The Validity of Shapiro's Cyclic Inequality

By B. A. TROESCH

Abstract. A cyclic sum $S_N(\mathbf{x}) = \sum x_i/(x_{i+1} + x_{i+2})$ is formed with N components of a vector $\mathbf{x}$, where in the sum $x_{N+1} = x_1$, $x_{N+2} = x_2$, and where all denominators are positive and all numerators are nonnegative. It is known that there exist vectors $\mathbf{x}$ for which $S_N(\mathbf{x}) < N/2$ if $N \geq 14$ and even, and if $N \geq 24$. It has been proved that the inequality $S_N(\mathbf{x}) \geq N/2$ holds for $N \leq 13$. Although it has been conjectured repeatedly that the inequality also holds for odd N between 15 and 23, this has apparently never been proved. Here we will confirm that the inequality indeed holds for all odd $N \leq 23$. This settles the question for all N .

1. Introduction. The problem suggested by H. S. Shapiro in 1954 [12] has attracted wide interest; the history of the problem up to 1970 is described vividly by D. S. Mitrinović in his book "Analytic Inequalities" [8, pp. 132ff.]. When the problem was published, it appeared very reasonable to conjecture that $N/2$ is the minimum that the cyclic sum S_N can attain. It came therefore as a surprise that for some N actually $S_N(\mathbf{x}) < N/2$ is possible ([5], reporting a result by Lighthill). This led to the considerable interest in the problem.

It has been proved that $S_N(\mathbf{x}) \geq N/2$ for all admissible vectors $\mathbf{x}$, if $N \leq 13$ [14]. On the other hand, there exist vectors $\mathbf{x}$ such that $S_N < N/2$, if $N \geq 14$ and even, and also for all $N \geq 24$ ([7] contains a slight misprint). The difference in behavior for N even against N odd is explained in [11].

In this investigation it will be shown that $S_N \geq N/2$ for the remaining cases, namely $15 \leq N \leq 23$ and odd. This settles the question of Shapiro's inequality for all N . From a result in [1], it follows that only the case $N = 23$ need to be investigated: if the inequality $S_N(\mathbf{x}) \geq N/2$ holds for $N = 23$, it automatically holds for all lower odd N .

Unfortunately, the only feasible method to show that $S_{23} \geq 23/2$ appears to be based on the discussion and some numerical computation of many different cases. This approach has been used in [9] for $N = 10$, in [6] for $N = 12$, and in [14] for $N = 13$. The largest N where a purely algebraic proof has been successful is $N = 8$ [3].

It is crucial to consider the cases separately depending on which components of $\mathbf{x}$ are zero, and which components are different from zero. The reason for this is clear: S_N is a function of the N variables $x_1, x_2, \dots, x_N$, where $x_k \geq 0$. At the stationary points of S_N we have $\partial S_N / \partial x_k = 0$ when $x_k > 0$, while at the boundary of the admissible domain where $x_k = 0$ the derivative of S_N need not vanish. Although no two consecutive components of $\mathbf{x}$ are permitted to vanish, the

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number of possibilities nevertheless grows very rapidly with N , and turns out to be over 2500 for $N = 23$. It seems very undesirable to let the computer investigate all these cases.

2. General Description of the Method. The approach, the results, and the notation described in [14] will be used. The number and the positions of the zero components in the vector $\mathbf{x}$ is essential; the string of consecutive nonzero components is called a *segment*. There are three observations that immediately reduce the number of cases to be considered down to 100 cases. First, it is shown in [14] that there is no loss of generality if the segments are rearranged, for instance in order of decreasing segment length. Furthermore, a case with $S_N < N/2$ must necessarily contain a segment of length 6 at least ([14, Section 4]). And last, segments of length 2 need not be considered, because it can be shown that there is always another case that has a lower sum S .

Let us denote by $(c_1, c_2, \dots, c_l)$ the case where c_1 is the length of the longest segment, down to c_l , the length of the shortest segment. The list of possibilities then starts out with (22), (20,1), (18,3), (18,1,1), (17,4), (16,5), (16,3,1) and ends with (6,3,3,4*1), (6,3,6*1), (6,8*1), namely a 6-segment followed by eight one-segments. It turns out that many additional cases can be eliminated from consideration, if the inequalities to be described below are taken into account, together with the restriction on the pivotal ratio u which is easily obtained for segments of odd length up to length 9.

