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# ON AN INTEGER'S INFINITARY DIVISORS

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ABSTRACT. The notions of unitary divisor and biunitary divisor are extended in a natural fashion to give *$k$-ary divisors*, for any natural number $k$. We show that we may sensibly allow $k$ to increase indefinitely, and this leads to *infinitary divisors*. The infinitary divisors of an integer are described in full, and applications to the obvious analogues of the classical perfect and amicable numbers and aliquot sequences are given.

## 1. INTRODUCTION

A divisor $d$ of a natural number $n$ is unitary if the greatest common divisor of $d$ and $n/d$ is 1, and is biunitary if the greatest common unitary divisor of $d$ and $n/d$ is 1. Unitary and biunitary divisors have been studied by several authors, often in terms analogous to those of the classical perfect and amicable numbers. Among these writers are E. Cohen [2], Hagis [4–6], Lal [7], Subbarao and Warren [9], Suryanarayana [11] (see also [12]) and Wall [15, 16].

It is easily seen that, for a prime power $p^y$, the unitary divisors are 1 and $p^y$, and the biunitary divisors are all the powers $1, p, p^2, \dots, p^y$, except for $p^{y/2}$ when $y$ is even.

There is no difficulty in extending this notion. Thus we may call $d$ a triunitary divisor of $n$ if the greatest common biunitary divisor of $d$ and $n/d$ is 1. We soon calculate that the triunitary divisors of $p^y$ are 1 and $p^y$, except if $y = 3$ or 6; those of $p^3$ are $1, p, p^2$, and $p^3$; and those of $p^6$ are $1, p^2$, $p^4$ and $p^6$. In this way, we may also define 4-ary divisors, 5-ary divisors, and so on. We shall speak in general of $k$-ary divisors. The lack of a pattern in the list of $k$-ary divisors of $p^y$ (for small values of $k$, not 1 or 2, and $y$) would have inhibited a study of these. But as we increase $k$, in fact a very striking pattern begins to appear.

Figure 1 shows the $k$-ary divisors of $p^y$ for $k = 1, 2, \dots, 6$, and $0 \le y \le 30$. The asterisks indicate those values of $x$ for which $p^x$ is a $k$-ary divisor of $p^y$. Figure 2 is the same for $k = 19$ and 20, and $0 \le y \le 80$. We notice that for small values of $y$, to be characterized later, the $k$-ary divisors remain fixed. The pattern for large $y$ is also fixed, and depends on whether $k$ is odd or even. Finally, in Figure 3, we show the 100-ary divisors of $p^y$ for $0 \le y \le 120$.

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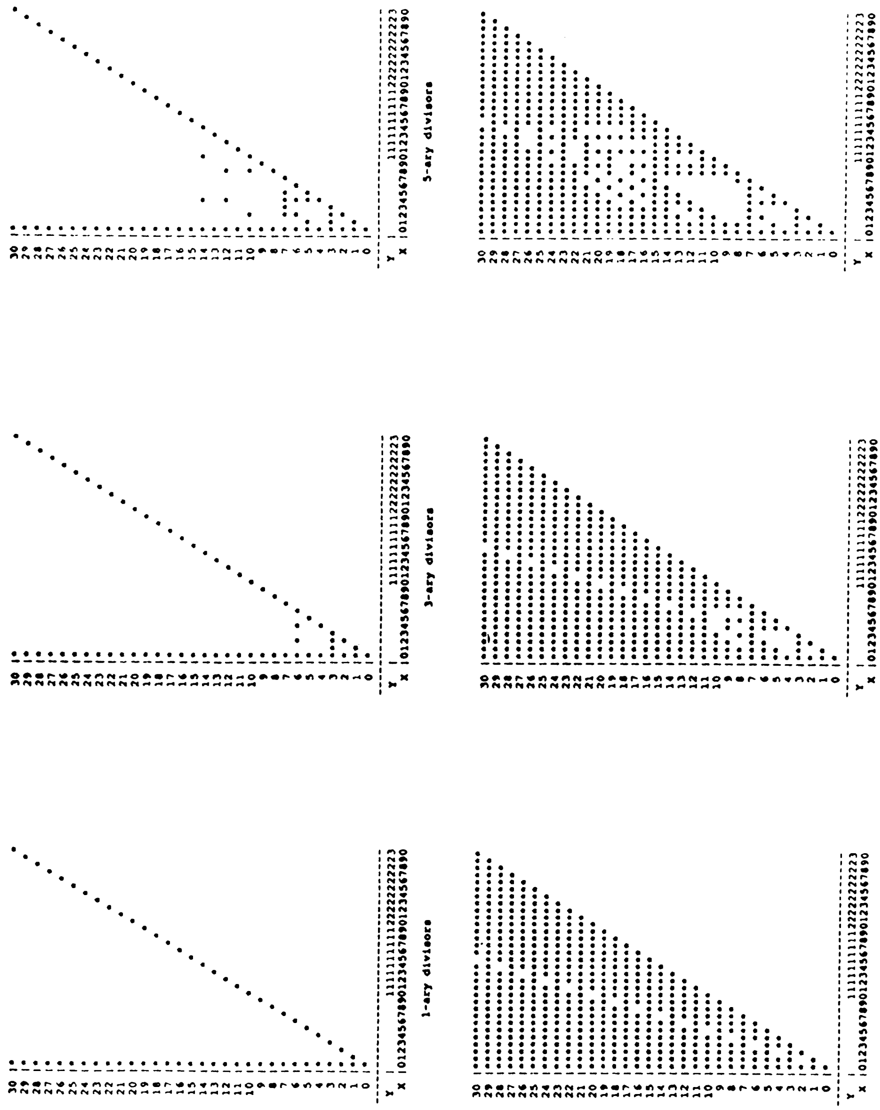

1-ary divisors
2-ary divisors
3-ary divisors
4-ary divisors
5-ary divisors
6-ary divisors

