$ is uniformly bounded for $\lambda \in [a - \varepsilon, b + \varepsilon] \subset [-1, 1]$, while (43) holds uniformly for $\lambda \in [a - \varepsilon, b + \varepsilon]$, then (32) holds uniformly in $[a, b]$.

Proof. By Jackson's Theorem,

$$E_k^{(2)}(f; w) \leq B_1 w_J(f; k^{-1}) \leq B_1 k^{-1/2} |\log k|^{-1-\eta}. \quad \square$$

3. Convergence Results for $Q_m^N(f; \lambda)$. We are now ready to prove our convergence theorems for $Q_m^N(f; \lambda)$. First a result on the iterated limit.

THEOREM 6. Assume that $f \in R[-1, 1]$, that $I(f; \lambda)$ exists and that $w \in \mathcal{A}$ and $\lambda \in [-1, 1]$ are such that (32) holds. Let $\{Q_m(\cdot)\}_{m=1}^\infty$ be a sequence of integration rules such that for all $g \in R[-1, 1]$,

$$\lim_{m \rightarrow \infty} Q_m(g) = I(g).$$

Then

$$(53) \quad \lim_{N \rightarrow \infty} \lim_{m \rightarrow \infty} Q_m^N(f; \lambda) = I(f; \lambda).$$

Proof. It suffices to show that for each fixed N ,

$$\lim_{m \rightarrow \infty} Q_m^N(f; \lambda) = S_N(f; \lambda),$$

since

$$I(f; \lambda) = S_N(f; \lambda) + \sum_{k=N+1}^{\infty} a_k q_k(\lambda) = S_N(f; \lambda) + o(1).$$

For fixed N , we choose m sufficiently large so that

$$|a_{km} - a_k| \leq \varepsilon \max_{0 \leq k \leq N} |q_k(\lambda)| / (N+1), \quad k = 0, 1, 2, \dots, N,$$

yielding the theorem. $\square$

Even though we have convergence of the iterated limit (53), we cannot in general have convergence of the double limit (that is the limit with m and $N \rightarrow \infty$ simultaneously), as illustrated by the following simple example:

Example 1. Let

$$w(x) := (1 - x^2)^{-1/2} \quad \text{and} \quad f(x) \equiv 1, \quad x \in (-1, 1),$$

and let $Q_m(\cdot)$ be the Gauss-Chebyshev rule

$$Q_m(g) := \frac{\pi}{m} \sum_{i=1}^m g \left(\cos \frac{2i-1}{2m} \pi \right).$$

Then, with $N = 2m$, we have that

$$Q_m^{2m}(f; \lambda) = \sum_{k=0}^{2m} Q_m(f p_k) q_k(\lambda) = \sum_{k=0}^{2m} Q_m(p_k) q_k(\lambda).$$

Since $Q_m(g)$ is exact for all $g \in \mathcal{P}_{2m-1}$,

$$Q_m(p_k) = I(p_k) = \int_{-1}^1 w(x) p_k(x) dx, \quad 0 \leq k \leq 2m-1,$$

so that

$$Q_m(p_0) = p_0 \pi \quad \text{and} \quad Q_m(p_k) = 0, \quad k = 1, 2, \dots, 2m-1.$$

Furthermore,

$$\begin{aligned} Q_m(p_{2m}) &= \frac{\pi}{m} \sum_{i=1}^m \left(\frac{2}{\pi}\right)^{1/2} T_{2m} \left(\cos \frac{2i-1}{2m} \pi \right) \\ &= (2\pi)^{1/2} m^{-1} \sum_{i=1}^m \cos(2i-1)\pi = -(2\pi)^{1/2}. \end{aligned}$$

Hence

$$Q_m^{2m}(f; \lambda) = \pi^{1/2} q_0(\lambda) - (2\pi)^{1/2} q_{2m}(\lambda).$$

But (see, for example, [8, p. 148])

$$q_0(\lambda) = I(f; \lambda) = 0,$$

so

$$Q_m^{2m}(f; \lambda) - I(f; \lambda) = -(2\pi)^{1/2} q_{2m}(\lambda),$$

which does not go to zero for any nonzero $\lambda \in (-1, 1)$ as $m \rightarrow \infty$, inasmuch as $q_{2m}(\lambda) = (2/\pi)^{1/2} U_{2m+1}(\lambda)$, where $U_{2m+1}(\lambda)$ is the Chebyshev polynomial of the second kind of degree $2m+1$.

