MATHEMATICS OF COMPUTATION
VOLUME 54, NUMBER 190
APRIL 1990, PAGES 661-669

# BEST $L^2$-APPROXIMATION OF CONVERGENT MOMENT SERIES

GERHARD BAUR AND BRUCE SHAWYER

ABSTRACT. The authors continue the investigation into the problem of finding the best approximation to the sum of a convergent series, $\sum_{n=0}^{\infty} x^n a_n$, where $\{a_n\}$ is a moment sequence.

The case considered is where $x = 1$. This requires a proper subset of the set of all moment series. Instead of having

$$a_n = \int_0^1 t^n \, d\phi(t) \quad \text{with} \quad \int_0^1 |d\phi(t)| = 1,$$

we have

$$a_n = \int_0^1 t^n (1-t)^\delta \psi(t)\, dt \quad \text{with} \quad \int_0^1 |\psi(t)|^2 dt = 1.$$

With this subset, the authors find the best sequence-to-sequence transformation and show that the error in this transformation of $(n+1)$ terms of the series is

$$\frac{1}{2\delta\sqrt{2\delta-1}} \frac{n+1}{\binom{n+2\delta}{n}} \sim \frac{\Gamma(2\delta)}{\sqrt{2\delta-1}} \frac{1}{n^{2\delta-1}} \quad \text{as } n \to \infty.$$

## 1. INTRODUCTION

In some recent papers [3-6], the second author, W. B. Jurkat, and H. Fiedler have considered the problem of finding the best approximation to the sum of a convergent series, $\sum_{n=0}^{\infty} x^n a_n$, where $\{a_n\}$ is a moment sequence, that is, where

$$a_n = \int_0^1 t^n \, d\phi(t) \quad \text{with} \quad \int_0^1 |d\phi(t)| = 1.$$

This has reduced to finding the best summability matrix to solve the approximation problems

$$\varepsilon_n^{(p)} = \varepsilon_n^{(p,x)} = \inf_{\gamma_n} \left\| \gamma_n(xt) - \frac{1}{1-xt} \right\|_p,$$

where $\gamma_n = \gamma_n(t) = \sum_{k=0}^{n} c_{n,k} t^k$ varies over the class of polynomials of degree $n$, $-1 \le x < 1$, $p \in \{1, 2, \infty\}$ and the norm is taken over $0 \le t \le 1$. Known

Received August 17, 1987; revised November 29, 1988.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 41A25, 41A50; Secondary 41A10, 40A25.

The second author was supported in part by the National Engineering and Sciences Research Council of Canada.

results are:

$$\begin{array}{lll} x=-1, & p=\infty & \text{with } \varepsilon_n^{(p,x)} = \frac{1}{4}\lambda^n \ (\lambda = 3-\sqrt{8}) \ [4]; \\ x=-1, & p=2 & \text{with } \varepsilon_n^{(p,x)} \sim \lambda\sqrt{\pi}/2\lambda^n \ [4]; \\ x=-1, & p=1 & \text{with } \varepsilon_n^{(p,x)} \sim 4\lambda^2\lambda^n \ [3]; \\ -1<x<1, & p\in\{1,2,\infty\} & \text{with } \varepsilon_n^{(p,x)} \sim c_p(x)\{\lambda(x)\}^n \end{array}$$

with $\lambda(x) = (2-x-2\sqrt{1-x})/|x|$ [5, 6]. For details, see the various papers.

There are interesting cases still to be investigated. For example, the case $x = 1$ gives the class of convergent moment series. The above analysis fails, since the singularity in the approximation problem is now at the end of the interval of approximation. It is interesting to note that

$$\lim_{x\to 1-} \lambda(x) = \lim_{x\to 1-} (2-x-2\sqrt{1-x})/|x| = 1$$

and

$$\lim_{x\to 1-} c_\infty(x) = \lim_{x\to 1-} \frac{|x|}{2(1-x)} = +\infty.$$

In order to have $\sum_{n=0}^\infty a_n$ convergent, we must have a proper subset of the set of all moment series. Therefore, we shall consider restrictions on the class of moment series. These will be clarified later.