FIGURE 1

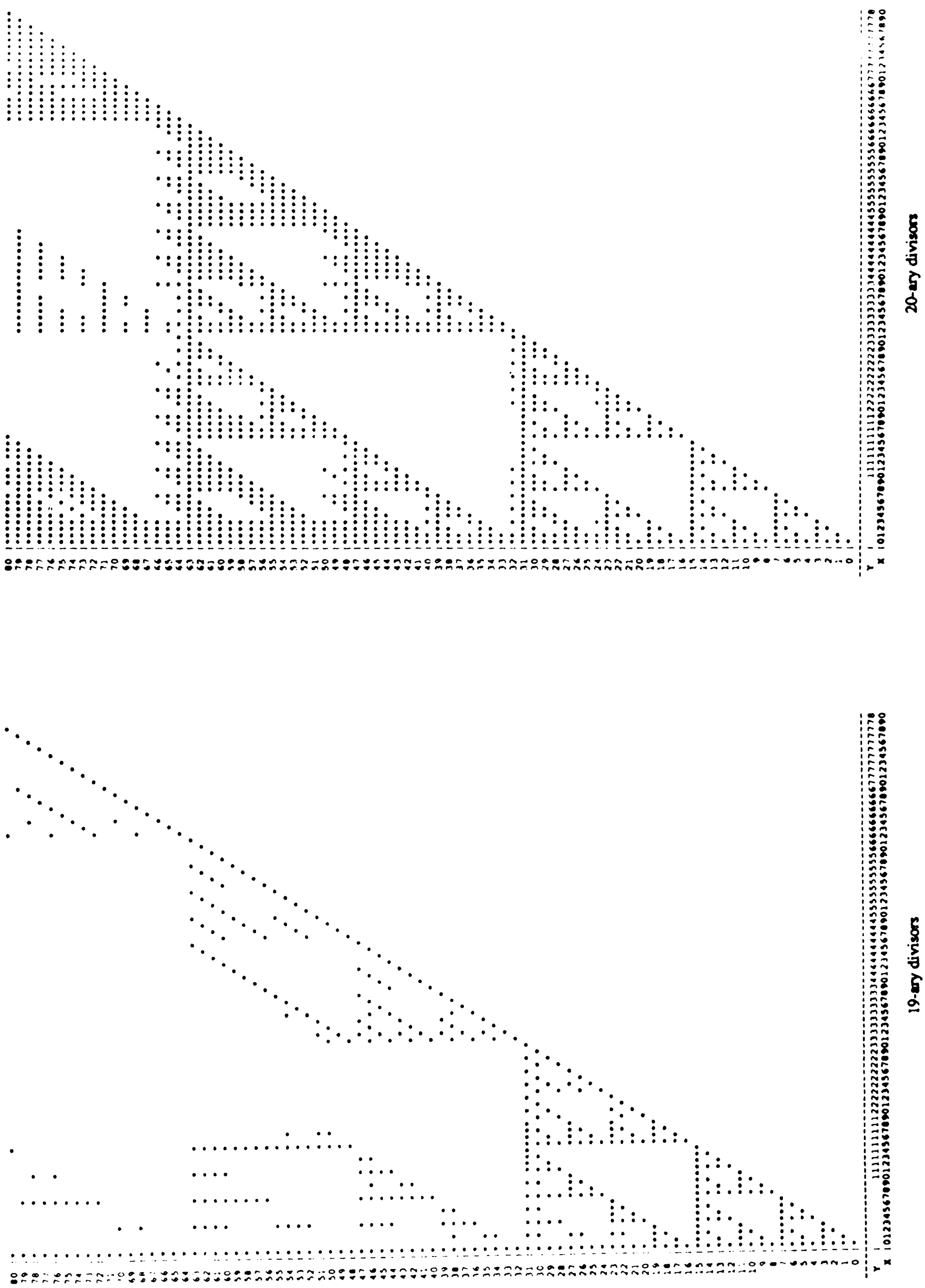
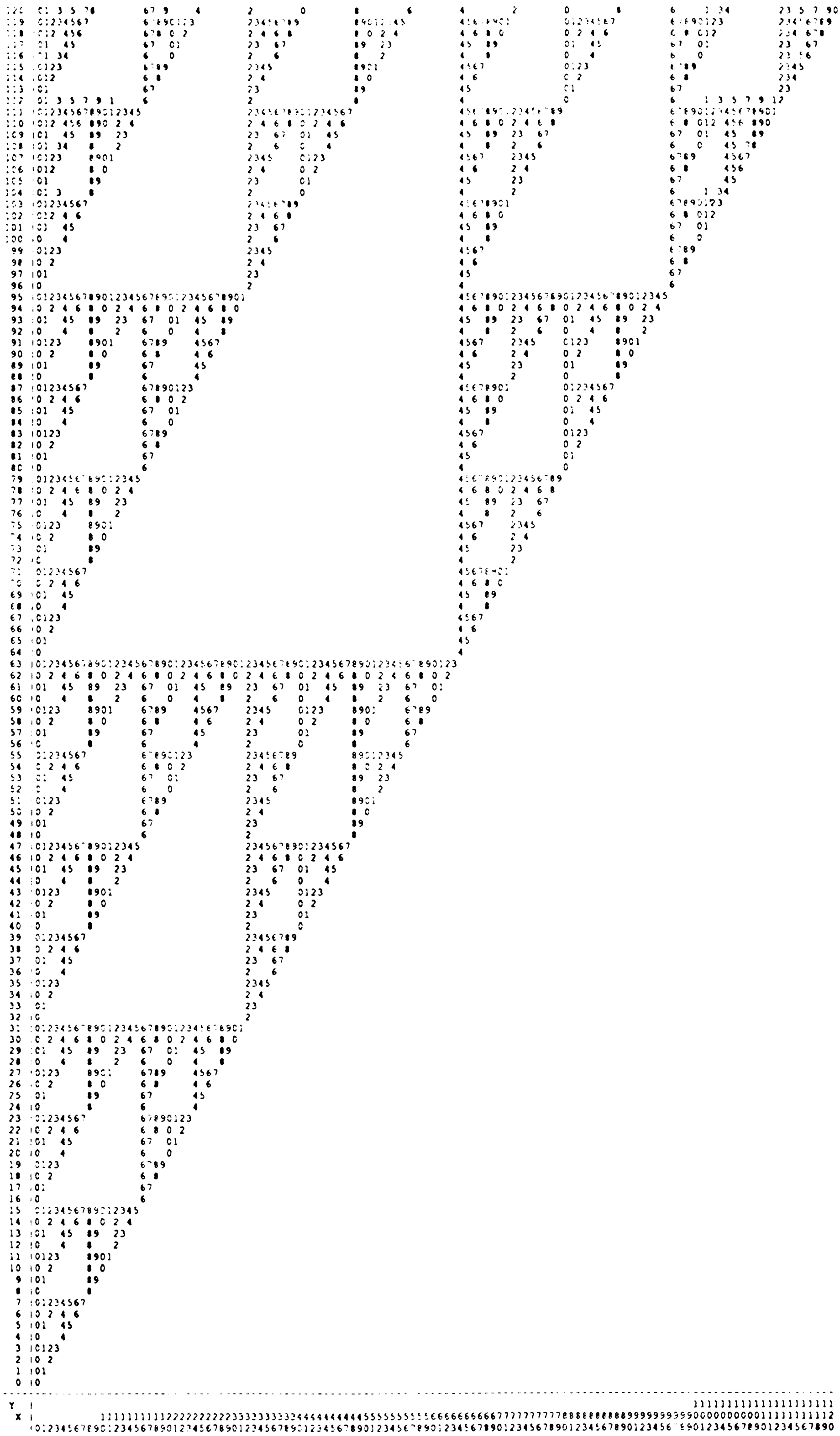



FIGURE 2



100-ary divisors

**FIGURE 3**

The pattern of divisors for the small values of $y$ would now appear to be established, whatever the value of $k$, and it is these which we shall be calling infinitary divisors. Note that the figure shows some collapse in the pattern for $y > 100$. The last (decimal) digit of $x$, where $p^x$ is a 100-ary divisor of $p^y$, is shown in this figure, so that the actual divisors may be read off; this will be useful later.

The pattern indicated by Figure 3 has the distinct appearance of a fractal. It may be compared with Sierpiński's "arrowhead" or "gasket" (Mandelbrot [8]). See also Sved [13], where the same fractal appears, also in a number-theoretic setting.

The pictures would appear to be worth a thousand words. The first aim of this paper is to describe this unexpected pattern in terms of our definition of $k$-ary divisors.

## 2. Infinitary divisors of prime powers

In the following, all letters denote nonnegative integers, with $p$ reserved for an arbitrary prime. To put the above on a formal footing, we begin with

**Definition 1.** A divisor $d$ of an integer $n$ is called a 1-ary divisor of $n$ if the greatest common divisor of $d$ and $n/d$ is 1; and $d$ is called a $k$-ary divisor of $n$ (for $k \geq 2$) if the greatest common $(k-1)$-ary divisor of $d$ and $n/d$ is 1.