The remaining cases are then investigated by a comprehensive search in a small region of a two-parameter plane (see Figure 1). The implementation of the search requires only a few lines of programming.

3. The Properties of a Segment. From the remark above it follows that each segment can be analyzed separately, and then segments with the same leading ratio u (see below) are concatenated to find the admissible stationary point. According to [9], there is at most one of them for each case.

Let us therefore analyze a segment of length m in more detail. We take as example m to be odd to enable us to be specific in the signs, where they alternate. Therefore, we set (the zero components are not included in the numbering)

$$\mathbf{x} = x_1 \ 0 \ x_2 \ x_3 \ \cdots \ x_m \ x_{m+1} \ 0 \ x_{m+2} \ \cdots$$

The sum for the m -segment is

$$S_m = \frac{x_2}{x_3 + x_4} + \frac{x_3}{x_4 + x_5} + \cdots + \frac{x_{m-1}}{x_m + x_{m+1}} + \frac{x_m}{x_{m+1}} + \frac{x_{m+1}}{x_{m+2}}.$$

A choice of new independent variables

$$y_1 = x_2, \ y_2 = x_3 + x_4, \dots, \ y_{m-1} = x_m + x_{m+1}, \ y_m = x_{m+1}, \ y_{m+1} = x_{m+2}$$

is used with success in [9], [6], and [14], and solving for $\mathbf{x}$,

$$x_{m+2} = y_{m+1}, \ x_{m+1} = y_m, \ x_m = y_{m-1} - y_m, \dots, \ x_3 = y_2 - y_3 + y_4 \cdots - y_m, \ x_2 = y_1,$$

leads to

$$\begin{aligned} S_m = & \frac{y_1}{y_2} + \frac{y_2 - y_3 \cdots - y_m}{y_3} + \frac{y_3 - y_4 \cdots + y_m}{y_4} \cdots \\ & + \frac{y_{m-2} - y_{m-1} + y_m}{y_{m-1}} + \frac{y_{m-1} - y_m}{y_m} + \frac{y_m}{y_{m+1}} \end{aligned}$$

or

$$S_m = c_2 + c_3 + \cdots + c_m + c_{m+1},$$

which defines the ratios c .

As in [14, Section 3], we set $r_k = y_k/y_{k+1}$, so that $c_2 = r_1$, $c_{m+1} = r_m$, $y_3c_3 - y_2 = -y_4c_4$, and quite generally, $y_kc_k - y_{k-1} = -y_{k+1}c_{k+1}$, $k = 3, 4, \dots, m$. In terms of the r_k 's this can be written as

$$(3.1) \quad c_{k+1} = r_k(r_{k-1} - c_k), \quad k = 3, 4, \dots, m.$$

For a stationary S_m , namely $\partial S_m / \partial y_k = 0$ for $k = 2, 3, \dots, m+1$, we obtain

$$\begin{aligned} & -\frac{y_1}{y_2} + \frac{y_2}{y_3} = 0, \\ & -\frac{y_2}{y_3} + 0 - \frac{y_4}{y_3} + \frac{y_5}{y_3} \dots - \frac{y_{m-1}}{y_3} + \frac{y_m}{y_3} + \frac{y_3}{y_4} = 0, \\ & + \frac{y_4}{y_3} - \frac{y_3}{y_4} + 0 - \frac{y_5}{y_4} \dots + \frac{y_{m-1}}{y_4} - \frac{y_m}{y_4} + \frac{y_4}{y_5} = 0, \\ & -\frac{y_5}{y_3} + \frac{y_5}{y_4} - \frac{y_4}{y_5} + 0 \dots - \frac{y_{m-1}}{y_5} + \frac{y_m}{y_5} + \frac{y_5}{y_6} = 0, \\ & \dots \dots \dots \\ & + \frac{y_{m-1}}{y_3} - \frac{y_{m-1}}{y_4} + \frac{y_{m-1}}{y_5} \dots 0 - \frac{y_m}{y_{m-1}} + \frac{y_{m-1}}{y_m} = 0, \\ & -\frac{y_m}{y_3} + \frac{y_m}{y_4} - \frac{y_m}{y_5} \dots - \frac{y_{m-1}}{y_m} + 0 + \frac{y_m}{y_{m+1}} = 0, \\ & -\frac{y_m}{y_{m+1}} + \frac{y_{m+1}}{y_{m+2}} = 0. \end{aligned}$$