## 2. Notations

Let $C = (c_{n,k})$ be a series-to-sequence triangular matrix, so that $c_{n,k} = 0$ whenever $k > n$. Define, for $|x| < 1$,

$$\sigma_n(x) = \sum_{k=0}^n c_{n,k} a_k x^k \quad \text{and} \quad \gamma_n(t) = \sum_{k=0}^n c_{n,k} t^k.$$

Thus

$$\sigma_n(x) = \int_0^1 \sum_{k=0}^n c_{n,k} (xt)^k \, d\phi(t) = \int_0^1 \gamma_n(xt)\, d\phi(t).$$

It is easy to show that

$$s(x) = \sum_{k=0}^\infty a_k x^k = \int_0^1 \frac{d\phi(t)}{1-xt}$$

whenever this exists. Thus,

$$\sigma_n(x) - s(x) = \int_0^1 \left(\gamma_n(xt) - \frac{1}{1-xt}\right) d\phi(t).$$

When $x = 1$, we have a difficulty in that the singularity is at the end of the interval. Therefore, we shall consider the following restriction on the type of moment: we shall assume henceforth that $a_n$ has the representation

$$a_n = \int_0^1 t^n (1-t)^\delta \psi(t)\, dt \quad \text{with} \quad \int_0^1 |\psi(t)|^p \, dt = 1.$$

This gives

$$\begin{aligned}\sigma_n(1)-s(1) &= \int_0^1\left(\gamma_n(t)-\frac{1}{1-t}\right)(1-t)^\delta\psi(t)\,dt\\ &= \int_0^1\left((1-t)^\delta\gamma_n(t)-(1-t)^{\delta-1}\right)\psi(t)\,dt\\ &\le \left(\int_0^1\left|(1-t)^\delta\gamma_n(t)-(1-t)^{\delta-1}\right|^q dt\right)^{1/q}\left(\int_0^1|\psi(t)|^p\,dt\right)^{1/p},\end{aligned}$$

where $1/p+1/q=1$. For this to be meaningful, it is necessary that $(\delta-1)q>-1$, that is, $\delta>1/p$.

As in [3], let $B=(b_{n,k})$ be the corresponding sequence-to-sequence matrix and let $\beta_n(t)$ be its row polynomial, so that $t\beta_n(t)=\gamma_n(0)-(1-t)\gamma_n(t)$.

## 3. Solution to the problem, when $p=2$

We have

$$\begin{aligned}\varepsilon_n^{(2)} &= \left(\int_0^1(1-t)^{2\delta}\left|\gamma_n(t)-\frac{1}{1-t}\right|^2 dt\right)^{1/2}\\ &= \left(2^{-2\delta-1}\int_{-1}^1(1-x)^{2\delta}\left|\gamma_n\left(\frac{x+1}{2}\right)-\frac{2}{1-x}\right|^2 dx\right)^{1/2}.\end{aligned}\tag{1}$$

In order to minimize $\varepsilon_n^{(2)}$ over all polynomials $\gamma_n(t)$ of degree less than or equal to $n$, we have to solve the following approximation problem:

$$\text{Find}\quad \min_{\substack{V\text{ polynomial}\\ \deg(V)\le n}}\left(\int_{-1}^1(1-x)^{2\delta}\left|V(x)-\frac{2}{1-x}\right|^2 dx\right)^{1/2},\tag{2}$$

that is, find the best $L^2_w$-polynomial approximation to $f(x)=2/(1-x)$, where $L^2_w=L^2_w[-1,1]$ denotes the class of all functions $g(x)$ such that $\int_{-1}^1 w(x)|g(x)|^2\,dx$ exists and is finite, with $w(x)$ being a given nonnegative weight function (in our case, $w(x)=(1-x)^{2\delta}$). Observe that $f(x)=2/(1-x)\in L^2_w$ if and only if $\delta>\frac{1}{2}$. It is known (see e.g. [1]) that problem (2) is solved by $V(x)=\sum_{k=0}^n c_k\phi_k(x)$, where $\{\phi_k(x)\}_{k=0,1,2,\dots}$ is an orthonormal system in $L^2_w$ with respect to the inner product $\langle f,g\rangle=\int_{-1}^1 w(x)f(x)g(x)\,dx$, and where $c_k$ are the Fourier coefficients, that is, $c_k=\langle 2/(1-x),\phi_k(x)\rangle$, $k=0,1,2,\dots$.