For convenience, we shall call $d$ a 0-ary divisor of $n$ if $d|n$. We write $d|_k n$ to indicate that $d$ is a $k$-ary divisor of $n$, and $(l, m)_k$ for the greatest common $k$-ary divisor of $l$ and $m$. It has become common to write $d\|n$ in place of $d|_1 n$.

It should be mentioned that different generalizations of unitary divisor have been given by Suryanarayana [10] (who also used the term "$k$-ary divisor") and Alladi [1].

The following observations are immediate and will be used later without special reference.

(i) For any $n$, $1|_k n$.

(ii) $p^x|_k p^y$ means $(p^x, p^{y-x})_{k-1} = 1$.

(iii) $p^x|_k p^y$ if and only if $p^{y-x}|_k p^y$.

The permanency of the pattern for the early $k$-ary divisors of $p^y$, described in §1, is accounted for in

**Theorem 1.** *For $k \geq y-1 \geq 0$, $p^x|_k p^y$ if and only if $p^x|_{y-1} p^y$.*

*Proof.* The proof is by induction. The result is true when $y = 1$, since $1|_k p$ for all $k$. We suppose now that it is true for $y \leq Y-1$, and consider $y = Y$. For $k = Y-1$, there is nothing to prove, so we suppose also that the result is true for $Y-1 \leq k \leq K-1$, and consider $k = K$.

Suppose $p^x|_K p^Y$. We must show that $p^x|_{Y-1} p^Y$. If this is not true, then $1 \leq x \leq Y-1$ and $(p^x, p^{Y-x})_{Y-2} = p^a$, $a \geq 1$. Since $p^a|_{Y-2} p^x$, the induction hypotheses show first that $p^a|_{x-1} p^x$, and then that $p^a|_{K-1} p^x$. Similarly,

$p^a|_{Y-2}p^{Y-x}$ and $Y - x \le Y - 1$, so $p^a|_{Y-x-1}p^{Y-x}$, and then $p^a|_{K-1}p^{Y-x}$. Hence $(p^x, p^{Y-x})_{K-1} \ge p^a > 1$, contradicting $p^x|_K p^Y$. Hence, $p^x|_{Y-1}p^Y$, as required.

Suppose next that $p^x|_{Y-1}p^Y$. We must show that $p^x|_K p^Y$. If this is not true, then $x \le Y - 1$ and $(p^x, p^{Y-x})_{K-1} = p^b, b \ge 1$. Then $p^b|_{K-1}p^x$, and the induction hypotheses give $p^b|_{x-1}p^x$ and then $p^b|_{Y-2}p^x$. Similarly, $p^b|_{Y-2}p^{Y-x}$, and we are contradicting $p^x|_{Y-1}p^Y$. The proof is complete. □

We are justified now in making the following

**Definition 2.** We call $p^x$ an infinitary divisor of $p^y$ $(y > 0)$ if $p^x|_{y-1}p^y$. We also define 1 to be an infinitary divisor of 1.

We write $p^x|_\infty p^y$ when $p^x$ is an infinitary divisor of $p^y$ (and $p^x \nmid_\infty p^y$ when it is not), and $(p^i, p^j)_\infty$ for the greatest common infinitary divisor of $p^i$ and $p^j$.

**Theorem 2.** *We have* $p^x|_\infty p^y$ *if and only if* $(p^x, p^{y-x})_\infty = 1$.

*Proof.* This is trivially true if $y = 0$ or 1, and generally if $x = 0$ or $x = y$, so assume now that $y \ge 2$ and that $1 \le x \le y - 1$. Then $x - 1 \le y - 2$ and $y - x - 1 \le y - 2$. If $p^x \nmid_\infty p^y$, then $p^x \nmid_{y-1} p^y$, so $(p^x, p^{y-x})_{y-2} = p^a > 1$. Then $p^a|_{y-2}p^x$, so that, by Theorem 1, $p^a|_{x-1}p^x$; similarly, $p^a|_{y-x-1}p^{y-x}$. But then $p^a|_\infty p^x$ and $p^a|_\infty p^{y-x}$, so $(p^x, p^{y-x})_\infty \ge p^a > 1$. For the converse, we assume that $(p^x, p^{y-x})_\infty = p^b > 1$ and essentially reverse the preceding argument. □

The theorems and corollaries which follow will lead to the complete characterization of the infinitary divisors of $p^y$. The pattern of Figure 3 (excluding the top nineteen lines) will thus be fully described, although the characterization we end with, in Theorem 8, will lead to a more efficient means of constructing tables of infinitary divisors.

**Theorem 3.** *We have* $p|_\infty p^y$ *if and only if* $y$ *is odd.*

*Proof.* Using Definition 2, we have $p|_\infty p^1$ and $p \nmid_\infty p^2$. Also, using Theorem 2, we have, for $y \ge 3$,

$$\begin{aligned} p|_\infty p^y &\iff (p, p^{y-1})_\infty = 1 \iff p \nmid_\infty p^{y-1} \\ &\iff (p, p^{y-2})_\infty = p \iff p|_\infty p^{y-2}. \end{aligned}$$

The result follows. □

**Theorem 4.** *If* $y$ *is even and* $p^x|_\infty p^y$, *then* $x$ *is even.*

*Proof.* Suppose $x$ is odd. Then $y - x$ is also odd and, using Theorem 3, $(p^x, p^{y-x})_\infty \ge p$. This contradicts the statement that $p^x|_\infty p^y$. □

**Theorem 5.** *We have* $p^x|_\infty p^y$ *if and only if* $p^{2x}|_\infty p^{2y}$.

*Proof*. We use induction on $y$. The result is trivially true if $y = 0$. Suppose the theorem is true for $y \leq Y - 1$, and consider $y = Y$. Clearly, we may assume $1 \leq x \leq Y - 1$.

Suppose $p^x |_\infty p^Y$, but $p^{2x} \nmid_\infty p^{2Y}$. The latter implies that $(p^{2x}, p^{2Y-2x})_\infty = p^a$, say, and, using Theorem 4, $a$ is even and positive. Put $a = 2b$. Since $p^{2b} |_\infty p^{2x}$ and $p^{2b} |_\infty p^{2Y-2x}$, the induction hypothesis implies that $p^b |_\infty p^x$ and $p^b |_\infty p^{Y-x}$. Then $(p^x, p^{Y-x})_\infty \geq p^b > 1$, contradicting the assumption that $p^x |_\infty p^Y$.