The first and last equation give $r_1 = r_2$, $r_m = r_{m+1}$, and adding all equations gives $u = r_2 = r_m = r_{m+1}$, where r_{m+1} is the leading element of the next segment. This shows, as mentioned in [14, Section 2], that at the stationary point all segments have the same pivotal element u , a fact which is very helpful in the investigation. With the notation above, the remaining equations become

$$\begin{aligned} & -c_3 - 1 + r_3 = 0, \\ & \frac{1}{r_3} - c_4 - 1 + r_4 = 0, \\ & -\frac{1}{r_3r_4} + \frac{1}{r_4} - c_5 - 1 + r_5 = 0, \\ & \dots \dots \dots, \\ & + \frac{1}{r_3r_4 \cdots r_{m-2}} - \frac{1}{r_4r_5 \cdots r_{m-2}} + \cdots + \frac{1}{r_{m-2}} - c_{m-1} - 1 + r_{m-1} = 0, \\ & -\frac{1}{r_3r_4 \cdots r_{m-1}} + \frac{1}{r_4r_5 \cdots r_{m-1}} \cdots + \frac{1}{r_{m-1}} - c_m - 1 + r_m = 0. \end{aligned} \quad (3.2)$$

Next, we leave the first equation as is, add the first equation to the second equation multiplied by r_3 , add the second equation to the third equation multiplied by r_4 ,

and so on:

$$\begin{aligned}c_3 &= r_3 - 1, \\c_3 &= r_3(r_4 - c_4), \\c_4 &= r_4(r_5 - c_5),\end{aligned}$$

or in general,

$$(3.3) \quad c_{k-1} = r_{k-1}(r_k - c_k), \quad k = 4, \dots, m.$$

By returning to Eq. (3.1) it is easy to show as follows that the c_k 's are symmetrical within a segment. Since $r_{m-1} = r_3$, $r_{m-2} = r_4, \dots$ (see [14, Section 3]) and $c_{m+1} = u$, the last equation in (3.1), namely $c_{m+1} = r_m(r_{m-1} - c_m)$, becomes $1 = r_3 - c_m$, and hence, $c_m = c_3$ from Eqs. (3.3). Next, again from Eq. (3.1), $c_m = r_{m-1}(r_{m-2} - c_{m-1})$ or $c_3 = r_3(r_4 - c_{m-1})$ shows that $c_{m-1} = c_4$, and so on. This reduces the number of independent variables by nearly a factor of two.

The equations can now be solved recursively by assuming values for u and r_3 , using Eqs. (3.3) and (3.1) in turn:

$$\begin{aligned}c_3 &= r_3 - 1, \\c_4 &= r_3(u - c_3), \\r_4 &= c_4 + \frac{c_3}{r_3}, \\c_5 &= r_4(r_3 - c_4), \\r_5 &= c_5 + \frac{c_4}{r_4},\end{aligned}$$

and so on. The symmetry in c requires that for m odd the condition

$$(3.4a) \quad c_{(m+1)/2} = c_{(m+5)/2},$$

and for m even,

$$(3.4b) \quad c_{(m+2)/2} = c_{(m+4)/2},$$

must hold. For a fixed r_3 , the values for u are changed to find the values for which this last condition is satisfied. Varying r_3 leads to curves in the $r_3 - u$ plane that have stationary values for S_m and are candidates for stationary values for the cyclic sum S_N .

There are two fortunate circumstances: the recursion formulas are identical for segments of any length, and the search can be restricted to a rather small region, as shown in Figure 1. To show this, we establish several bounds.

4. Some Inequalities. The following inequalities are all based on the fact that the c 's and the r 's must be positive.

a. Since $c_3 = r_3 - 1$, it follows that

$$(4.1) \quad r_3 > 1.$$

b. Next,

$$c_3 = \frac{y_2 - y_3 + \dots - y_m}{y_3}$$

can also be written as

$$c_3 = u - 1 + \frac{1}{r_3} - \frac{y_6 c_6}{y_3},$$

and hence

$$(4.2) \quad u + \frac{1}{r_3} - r_3 - \frac{y_6 c_6}{y_3} = 0,$$

or

$$(4.3) \quad u > r_3 - \frac{1}{r_3}.$$

c. Similarly, c_4 can be written from its definition in two ways:

$$c_4 = r_3 - 1 + \frac{y_6 c_6}{y_4} = r_3 - 1 + \frac{1}{r_4} - \frac{y_7 c_7}{y_4}.$$