Given $\alpha,\beta\in\mathbf{R}$, then the *Jacobi polynomials* $P_n^{(\alpha,\beta)}(x)$ are defined by

$$P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{n!2^n}(1-x)^{-\alpha}(1+x)^{-\beta}\left(\frac{d}{dx}\right)^n(1-x)^{n+\alpha}(1+x)^{n+\beta}.\tag{3}$$

The following well-known properties of $P_n^{(\alpha,\beta)}(x)$ can be found, for example, in [2, pp. 168–173, formulae (1), (3), (4), (13), (17), (32)]:

(4) If $\alpha, \beta > -1$, then the polynomials $P_n^{(\alpha,\beta)}(x)$ form an orthogonal system in $L_w^2[-1,1]$ with respect to the weight function $w(x) = (1-x)^\alpha(1+x)^\beta$;

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}; \tag{5}$$

$$\begin{aligned}\langle P_n^{(\alpha,\beta)}, P_n^{(\alpha,\beta)}\rangle &= \int_{-1}^{1}(1-x)^\alpha(1+x)^\beta|P_n^{(\alpha,\beta)}(x)|^2\,dx\\ &= 2^{\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)};\end{aligned} \tag{6}$$

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x); \tag{7}$$

$$\begin{aligned}&2^m\left(\frac{d}{dx}\right)^m P_n^{(\alpha,\beta)}(x)\\ &\quad= \frac{\Gamma(n+m+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)}P_{n-m}^{(\alpha+m,\beta+m)}(x), \qquad m = 0, 1, \ldots, n;\end{aligned} \tag{8}$$

$$\begin{aligned}&\left(n+\frac{\alpha}{2}+\frac{\beta}{2}+1\right)(1-x)P_n^{(\alpha+1,\beta)}(x)\\ &\quad= (n+\alpha+1)P_n^{(\alpha,\beta)}(x) - (n+1)P_{n+1}^{(\alpha,\beta)}(x).\end{aligned} \tag{9}$$

**Lemma 1.** *Let $\delta > \frac{1}{2}$. Then the solution of* (2), *that is, the best polynomial approximation to $f(x) = 2/(1-x)$ in $L_w^2[-1,1]$ with $w(x) = (1-x)^{2\delta}$, is given by*

$$\begin{aligned}V(x) &= \frac{1}{2\delta}\sum_{k=0}^{n}\frac{2k+2\delta+1}{\binom{k+2\delta}{k}}P_k^{(2\delta,0)}(x)\\ &= \frac{2}{1-x} - \frac{1}{\delta(1-x)}\frac{n+1}{\binom{n+2\delta}{n}}P_{n+1}^{(2\delta-1,0)}(x).\end{aligned}$$

*Proof.* From (4) and (6) we conclude that

$$\phi_n = \frac{\sqrt{2n+2\delta+1}}{2^{\delta+1/2}}P_n^{(2\delta,0)}(x), \qquad n = 0, 1, 2, \ldots,$$

forms an orthonormal system in $L_w^2[-1,1]$. Thus, we have

$$V(x) = \sum_{k=0}^{n} c_k\phi_k(x),$$

where

$$c_k = \left\langle\frac{2}{1-x}, \phi_k(x)\right\rangle = \frac{\sqrt{2k+2\delta+1}}{2^{\delta+1/2}}\int_{-1}^{1}2(1-x)^{2\delta-1}P_k^{(2\delta,0)}(x)\,dx.$$