Suppose next that $p^x \nmid_\infty p^Y$. Then $(p^x, p^{Y-x})_\infty = p^c > 1$, from which, by hypothesis, $p^{2c} |_\infty p^{2x}$ and $p^{2c} |_\infty p^{2Y-2x}$. It follows that $p^{2x} \nmid_\infty p^{2Y}$. This completes the proof for $y = Y$, and thus for all $y$. □

**Theorem 6.** *If $p^x |_\infty p^y$ and $y$ is divisible by $2^j$ for some $j \geq 0$, then $x$ is divisible by $2^j$.*

*Proof*. The result is trivial when $j = 0$. Suppose it is true when $j = J$, and consider $j = J + 1$. Put $y = 2^{J+1} a$. By Theorem 4, $x$ is even, say $x = 2w$. Then $p^{2w} |_\infty p^{2^{J+1} a}$, so that, by Theorem 5, $p^w |_\infty p^{2^J a}$. Then, by the induction hypothesis, $w$ is divisible by $2^J$, and the result follows. □

**Corollary 1.** *The infinitary divisors of $p^{2^a}$ are 1 and $p^{2^a}$.*

*Proof*. This is immediate. □

**Corollary 2.** *For $0 \leq k < 2^j$, $p^{2^j} |_\infty p^{2^j + k}$; for $2^j \leq k < 2^{j+1}$, $p^{2^j} \nmid_\infty p^{2^j + k}$.*

*Proof*. The first statement follows from Corollary 1 since, for these $k$, $(p^{2^j}, p^k)_\infty = 1$. The second statement follows from the first. □

**Theorem 7.** *We have $p^{2^j} |_\infty p^y$ if and only if $y \equiv 2^j$ or $2^j + 1$ or $2^j + 2$ or $\cdots$ or $2^{j+1} - 1 \pmod{2^{j+1}}$.*

*Proof*. We have

$$\begin{aligned} p^{2^j} |_\infty p^y &\iff (p^{2^j}, p^{y-2^j})_\infty = 1 \iff p^{2^j} \nmid_\infty p^{y-2^j} \quad \text{( by Corollary 1)} \\ &\iff (p^{2^j}, p^{y-2^{j+1}})_\infty > 1 \iff p^{2^j} |_\infty p^{y-2^{j+1}} \quad \text{(by Corollary 1)} \\ &\iff \cdots \iff p^{2^j} |_\infty p^{y-2^{j+1} l}, \end{aligned}$$

where $l$ is chosen to be the largest integer such that $y - 2^{j+1} l \geq 2^j$. Then $2^j \leq y - 2^{j+1} l < 2^j + 2^{j+1}$, and the result follows from Corollary 2. □

The remainder of the identification process for infinitary divisors is carried out mainly in terms of the binary representations of the exponents on the prime $p$. We write a binary representation in general fashion as $\sum q_j 2^j$; the sum is finite, $j \geq 0$, each $q_j$ is 0 or 1, and trailing zeros are allowed where required. A little reflection gives us the following alternative statement of Theorem 7.

**Theorem 7′.** *Let* $y = \sum y_j 2^j$. *Then* $p^{2^j} |_\infty p^y$ *if and only if* $y_j = 1$.

**Theorem 8.** *Let* $x = \sum x_j 2^j$ *and* $y - x = \sum z_j 2^j$. *Then*

$$p^x |_\infty p^y \quad \text{if and only if} \quad \sum x_j z_j = 0.$$

*Proof*. Suppose first that $\sum x_j z_j \neq 0$. Then $x_j = z_j = 1$ for some $j$, so $p^{2^j} |_\infty p^x$ and $p^{2^j} |_\infty p^{y-x}$, by Theorem 7′. Hence $(p^x, p^{y-x})_\infty \geq p^{2^j} \geq p$, so $p^x \nmid_\infty p^y$.

For the converse, suppose $p^x \nmid_\infty p^y$, so that $(p^x, p^{y-x})_\infty = p^a$, say, with $a \geq 1$. Put

$$a = \sum a_j 2^j, \quad x - a = \sum b_j 2^j, \quad y - x - a = \sum c_j 2^j.$$

Since $a \geq 1$, we have $a_i = 1$ for some $i$. Since $p^a |_\infty p^x$, we have $\sum a_j b_j = 0$ (using the part of this theorem already proved) and it follows that $b_i = 0$ and that $x_j = a_j + b_j$ for each $j$. Hence $x_i = 1$. Similarly, $p^a |_\infty p^{y-x}$, so $\sum a_j c_j = 0$, from which $c_i = 0$ and $z_i = a_i + c_i = 1$. Thus $\sum x_j z_j \neq 0$. □

**Corollary 3.** *The infinitary divisors of* $p^{2^a - 1}$ *are all* $p^x$, $0 \leq x \leq 2^a - 1$.

*Proof*. Taking $y = 2^a - 1$ in Theorem 8, we see there that if $x_j = 0$ or 1, then $z_j = 1$ or 0, respectively, and $\sum x_j z_j = 0$. □

In the next theorem, we prove a very pleasing and useful property, namely, that infinitary divisors are transitive. This is not true of $k$-ary divisors in general. For example, $p|_5 p^3$ and $p^3|_5 p^7$, but $p \nmid_5 p^7$.

**Theorem 9.** *If* $p^x |_\infty p^y$ *and* $p^y |_\infty p^z$, *then* $p^x |_\infty p^z$.

*Proof*. The result is trivial if $x = 0$, so suppose $x \geq 1$. Write $x = \sum x_j 2^j$, $y = \sum y_j 2^j$, $y - x = \sum r_j 2^j$, $z - y = \sum s_j 2^j$, and $z - x = \sum t_j 2^j$. We must show that $\sum x_j t_j = 0$, given that $\sum x_j r_j = 0$ and $\sum y_j s_j = 0$. Consider any particular value of $j$, say $j = k$, for which $x_k = 1$. Since $\sum x_j r_j = 0$, we then have $r_k = 0$ and $y_j = x_j + r_j$ for each $j$, so $y_k = 1$. Then $s_k = 0$. We note that $z - x = (z - y) + (y - x)$. If $k = 0$, then $t_0 = s_0 + r_0 = 0$. If $k > 0$, then $t_k = s_k + r_k$ unless $s_i = r_i = 1$ for some $i < k$. In that case, $y_i = 0$, since $\sum y_j s_j = 0$, and we cannot have $y_i = x_i + r_i$. Hence $t_k = s_k + r_k = 0$, so $x_j t_j = 0$ for all $j$, and the proof is finished. □

**Theorem 10.** *Suppose* $2^a \leq y < 2^{a+1}$. *If* $p^x |_\infty p^{y-2^a}$, *then* $p^x |_\infty p^y$ *and* $p^{2^a + x} |_\infty p^y$; *if* $x \leq y - 2^a$ *and* $p^x |_\infty p^y$, *then* $p^x |_\infty p^{y-2^a}$.

*Proof*. Assume $p^x |_\infty p^{y-2^a}$. Since $2^a \leq y < 2^{a+1}$, Theorem 7′ implies that $p^{2^a} |_\infty p^y$, so $p^{y-2^a} |_\infty p^y$, and so $p^x |_\infty p^y$, by Theorem 9.

Now put $x = \sum x_j 2^j$ and $y - 2^a - x = \sum z_j 2^j$. We have $x \leq y - 2^a < 2^{a+1} - 2^a = 2^a$ and $2^a \leq 2^a + x < 2^{a+1}$, so $2^a + x$ has the proper binary representation $2^a + \sum x_j 2^j$. By Theorem 8, $\sum x_j z_j = 0$, and so, by the same theorem, $p^{2^a+x} |_\infty p^y$.