On the other hand, it follows from the second Eq. (3.2) that

$$(4.4) \quad c_4 = r_4 + \frac{1}{r_3} - 1,$$

and therefore

$$\begin{aligned} r_3 + \frac{1}{r_4} - r_4 - \frac{1}{r_3} &= \frac{y_7 c_7}{y_4} > 0, \\ (r_3 - r_4) \left(1 + \frac{1}{r_3 r_4} \right) &> 0, \end{aligned}$$

and finally

$$(4.5) \quad r_4 < r_3.$$

d. A useful inequality is obtained by the other representation for c_4 above:

$$r_3 - r_4 - \frac{1}{r_3} + \frac{y_6 c_6}{y_4} = 0$$

dividing it by r_3 ,

$$1 - \frac{r_4}{r_3} - \frac{1}{r_3^2} + \frac{y_6 c_6}{y_3} = 0,$$

and adding it to Eq. (4.2) gives

$$(4.6) \quad u = r_3 - \frac{1}{r_3} + \frac{1}{r_3^2} - \left(1 - \frac{r_4}{r_3} \right).$$

The desired inequality is

$$(4.7) \quad u < r_3 - \frac{1}{r_3} + \frac{1}{r_3^2}.$$

e. Equation (4.6) gives also the result

$$u - r_4 = r_3 - r_4 - \frac{1}{r_3} + \frac{1}{r_3^2} - 1 + \frac{r_4}{r_3} = \left(1 - \frac{1}{r_3} \right) \left(r_3 - \frac{1}{r_3} - r_4 \right).$$

If, as we will show next, the second factor is negative, then

$$(4.8) \quad r_4 > u.$$

To this end, we write Eq. (3.1) for $k = 4$ and $k = 5$:

$$\frac{c_5}{r_4} = (r_3 - c_4), \quad c_6 = r_5(r_4 - c_5);$$

therefore $r_4 > c_5$, or $c_5/r_4 < 1$, so that $r_3 < c_4 + 1$, and then from Eq. (4.4), $r_3 < r_4 + 1/r_3$, as claimed above.

f. Furthermore, by similar considerations, one can show that, after some algebra,

$$(4.9) \quad r_5 > 1 \quad \text{if } r_4 > 1,$$

since $r_5 - 1 = (r_4 - 1/r_4)(1 - 1/r_3) + r_4(r_3 - r_4)$.

Therefore, if $u > 1$, then $r_3 > 1$, $r_4 > 1$, and $r_5 > 1$ follows from Eqs. (4.1) and (4.8). This result then eliminates analytically many cases if the longest segment is a 9-segment.

5. The Curves in the $r_3 - u$ Plane. The recursion formulas and Eq. (3.4) show that an admissible segment of length m is completely determined by r_3 and u . Of particular interest are the values of

$$p_m = \prod_{j=1}^m r_j,$$

since from the definition of the r_j 's in any particular case the product of all p_k 's must equal 1 ([14, Section 3]), and the values of

$$S_m = \sum_{j=2}^{m+1} c_j.$$

The final goal is to show that

$$S_{23} = \sum S_m \geq 23/2$$

in all cases.

As mentioned above, the search in the $r_3 - u$ plane can be restricted to a small region because of the inequalities (4.1), (4.3), and (4.7). Furthermore, segments need only be considered if

$$(5.1) \quad u < 2.2.$$

Otherwise, it follows from Eqs. (4.7), (4.8), and (4.9) that $r_3 > 2.4$, $r_4 > 2.2$, and $r_5 > 1$. A simple computation then shows that $S_7 > 11.8$, exceeding the allowed limit already. All longer segments have an even larger sum. Similar considerations show that $u < 1.4$ must hold, except in four cases.

The search for cases with possibly $S_{23} < 23/2$ can therefore be restricted to the small region shown in Figure 1. The admissible values for an individual segment lie on smooth curves; in Figure 1, the curves for the 8-segment and for the 11-segment are drawn as examples. Segments up to length 9 have just one curve, as can be proved by Descartes's rule of signs, whereas longer segments have one or two curves, with the exception of the 19-segment, which has three curves.

The computation starts with the longest segment in the case being considered. To find a point P on the $r_3 - u$ curve, the r_3 is kept constant and u is changed until Eq. (3.4) is satisfied. The search in r_3 with fixed u is less desirable because of the shape of some curves, like the 11-segment curve. The point P can be ignored, if any of the r 's or c 's turn out to be negative, or if the sum $S_m \geq 23/2$. For segments which are no longer than length 4, the explicit formulas for p_k and S_k , given in [14],

FIGURE 1

Region of admissible solutions, bounded by Eqs. (4.1), (4.3), (4.7), (5.1).

can be computed simultaneously and added to S_m . Only the points where this sum is smaller than $23/2$ need to be analyzed further.