Example 1 shows that at least in general, converting the iterated limit to a single limit does not lead to convergence. However, there are cases where this procedure will work. One simple example occurs when $m = N+1$ and $Q_m(\cdot)$ is the Gauss integration rule with respect to w . In this case, it turns out that

$$(54) \quad Q_{N+1}^N(f; \lambda) = I(L_{N+1}; \lambda),$$

where L_{N+1} is the Lagrange interpolation polynomial of degree $\leq N$ interpolating f at the zeros of p_{N+1} . This follows since

$$\begin{aligned} (55) \quad I(L_{N+1}; \lambda) &= \sum_{k=0}^N (L_{N+1}, p_k) q_k(\lambda) = \sum_{k=0}^N Q_{N+1}(L_{N+1} p_k) q_k(\lambda) \\ &= \sum_{k=0}^N Q_{N+1}(f p_k) q_k(\lambda) = Q_{N+1}^N(f; \lambda) \end{aligned}$$

(see, for example, [16, pp. 1250–1251]). Since it has been shown in [1] that for $w \in \text{GSJ}$,

$$\lim_{N \rightarrow \infty} I(L_{N+1}, \lambda) = I(f; \lambda),$$

we have that for the sequence of Gauss rules $\{Q_m(\cdot)\}_{m=1}^\infty$ associated with $w \in \text{GSJ}$,

$$\lim_{N \rightarrow \infty} Q_{N+1}^N(f; \lambda) = I(f; \lambda).$$

We can generalize this result to any sequence of integration rules $\{Q_m(\cdot)\}_{m=1}^\infty$ that is ultimately exact for all polynomials, that is $Q_m(g) = I(g)$ for all $g \in \mathcal{P}_n$ and all $m \geq m(n)$. A particular instance of this, that allows points to be concentrated in regions where the behaviour of f is problematic, is rules exact for piecewise polynomials of increasing degree.

In the general situation, if the weights w_{im} and the points x_{im} in a sequence of rules $\{Q_m(\cdot)\}_{m=1}^\infty$ are such that

$$(56) \quad \sum_{i=1}^m |w_{im}^N(\lambda)| = O(\log N),$$

if $f \in DT[-1, 1]$ and if $w \in \text{GSJ}$, then we have that

$$(57) \quad \lim_{N \rightarrow \infty} Q_{m(2N)}^N(f; \lambda) = I(f; \lambda).$$

Here $m(2N)$ denotes the least integer m such that $Q_m(g) = I(g)$ for all $g \in \mathcal{P}_{2N}$. The proof follows standard lines, namely

$$(58) \quad I(f; \lambda) = I(P_N^*; \lambda) + I(r_N; \lambda),$$

where, as above, $P_N^* \in \mathcal{P}_N$ is the polynomial of best approximation to f in the uniform norm and $r_N := f - P_N^*$. Since, by hypothesis, $Q_m(gp_k) = I(gp_k)$ for all $k \leq N$, all $m \geq m(2N)$ and all $g \in \mathcal{P}_N$, it follows that

$$(59) \quad Q_m^N(f; \lambda) = I(P_N^*; \lambda) + Q_m^N(r_N; \lambda).$$

Hence

$$\begin{aligned} |I(f; \lambda) - Q_m^N(f; \lambda)| &\leq |I(r_N; \lambda)| + |Q_m^N(r_N; \lambda)| \\ &\leq |I(r_N; \lambda)| + \sum_{i=1}^m |w_{im}^N(\lambda)| \|r_N\|. \end{aligned}$$

As in the proof of Theorem 3 above, $I(r_N; \lambda) = o(1)$, and since f satisfies (28), the second term is also $o(1)$ from (56), proving (57).

What about conditions on w_{im} and x_{im} that ensure (56)? We shall prove

LEMMA 3. *With the above notation, if*

$$(60) \quad h_{im} := x_{i+1,m} - x_{im} \leq C_1/N$$

for some $C_1 > 0$, uniformly for all i and $m \geq m(2N)$, while

$$(61) \quad |w_{im}|/w(x_{im}) \leq C_2(h_{i-1,m} + h_{im}),$$

then (56) holds whenever $w \in \text{GSJ}$ and $\lambda \in \mathcal{D}$.