For the second part, suppose $p^x \nmid_\infty p^{y-2^a}$, and let $x$ and $y - 2^a - x$ be as before. Then $x_j = z_j = 0$ for $j \geq a$ and $x_k = z_k = 1$ for some $k < a$ (by Theorem 8). But then $y - x = 2^a + \sum z_j 2^j$ and the right-hand side is a proper binary representation; since $x_k = z_k = 1$, we have $p^x \nmid_\infty p^y$, as required. □

**Theorem 11.** *If* $2^a \leq y < 2^{a+1}$ *and* $y - 2^a < x < 2^a$, *then* $p^x \nmid_\infty p^y$.

*Proof*. Since $y - 2^a < x < 2^a$, we also have $y - 2^a < y - x < 2^a$. Then, putting $x = \sum x_j 2^j$ and $y - x = \sum z_j 2^j$, we may assume in each sum that $j \leq a - 1$. Put also $y = \sum y_j 2^j$. If $x_j z_j = 0$ for all $j$, then $y_j = x_j + z_j$ for all $j$, and it is impossible to have $y_a = 1$, which we require since $2^a \leq y < 2^{a+1}$. Hence $x_j = z_j = 1$ for some $j$, implying, by Theorem 8, that $p^x \nmid_\infty p^y$. □

Theorems 10 and 11 imply the "arrowhead" of Figure 3. In particular, Theorem 11 accounts for the large empty triangles.

We can use Theorem 10 to find the infinitary divisors of prime powers very quickly (that is, in polynomial time). For example, the infinitary divisors of $p^{150}$ are the infinitary divisors $p^x$ of $p^{150-128}$, i.e., $p^{22}$, and each $p^{128+x}$. Use Figure 3 for the infinitary divisors of $p^{22}$ or calculate them from those of $p^{22-16}$, i.e., $p^6$. The infinitary divisors of $p^6$ are $p^x$ for $x = 0, 2, 4, 6$; so those of $p^{22}$ have $x = 0, 2, 4, 6, 16, 18, 20, 22$. Then the infinitary divisors of $p^{150}$ are $p^x$ for $x = 0, 2, 4, 6, 16, 18, 20, 22, 128, 130, 132, 134, 144, 146, 148, 150$.

The simplest means of constructing the Sierpiński arrowhead is by means of Pascal's triangle, where only the parity of the binomial coefficients need be noted (Sved [13]). This gives immediately the following unexpected characterization of infinitary divisors.

**Theorem 12.** *We have* $p^x |_\infty p^y$ *if and only if* $\binom{y}{x}$ *is odd.*

## 3. Infinitary divisors of integers

The simplest and quickest way to introduce infinitary divisors in general is as follows.

**Definition 3.** Let $d$ be a divisor of $n$ and write $n = \prod_{j=1}^t p_j^{y_j}$, for distinct primes $p_1, p_2, \ldots, p_t$, and $d = \prod_{j=1}^t p_j^{x_j}$ (where $0 \leq x_j \leq y_j$, $j = 1, 2, \ldots, t$). Then $d$ is an infinitary divisor of $n$ if $p_j^{x_j} |_\infty p_j^{y_j}$ for each $j = 1, 2, \ldots, t$.

We write $d |_\infty n$ if $d$ is an infinitary divisor of $n$.

A more fundamental approach, parallel to what has been done for prime powers, would be to write, say,

$$h(n) = \max_{p^y \| n} y,$$

and to define $d$ to be an infinitary divisor of $n$ if $d|_{h(n)-1}n$. It could then be shown that $d|_k n$ for any $k \geq h(n)-1$ and after some work we would obtain the result assumed by Definition 3. Conversely, the results just alluded to can be shown to be a consequence of our definition.

## 4. Functions of infinitary divisors

We denote the number of infinitary divisors of $n$ by $\tau_\infty(n)$ and their sum by $\sigma_\infty(n)$. Essentially the same discussion as that for the example following Theorem 11 gives us

**Theorem 13.** *Let* $y = \sum y_j 2^j$. *Then*

$$\tau_\infty(p^y) = 2^{\sum y_j}, \qquad \sigma_\infty(p^y) = \prod_{y_j=1}\left(1 + p^{2^j}\right).$$

*Proof.* Suppose $2^a \leq y < 2^{a+1}$. Then, by Theorem 10,

$$\tau_\infty(p^y) = 2\tau_\infty(p^{y-2^a}), \qquad \sigma_\infty(p^y) = \sigma_\infty(p^{y-2^a}) + p^{2^a}\sigma_\infty(p^{y-2^a}).$$

Applying the same argument to the infinitary divisors of $p^{y-2^a}$, and repeating it as often as necessary, gives the theorem. □

This theorem in fact gives a direct means of finding the infinitary divisors of $p^y$. For example, since $150 = 128 + 16 + 4 + 2$, we have

$$\sigma_\infty(p^{150}) = (1+p^2)(1+p^4)(1+p^{16})(1+p^{128}).$$

The terms in the sum, after the product on the right is multiplied out, are the infinitary divisors of $p^{150}$.

The functions $\tau_\infty$ and $\sigma_\infty$ are easily seen to be multiplicative, so general expressions for $\tau_\infty(n)$ and $\sigma_\infty(n)$ may be written down with the aid of Theorem 13.

## 5. Infinitary perfect and multiperfect numbers

We define an integer $n$ to be infinitary perfect if $\sigma_\infty(n) = 2n$ and infinitary multiperfect if $\sigma_\infty(n) = sn$ for some $s \geq 2$.

It is apparent from Theorem 13 that for values of $n$ which are not, to take the extreme case, products of powers of primes of the form $p^{2^a}$, there is generally a rich algebraic factorization of $\sigma_\infty(n)$, so that more freedom is to be expected in searching for infinitary perfect numbers than is the case for $k$-ary perfect numbers for particular (small) $k$. (We say $n$ is $k$-ary perfect if the sum of all $k$-ary divisors of $n$ is $2n$.) The only biunitary perfect numbers are 6, 60, and 90 (Wall [15]) and only five unitary perfect numbers are known (Wall [16]).

Without too intensive a search, we have found the following infinitary perfect numbers:

$$
\begin{array}{ll}
2\cdot 3, & 2^6 3^4 5^3 7^2 13\cdot 17\cdot 41, \\
2\cdot 3^2 5, & 2^8 3^3 5\cdot 11\cdot 43\cdot 257, \\
2^2 3\cdot 5, & 2^{10} 3^2 5^2 7\cdot 11\cdot 13\cdot 43\cdot 257, \\
2^4 3^3 5\cdot 17, & 2^{10} 3^4 5^3 7^2 11\cdot 13\cdot 41\cdot 43\cdot 257, \\
2^5 3^4 7\cdot 17\cdot 41, & 2^{12} 3^5 7\cdot 11\cdot 17\cdot 41\cdot 43\cdot 257, \\
2^6 3^2 5^2 7\cdot 13\cdot 17, & 2^{12} 3^6 5\cdot 7\cdot 11\cdot 17\cdot 41\cdot 43\cdot 257, \\
2^6 3^4 5\cdot 7\cdot 17\cdot 41, & 2^{12} 3^6 5^3 7^2 11\cdot 13\cdot 17\cdot 41\cdot 43\cdot 257.
\end{array}
$$

Assuming the validity of the comments following the statement of Definition 3, it will be observed, for example, that the last of the above numbers is $k$-ary perfect for all $k \geq 11$.