Advantage can also be taken of the fact that for 7- and 9-segments, $u \geq .922$, and that $S_5 \geq 3.0$, $S_7 \geq 4.0$, and $S_9 \geq 5.0$.

Among the about twenty cases left with the possibility that $S_{23} < 23/2$, most are resolved by casual inspection of the numerical results. The cases with the smallest sum S_{23} are listed in Table 1, and all other cases have a larger sum, except for the trivial case with all $x_k = 1$.

In order to check the results and the numerical approach, several cases between $N = 14$ and $N = 22$ were computed by the method described above and the same programming implementation, and indeed the values for $S_N < N/2$ were found, for instance, the case $(11, 1, 1)$ led to $S_{16} < 7.989$.

TABLE 1

Case	(20, 1)	(18, 1, 1)	(16, 1, 1, 1)	(14, 4 * 1)	(12, 5 * 1)
min S_{23}	11.513	11.512	11.513	11.520	11.533

6. A Remark. Since $\inf S_N < N/2$ occurs already for $N = 14$, it might be reasonable to expect that for very large N the ratio S_N/N could fall well below the value $1/2$. The result in [10] that $S_N/N \geq 0.3307\dots$ and in [2] that $S_N/N \geq 0.461238\dots$ for any N were therefore significant. However, in a remarkable paper, Drinfeld [4] proved that $\inf_N (S_N/N) = 0.4945668$. Without the knowledge of Drinfeld's proof, the same result was obtained in [13], including the

formulas identical to those in [4]. But this did not constitute a proof, but rather an example of [4], because a definite distribution of the zero-components of $\mathbf{x}$ was assumed. The assumption appeared reasonable, based on previous experience. It would be desirable to prove that for any N this particular distribution of nonzero components always gives the lowest sum S_N , except of course for the case with all components equal to 1. A result of this kind would make the investigation reported here essentially trivial.

It seems astounding that S_N/N , which can be made easily as low as $1/2$ for any $N \geq 3$ by choosing all $x_k = 1$, can never fall below that value by more than about 1%.

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1. P. H. DIANANDA, "On a cyclic sum," *Proc. Glasgow Math. Assoc.*, v. 6, 1963, pp. 11-13.
2. P. H. DIANANDA, "A cyclic inequality and an extension of it. II," *Proc. Edinburgh Math. Soc.* (2), v. 13, 1962, pp. 143-152.
3. D. Ž. DJOKOVIĆ, "Sur une inégalité," *Proc. Glasgow Math. Assoc.*, v. 6, 1963, pp. 1-10.
4. V. G. DRINFELD, "A cyclic inequality," *Math. Notes*, v. 9, 1971, pp. 68-71.
5. C. V. DURELL, "Query," *Math. Gaz.*, v. 40, 1956, p. 266.
6. E. K. GODUNOVA & V. I. LEVIN, "A cyclic sum with 12 terms," *Math. Notes*, v. 19, 1976, pp. 510-517.
7. J. C. LAGARIAS, "The van der Waerden conjecture: Two Soviet solutions," *Notices Amer. Math. Soc.*, v. 29, 1982, pp. 130-133.
8. D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, New York, 1970.
9. P. NOVOSAD, "Isoperimetric eigenvalue problems in algebra," *Comm. Pure Appl. Math.*, v. 21, 1968, pp. 401-465.
10. R. A. RANKIN, "A cyclic inequality," *Proc. Edinburgh Math. Soc.* (2), v. 12, 1961, pp. 139-147.
11. J. L. SEARCY & B. A. TROESCH, "A cyclic inequality and a related eigenvalue problem," *Pacific J. Math.*, v. 81, 1979, pp. 217-226.
12. H. S. SHAPIRO, "Problem 4603," *Amer. Math. Monthly*, v. 61, 1954, p. 571.
13. B. A. TROESCH, "The shooting method applied to a cyclic inequality," *Math. Comp.*, v. 34, 1980, pp. 175-184.
14. B. A. TROESCH, "On Shapiro's cyclic inequality for $N = 13$," *Math. Comp.*, v. 45, 1985, pp. 199-207.