Proof. As in the proof of Theorem 3, we consider first the case $m = 0$ in (34), and we decompose the sum on the left-hand side of (56) into five sums:

$$\begin{aligned} \sum_{i=1}^m |w_{im}^N(\lambda)| &= \sum_{x_{im} \leq -1+\delta} + \sum_{\substack{\lambda - x_{im} > 2C_1/N \\ x_{im} > -1+\delta}} + \sum_{|x_{im} - \lambda| \leq 2C_1/N} \\ &\quad + \sum_{\substack{x_{im} - \lambda > 2C_1/N \\ x_{im} < 1-\delta}} + \sum_{x_{im} \geq 1-\delta} \\ &=: \sum_1 + \sum_2 + \sum_3 + \sum_4 + \sum_5, \end{aligned}$$

where δ is some sufficiently small positive number. Now by (17) and (21), and the uniform boundedness of $\{q_k(\lambda)\}_0^\infty$,

$$\begin{aligned} \sum_1 &\leq (1 - \delta + \lambda)^{-1} O(1) \sum w_{im} \{|p_N(x_{im})| + |p_{N+1}(x_{im})|\} \\ &\leq O(1) \int_{-1}^{-1+\delta+C_1/N} w(x) \{|p_N(x)| + |p_{N+1}(x)|\} dx \\ &\quad \text{(by Theorem 5 in [10, p. 534])} \\ &\leq O(1) \left\{ \int_{-1}^1 w(x) dx \right\}^{1/2} = O(1). \end{aligned}$$

Similarly, $\sum_5 = O(1)$. Next,

$$\begin{aligned}\sum_2 &= O(1) \sum |w_{im}|/(\lambda - x_{im}) \\ &\leq O(1) \int_{-1}^{\lambda - C_1/N} dx/(\lambda - x) = O(\log N),\end{aligned}$$

by uniform boundedness of p_N , q_N and w . Similarly, $\sum_4 = O(\log N)$. Finally,

$$\begin{aligned}\sum_3 &= O(1) \sum_{|x_{im} - \lambda| \leq 2C_1/N} |w_{im}| \sum_{k=0}^N |p_k(x_{im})q_N(\lambda)| \\ &= O(1)(N+1) \sum_{|x_{im} - \lambda| \leq 2C_1/N} |w_{im}| \\ &= O(1),\end{aligned}$$

which proves the lemma for the case $m = 0$. For the general case, we enclose each of the interior singularities of w in a small interval avoiding λ and treat the w_{im} associated with these intervals in the same manner as $\sum_1$. $\square$

The assertion (57) is a special case of the following theorem:

THEOREM 7. *Suppose that for $m = 1, 2, 3, \dots$, the rule $Q_m(\cdot)$ has precision $\pi_m > N_m$, that $t_m := \min\{N_m, \pi_m - N_m\}$ satisfies*

$$(62) \quad \lim_{m \rightarrow \infty} t_m = \infty,$$

and that

$$(63) \quad \sum_{i=1}^m |w_{im}^{N_m}(\lambda)| \leq C \log t_m, \quad m = 1, 2, 3, \dots$$

Assume that $f \in C[-1, 1]$ satisfies (28) with $I = [-1, 1]$, that $I(f; \lambda)$ exists, that $q_0(\lambda)$ is finite and that $w(x)$ is bounded above in a neighborhood of λ . Then

$$(64) \quad \lim_{m \rightarrow \infty} Q_m^{N_m}(f; \lambda) = I(f; \lambda).$$