The next thirteen numbers satisfy $\sigma_\infty(n) = 3n$:

$$
\begin{array}{ll}
2^3 3\cdot 5, & 2^{11} 3^4 5^3 7^2 11\cdot 13\cdot 41\cdot 43\cdot 257, \\
2^5 3^3 5\cdot 17, & 2^{13} 3^5 7\cdot 11\cdot 17\cdot 41\cdot 43\cdot 257, \\
2^7 3^2 5^2 7\cdot 13\cdot 17, & 2^{13} 3^6 5\cdot 7\cdot 11\cdot 17\cdot 41\cdot 43\cdot 257, \\
2^7 3^4 5\cdot 7\cdot 17\cdot 41, & 2^{13} 3^6 5^3 7^2 11\cdot 13\cdot 17\cdot 41\cdot 43\cdot 257, \\
2^7 3^4 5^3 7^2 13\cdot 17\cdot 41, & 2^{14} 3^5 5\cdot 7\cdot 11\cdot 17\cdot 41\cdot 43\cdot 257, \\
2^9 3^3 5\cdot 11\cdot 43\cdot 257, & 2^{14} 3^5 5^3 7^2 11\cdot 13\cdot 17\cdot 41\cdot 43\cdot 257. \\
2^{11} 3^2 5^2 7\cdot 11\cdot 13\cdot 43\cdot 257, &
\end{array}
$$

The next seven numbers satisfy $\sigma_\infty(n) = 4n$:

$$
\begin{array}{ll}
2^7 3^3 5^2 7\cdot 13\cdot 17, & 2^{11} 3^5 5^3 7^2 11\cdot 13\cdot 41\cdot 43\cdot 257, \\
2^7 3^5 5\cdot 7\cdot 17\cdot 41, & 2^{13} 3^7 5\cdot 7\cdot 11\cdot 17\cdot 41\cdot 43\cdot 257, \\
2^7 3^5 5^3 7^2 13\cdot 17\cdot 41, & 2^{13} 3^7 5^3 7^2 11\cdot 13\cdot 17\cdot 41\cdot 43\cdot 257. \\
2^{11} 3^3 5^2 7\cdot 11\cdot 13\cdot 43\cdot 257, &
\end{array}
$$

The next two numbers satisty $\sigma_\infty(n) = 5n$:

$$2^{15} 3^7 5\cdot 7\cdot 11\cdot 17\cdot 41\cdot 43\cdot 257, \quad 2^{15} 3^7 5^3 7^2 11\cdot 13\cdot 17\cdot 41\cdot 43\cdot 257.$$

There is no prize for finding further examples of infinitary multiperfect numbers. The above examples are all even: a simple adjustment of the proof of Theorem 1 in Hagis [6] shows that there are no odd infinitary multiperfect numbers. We conjecture further that there are no infinitary multiperfect numbers not divisible by 3.

It is not difficult to devise methods of generating new infinitary multiperfect numbers from known ones. The following are two results in this direction.

**Theorem 14.** *Suppose* $\sigma_\infty(n) = qn$, *where* $q$ *is prime, and that* $q^{2a} \| n$, *for some* $a$. *Then* $\sigma_\infty(qn) = (q+1)qn$.

*Proof.* Using Theorem 13 and the multiplicativity of $\sigma_\infty$, we have

$$\begin{aligned}\sigma_\infty(qn) &= \sigma_\infty\left(q^{2a+1} \cdot \frac{n}{q^{2a}}\right) = (q+1)\sigma_\infty(q^{2a})\sigma_\infty\left(\frac{n}{q^{2a}}\right) \\ &= (q+1)\sigma_\infty(n) = (q+1)qn,\end{aligned}$$

as required. □

For example, given that $n = 2^6 3^2 5^2 7 \cdot 13 \cdot 17$ is infinitary perfect (it appears in the first list above), we immediately expect to find $2n$ in the second list and $6n$ in the third list, as is the case.

**Theorem 15.** *Suppose* $\sigma_\infty(n) = sn$, *and that* $l$ *and* $m$ *satisfy*

$$l\sigma_\infty(m) = m\sigma_\infty(l), \quad l\|n, \quad (m, n/l) = 1.$$

*Then* $\sigma_\infty(mn/l) = s(mn/l)$.

*Proof.* We have

$$\sigma_\infty\left(\frac{mn}{l}\right) = \sigma_\infty(m)\sigma_\infty\left(\frac{n}{l}\right) = \frac{\sigma_\infty(m)}{\sigma_\infty(l)}\sigma_\infty(n) = \frac{m}{l}\sigma_\infty(n) = s\frac{mn}{l}. \quad \square$$

Numbers $l$ and $m$ to satisfy the conditions of this theorem may be obtained as follows. Suppose $\sigma_\infty(u) = tu$ and $\sigma_\infty(v) = tv$ for some $t$, and that $u|v$. Set $w = (u, v)_1$, $l = u/w$, $m = v/w$. Since $w$ is a unitary divisor of $u$, we have $(w, u/w) = 1$; that is, $(l, w) = 1$ and similarly $(m, w) = 1$. Then

$$\frac{l}{m} = \frac{u}{v} = \frac{\sigma_\infty(u)}{\sigma_\infty(v)} = \frac{\sigma_\infty(lw)}{\sigma_\infty(mw)} = \frac{\sigma_\infty(l)}{\sigma_\infty(m)}.$$

If there is some number $n$ with $\sigma_\infty(n) = sn$, $l\|n$, and $(m, n/l) = 1$, then Theorem 15 implies that $mn/l$ is also infinitary multiperfect.

For example, the infinitary perfect numbers $2^6 3^4 5 \cdot 7 \cdot 17 \cdot 41$ and $2^6 3^4 5^3 7^2 \cdot 13 \cdot 17 \cdot 41$ may be taken as $u$ and $v$. Then $w = 2^6 3^4 17 \cdot 41$, $l = 5 \cdot 7$, and $m = 5^3 7^2 13$. In the above lists, there are seven later occurrences of infinitary multiperfect numbers $n$ such that $l\|n$ and $(m, n/l) = 1$, and consequently there are seven corresponding infinitary multiperfect numbers $mn/l = 5^2 7 \cdot 13 \cdot n$.

Despite the apparent ease of finding infinitary multiperfect numbers, it seems to be difficult to show that all such numbers of a desired shape have been found. We do not know, for example, if there are any infinitary perfect numbers divisible by 8 but not 16. We can, however, prove

**Theorem 16.** *The only infinitary perfect numbers not divisible by* 8 *are* 6, 60, *and* 90.

*Proof.* Let $n$ be an infinitary perfect number. If $n = 2m$ and $m$ is odd, then the proof that $n = 6$ or 90 is similar to what follows, but easier, and is omitted.

Suppose $n = 4m$, with $m$ odd. Since $\sigma_\infty$ is multiplicative and $\sigma_\infty(n) = 2n$, we have

$$5\sigma_\infty(m) = 8m. \tag{1}$$

Then $5|m$ and $8\|\sigma_\infty(m)$. The latter implies, by Theorem 13, that $m$ can have at most three distinct prime factors. There are thus three possibilities for the shape of $m$, and we consider them in turn.

*Case* 1: $m = 5^a$. From (1), $\sigma_\infty(5^a) = 8 \cdot 5^{a-1}$. Since the left-hand side is not divisible by 5, we must have $a = 1$. But then we have no solution.

*Case* 2: $m = 5^a q^b$, where $q$ is a prime, not 2 or 5. By Theorem 13, (1) must take one of the following forms:

$$(5^a + 1)(q^b + 1) = 8 \cdot 5^{a-1} q^b, \qquad ab \geq 1, \tag{2}$$

$$(5^a + 1)(q^c + 1)(q^d + 1) = 8 \cdot 5^{a-1} q^{c+d}, \qquad a \geq 1,\ d > c \geq 1, \tag{3}$$

$$(5^c + 1)(5^d + 1)(q^b + 1) = 8 \cdot 5^{c+d-1} q^b, \qquad b \geq 1,\ d > c \geq 1. \tag{4}$$

If (2) holds, then $5^a + q^b + 1 = 3 \cdot 5^{a-1} q^b$, and so, since $a \geq 1$,

$$q^b = \frac{5^a + 1}{3 \cdot 5^{a-1} - 1} \leq 3.$$

Then $q^b = 3$ and, from (2), $a = 1$. We thus obtain the solution $n = 2^2 3 \cdot 5 = 60$, and this is the only solution to arise this way.

Suppose (3) holds. Neither $q^c + 1$ nor $q^d + 1$ can be divisible by 4, since the right-hand side of (3) is not divisible by 16, so we must have $q^c \geq 9$ and $q^d \geq 81$. Then

$$\begin{aligned} \frac{4}{3} \leq \frac{8 \cdot 5^{a-1}}{5^a + 1} &= \frac{(q^c + 1)(q^d + 1)}{q^{c+d}} = 1 + \frac{1}{q^c} + \frac{1}{q^d} + \frac{1}{q^{c+d}} \\ &\leq 1 + \frac{1}{9} + \frac{1}{81} + \frac{1}{729} = \frac{820}{729}. \end{aligned}$$

This is a contradiction.

Next, suppose (4) holds. Then $q^b + 1$ cannot be divisible by 4, so $q^b \geq 9$. In that case,

$$\begin{aligned} \frac{9}{10} \leq \frac{q^b}{q^b + 1} &= \frac{(5^c + 1)(5^d + 1)}{8 \cdot 5^{c+d-1}} = \frac{5}{8}\left(1 + \frac{1}{5^c} + \frac{1}{5^d} + \frac{1}{5^{c+d}}\right) \\ &\leq \frac{5}{8}\left(1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125}\right) = \frac{39}{50}, \end{aligned}$$

which is a contradiction.

*Case* 3: $m = 5^a q^b r^c$, where $q$ and $r$ are distinct primes, not 2 or 5. Now (1) takes the form

$$(5^a + 1)(q^b + 1)(r^c + 1) = 8 \cdot 5^{a-1} q^b r^c. \tag{5}$$

Neither $q^b+1$ nor $r^c+1$ can be divisible by 4, so we may take $q^b \geq 9$ and $r^c \geq 13$. Then

$$\begin{aligned}\frac{4}{3} \leq \frac{8\cdot 5^{a-1}}{5^a+1} = \frac{(q^b+1)(r^c+1)}{q^b r^c} &= 1+\frac{1}{q^b}+\frac{1}{r^c}+\frac{1}{q^b r^c} \\ \leq 1+\frac{1}{9}+\frac{1}{13}+\frac{1}{117} &= \frac{140}{117}.\end{aligned}$$

This is a contradiction.

With the comment above that all infinitary multiperfect numbers are even, the proof is now complete. □

## 6. Infinitary amicable pairs and aliquot cycles

We call two integers $m$ and $n$ infinitary amicable if $\sigma_\infty(m) = m+n = \sigma_\infty(n)$. A more general notion is that of an infinitary aliquot sequence $\{n_j\}_{j=0}^{\infty}$: given the "leader" $n_0$, we define $n_j$, for $j \geq 1$, by $n_j = \sigma_\infty(n_{j-1}) - n_{j-1}$. An infinitary aliquot cycle of order $r$ is a subsequence $n_k, n_{k+1}, \ldots, n_{k+r-1}$ with the property that $n_{k+r} = n_k$. Such cycles of order 1 are infinitary perfect numbers, and cycles of order 2 are infinitary amicable pairs.

A computer run, in which each integer less than $10^6$ was considered in turn as leader, found 62 infinitary amicable pairs, eight infinitary aliquot cycles of order 4, three of order 6, and one of order 11. These are all given below. In this search, there were 36172 infinitary aliquot sequences whose eventual behavior was unknown in that a term of the sequence exceeded the imposed bound of $9\cdot 10^{12}$. Of the remaining sequences, many terminated in cycles with smallest member greater than $10^6$. There was no systematic search for these, so they are not listed, but the longest observed infinitary aliquot cycle was of order 23 and had smallest member 12647808. The computations showed that there are no other cycles of order less than 17 which have smallest member less than $10^6$.

Most of the theorems of Hagis [4, 6] concerned with the corresponding notions for unitary and biunitary divisors may be easily adjusted to apply also to infinitary divisors. These give means of obtaining new amicable pairs and aliquot cycles from known ones. A survey of the extensive literature on the corresponding topic for ordinary and unitary divisors will be found in Guy [3].

The following is a list of all infinitary amicable pairs with smaller member less than $10^6$:

| | |
|---|---|
| $114 = 2\cdot 3\cdot 19$ | $126 = 2\cdot 3^2 7$ |
| $594 = 2\cdot 3^3 11$ | $846 = 2\cdot 3^2 47$ |
| $1140 = 2^2 3\cdot 5\cdot 19$ | $1260 = 2^2 3^2 5\cdot 7$ |
| $4320 = 2^5 3^3 5$ | $7920 = 2^4 3^2 5\cdot 11$ |
| $5940 = 2^2 3^3 5\cdot 11$ | $8460 = 2^2 3^2 5\cdot 47$ |
| $8640 = 2^6 3^3 5$ | $11760 = 2^4 3\cdot 5\cdot 7^2$ |
| $10744 = 2^3 17\cdot 79$ | $10856 = 2^3 23\cdot 59$ |