Proof. If $P \in \mathcal{P}_{t_m}$, then

$$\begin{aligned}Q_m^{N_m}(P; \lambda) &= \sum_{k=0}^{N_m} Q_m(Pp_k)q_k(\lambda) = \sum_{k=0}^{N_m} (P, p_k)q_k(\lambda) \\ &= \sum_{k=0}^{t_m} (P, p_k)q_k(\lambda) = I(P; \lambda),\end{aligned}$$

since $t_m \leq N_m$. Then, if $P_m^* \in \mathcal{P}_{t_m}$ is the polynomial of best approximation to f in the uniform norm, and if $r_m := f - P_m^*$, then as above, for m sufficiently large

so that $[\lambda - 1/t_m, \lambda + 1/t_m] \subset [-1, 1]$,

$$\begin{aligned}
 |Q_m^{N_m}(f; \lambda) - I(f; \lambda)| &= |Q_m^{N_m}(r_m; \lambda) - I(r_m; \lambda)| \\
 &\leq \sum_{i=1}^m |w_i^{N_m}(\lambda)| \|r_m\| + \int_{|\lambda-x| \geq 1/t_m} w(x) \frac{|r_m(x)|}{|x-\lambda|} dx \\
 &\quad + \left| \int_{|\lambda-x| \leq 1/t_m} \frac{w(x)r_m(x)}{x-\lambda} dx \right| \\
 &\leq C \log t_m \omega(f; t_m^{-1}) + C_1 \|r_m\| \log t_m \\
 &\quad + \left| \int_{|\lambda-x| \leq 1/t_m} w(x) \frac{f(x) - f(\lambda)}{x-\lambda} dx \right| \\
 &\quad + \left| \int_{|\lambda-x| \leq 1/t_m} w(x) \frac{P_m^*(x) - P_m^*(\lambda)}{x-\lambda} dx \right| \\
 &\quad + |r_m(\lambda)| \left| \int_{|\lambda-x| \leq 1/t_m} \frac{w(x)}{x-\lambda} dx \right| \\
 &\leq o(1) + o(1) + o(1) + o(1) + o(1),
 \end{aligned}$$

by the arguments used in the proof of Theorem 3 and the fact that w is bounded above near λ . $\square$

We conclude with an almost trivial theorem that gives necessary and sufficient conditions for the convergence of a sequence of approximations $\{Q_m^{N_m}(f; \lambda)\}_{m=1}^\infty$. It shows that we must choose N_m in such a way that $Q_m(fp_k)$ is small for all k large enough with $k \leq N_m$:

THEOREM 8. *Assume that for all $g \in R[-1, 1]$,*

$$\lim_{m \rightarrow \infty} Q_m(g) = I(g),$$

that $I(f; \lambda)$ exists and that (32) holds. Then, given a sequence $\{(m, N_m)\}_{m=1}^\infty$ of pairs of positive integers with

$$\lim_{m \rightarrow \infty} N_m = \infty,$$

we have that

$$\lim_{m \rightarrow \infty} Q_m^{N_m}(f; \lambda) = I(f; \lambda)$$

if and only if for every $\varepsilon > 0$ we can find a positive integer K such that for all large enough m ,

$$(65) \quad \left| \sum_{k=K}^{N_m} Q_m(fp_k)q_k(\lambda) \right| < \varepsilon.$$

Proof. For any fixed J and all m large enough,

$$\begin{aligned}
 Q_m^{N_m}(f; \lambda) - I(f; \lambda) &= \sum_{k=0}^{N_m} Q_m(fp_k)q_k(\lambda) - \sum_{k=0}^{\infty} (f, p_k)q_k(\lambda) \\
 &= \sum_{k=0}^{K-1} \{Q_m(fp_k) - (f, p_k)\}q_k(\lambda) \\
 &\quad - \sum_{k=K}^{\infty} (f, p_k)q_k(\lambda) + \sum_{k=K}^{N_m} Q_m(fp_k)q_k(\lambda).
 \end{aligned}$$

Here, as in the proof of Theorem 6, the first term in this last right-hand side is $o(1)$ as $m \rightarrow \infty$. Further, given $\varepsilon > 0$, we can find a K such that the absolute value of the second term in this last right-hand side is bounded above by ε . Hence for m large enough,

$$\left| Q_m^{N_m}(f; \lambda) - I(f; \lambda) - \sum_{k=K}^{N_m} Q_m(fp_k)q_k(\lambda) \right| < 2\varepsilon,$$

which proves the theorem. $\square$

Department of Applied Mathematics
Weizmann Institute of Science
P.O. Box 26
Rehovot 76100, Israel
E-mail: maweintr@weizmann.bitnet

Centre for Advanced Computing and Decision Support
C.S.I.R.
P.O. Box 395
Pretoria 0001, Republic of South Africa

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