| | |
|---|---|
| $12285 = 3^3 5 \cdot 7 \cdot 13$ | $14595 = 3 \cdot 5 \cdot 7 \cdot 139$ |
| $13500 = 2^2 3^3 5^3$ | $17700 = 2^2 3 \cdot 5^2 59$ |
| $25728 = 2^7 3 \cdot 67$ | $43632 = 2^4 3^3 101$ |
| $35712 = 2^7 3^2 31$ | $45888 = 2^6 3 \cdot 239$ |
| $44772 = 2^2 3 \cdot 7 \cdot 13 \cdot 41$ | $49308 = 2^2 3 \cdot 7 \cdot 587$ |
| $60858 = 2 \cdot 3^3 7^2 23$ | $83142 = 2 \cdot 3^2 31 \cdot 149$ |
| $62100 = 2^2 3^3 5^2 23$ | $62700 = 2^2 3 \cdot 5^2 11 \cdot 19$ |
| $67095 = 3^3 5 \cdot 7 \cdot 71$ | $71145 = 3^3 5 \cdot 17 \cdot 31$ |
| $67158 = 2 \cdot 3^2 7 \cdot 13 \cdot 41$ | $73962 = 2 \cdot 3^2 7 \cdot 587$ |
| $74784 = 2^5 3 \cdot 19 \cdot 41$ | $96576 = 2^6 3 \cdot 503$ |
| $79296 = 2^6 3 \cdot 7 \cdot 59$ | $83904 = 2^6 3 \cdot 19 \cdot 23$ |
| $79650 = 2 \cdot 3^3 5^2 59$ | $107550 = 2 \cdot 3^2 5^2 239$ |
| $79750 = 2 \cdot 5^3 11 \cdot 29$ | $88730 = 2 \cdot 5 \cdot 19 \cdot 467$ |
| $86400 = 2^7 3^3 5^2$ | $178800 = 2^4 3 \cdot 5^2 149$ |
| $92960 = 2^5 5 \cdot 7 \cdot 83$ | $112672 = 2^5 7 \cdot 503$ |
| $118500 = 2^2 3 \cdot 5^3 79$ | $131100 = 2^2 3 \cdot 5^2 19 \cdot 23$ |
| $118944 = 2^5 3^2 7 \cdot 59$ | $125856 = 2^5 3^2 19 \cdot 23$ |
| $142310 = 2 \cdot 5 \cdot 7 \cdot 19 \cdot 107$ | $168730 = 2 \cdot 5 \cdot 47 \cdot 359$ |
| $143808 = 2^6 3 \cdot 7 \cdot 107$ | $149952 = 2^6 3 \cdot 11 \cdot 71$ |
| $177750 = 2 \cdot 3^2 5^3 79$ | $196650 = 2 \cdot 3^2 5^2 19 \cdot 23$ |
| $185368 = 2^3 17 \cdot 29 \cdot 47$ | $203432 = 2^3 59 \cdot 431$ |
| $204512 = 2^5 7 \cdot 11 \cdot 83$ | $206752 = 2^5 7 \cdot 13 \cdot 71$ |
| $215712 = 2^5 3^2 7 \cdot 107$ | $224928 = 2^5 3^2 11 \cdot 71$ |
| $298188 = 2^2 3^3 11 \cdot 251$ | $306612 = 2^2 3^3 17 \cdot 167$ |
| $308220 = 2^2 3 \cdot 5 \cdot 11 \cdot 467$ | $365700 = 2^2 3 \cdot 5^2 23 \cdot 53$ |
| $356408 = 2^3 13 \cdot 23 \cdot 149$ | $399592 = 2^3 199 \cdot 251$ |
| $377784 = 2^3 3^4 11 \cdot 53$ | $419256 = 2^3 3^4 647$ |
| $420640 = 2^5 5 \cdot 11 \cdot 239$ | $460640 = 2^5 5 \cdot 2879$ |
| $462330 = 2 \cdot 3^2 5 \cdot 11 \cdot 467$ | $548550 = 2 \cdot 3^2 5^2 23 \cdot 53$ |
| $476160 = 2^{10} 3 \cdot 5 \cdot 31$ | $510720 = 2^8 3 \cdot 5 \cdot 7 \cdot 19$ |
| $482720 = 2^5 5 \cdot 7 \cdot 431$ | $574816 = 2^5 11 \cdot 23 \cdot 71$ |
| $487296 = 2^7 3^4 47$ | $516384 = 2^5 3^2 11 \cdot 163$ |

| | |
|---|---|
| $545238 = 2 \cdot 3^3 23 \cdot 439$ | $721962 = 2 \cdot 3^2 19 \cdot 2111$ |
| $576882 = 2 \cdot 3^5 1187$ | $592110 = 2 \cdot 3^4 5 \cdot 17 \cdot 43$ |
| $600392 = 2^3 13 \cdot 23 \cdot 251$ | $669688 = 2^3 97 \cdot 863$ |
| $608580 = 2^2 3^3 5 \cdot 7^2 23$ | $831420 = 2^2 3^2 5 \cdot 31 \cdot 149$ |
| $609928 = 2^3 11 \cdot 29 \cdot 239$ | $686072 = 2^3 191 \cdot 449$ |
| $624184 = 2^3 11 \cdot 41 \cdot 173$ | $691256 = 2^3 71 \cdot 1217$ |
| $635624 = 2^3 11 \cdot 31 \cdot 233$ | $712216 = 2^3 127 \cdot 701$ |
| $643336 = 2^3 29 \cdot 47 \cdot 59$ | $652664 = 2^3 17 \cdot 4799$ |
| $643776 = 2^6 3 \cdot 7 \cdot 479$ | $661824 = 2^6 3^3 383$ |
| $669900 = 2^2 3 \cdot 5^2 7 \cdot 11 \cdot 29$ | $827700 = 2^2 3 \cdot 5^2 31 \cdot 89$ |
| $671580 = 2^2 3^2 5 \cdot 7 \cdot 13 \cdot 41$ | $739620 = 2^2 3^2 5 \cdot 7 \cdot 587$ |
| $726104 = 2^3 17 \cdot 19 \cdot 281$ | $796696 = 2^3 53 \cdot 1879$ |
| $784224 = 2^5 3^2 7 \cdot 389$ | $806976 = 2^6 3^3 467$ |
| $785148 = 2^2 3 \cdot 7 \cdot 13 \cdot 719$ | $827652 = 2^2 3 \cdot 7 \cdot 59 \cdot 167$ |
| $796500 = 2^2 3^3 5^3 59$ | $1075500 = 2^2 3^2 5^3 239$ |
| $815100 = 2^2 3 \cdot 5^2 11 \cdot 13 \cdot 19$ | $932100 = 2^2 3 \cdot 5^2 13 \cdot 239$ |
| $863360 = 2^7 5 \cdot 19 \cdot 71$ | $1339840 = 2^6 5 \cdot 53 \cdot 79$ |
| $898216 = 2^3 11 \cdot 59 \cdot 173$ | $980984 = 2^3 47 \cdot 2609$ |
| $916200 = 2^3 3^2 5^2 509$ | $1072800 = 2^5 3^2 5^2 149$ |
| $947835 = 3^3 5 \cdot 7 \cdot 17 \cdot 59$ | $1125765 = 3^3 5 \cdot 31 \cdot 269$ |
| $974400 = 2^6 3 \cdot 5^2 7 \cdot 29$ | $1147200 = 2^6 3 \cdot 5^2 239$ |
| $988038 = 2 \cdot 3^5 19 \cdot 107$ | $1137402 = 2 \cdot 3^4 7 \cdot 17 \cdot 59$ |
| $998104 = 2^3 17 \cdot 41 \cdot 179$ | $1043096 = 2^3 23 \cdot 5669$ |

A scanning of this list suggests that it would be interesting to investigate why the two members of an infinitary amicable pair often have such similar prime factorizations. The analogues of the theorems in Hagis [6] and the methods of te Riele [14] go part of the way in explaining this.

The eight infinitary aliquot cycles of order 4 with smallest member less than $10^6$ are:

$$(1026, 1374, 1386, 1494),$$
$$(10098, 15822, 19458, 15102),$$
$$(10260, 13740, 13860, 14940),$$
$$(41800, 51800, 66760, 83540),$$
$$(45696, 101184, 94656, 88944),$$

$$(100980, 158220, 194580, 151020),$$
$$(241824, 321216, 331584, 313056),$$
$$(685440, 1517760, 1419840, 1334160).$$

The three of order 6 are:

$$(12420, 16380, 17220, 23100, 26820, 18180),$$
$$(512946, 869454, 891906, 933918, 933930, 769374),$$
$$(830568, 1245912, 1868928, 3288192, 5447088, 1076832).$$

Finally, the only other infinitary aliquot cycle of order less than 17 with least member less than $10^6$ is:

$$(448800, 696864, 1124448, 1651584, 3636096, 6608784, 5729136, 3736464, 2187696, 1572432, 895152).$$

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