Now, successive integration by parts yields

$$\begin{aligned} c_k &= \frac{\sqrt{2k+2\delta+1}}{2^{k+\delta-1/2}} \int_{-1}^{1} (1-x)^{2\delta-1}(1+x)^k \, dx \\ &= \frac{\sqrt{2k+2\delta+1}}{2^{k+\delta-1/2}} \frac{\Gamma(2\delta)\Gamma(k+1)}{\Gamma(2\delta+k+1)} 2^{k+2\delta} \\ &= 2^{\delta+1/2} \frac{1}{2\delta} \frac{\sqrt{2k+2\delta+1}}{\binom{k+2\delta}{k}}. \end{aligned}$$

In order to obtain the second formula, we use (9) with $\alpha = 2\delta - 1$, $\beta = 0$, $n = k$, so that

$$V(x) = \frac{1}{2\delta} \cdot \frac{1}{1-x} \sum_{k=0}^{n} \left( 2 \frac{k+2\delta}{\binom{k+2\delta}{k}} P_k^{(2\delta-1,0)}(x) - 2 \frac{k+2\delta+1}{\binom{k+2\delta+1}{k+1}} P_{k+1}^{(2\delta-1,0)}(x) \right).$$

This sum being a telescoping sum, we conclude that

$$V(x) = \frac{2}{1-x} - \frac{1}{\delta(1-x)} \frac{n+1+2\delta}{\binom{n+1+2\delta}{n+1}} P_{n+1}^{(2\delta-1,0)}(x),$$

from which the result follows (observe $P_0(x) = 1$). □

**Lemma 2.** *Let $V(x)$ be the best polynomial approximation to $f(x) = 2/(1-x)$ in $L_w^2[-1,1]$ as in Lemma 1. Then the minimal value in (2) is*

$$\left( \int_{-1}^{1} w(x) |V(x) - f(x)|^2 \, dx \right)^{1/2} = \frac{2^\delta}{\delta} \frac{1}{\sqrt{2(2\delta-1)}} \frac{n+1}{\binom{n+2\delta}{n}}.$$

*Proof.* It is known that

$$\int_{-1}^{1} w(x) |V(x) - f(x)|^2 \, dx = \int_{-1}^{1} w(x) |f(x)|^2 \, dx - \sum_{k=0}^{n} |c_k|^2.$$

Now,

$$\int_{-1}^{1} w(x) |f(x)|^2 \, dx = \frac{2^{2\delta+1}}{2\delta-1},$$

and

$$\begin{aligned} \sum_{k=0}^{n} |c_k|^2 &= \frac{2^{2\delta-1}}{\delta^2} \sum_{k=0}^{n} \frac{2k+2\delta+1}{\binom{k+2\delta}{k}^2} \\ &= \frac{2^{2\delta-1}}{\delta^2(2\delta-1)} \sum_{k=0}^{n} \left( \frac{(k+2\delta)^2}{\binom{k+2\delta}{k}^2} - \frac{(k+1+2\delta)^2}{\binom{k+1+2\delta}{k+1}^2} \right) \\ &= \frac{2^{2\delta-1}}{\delta^2(2\delta-1)} \left( 4\delta^2 - \frac{(n+1+2\delta)^2}{\binom{n+1+2\delta}{n+1}^2} \right) \\ &= \frac{2^{2\delta+1}}{2\delta-1} - \frac{2^{2\delta-1}}{\delta^2(2\delta-1)} \cdot \frac{(n+1)^2}{\binom{n+2\delta}{n}^2}, \end{aligned}$$

which yields the lemma. □

Now, our main result reads as follows:

**Theorem 1.** *We have*

$$\varepsilon_n^{(2)} = \frac{1}{2\delta\sqrt{2\delta-1}} \frac{n+1}{\binom{n+2\delta}{n}},$$

*and the optimal polynomial is given by*

$$\gamma_n(t) = \frac{1}{1-t} - \frac{1}{2\delta(1-t)} \frac{n+1}{\binom{n+2\delta}{n}} P_{n+1}^{(2\delta-1,0)}(2t-1).$$

This follows immediately from (1) and the preceding lemmas. (Observe that $\gamma_n(t) = V(2t-1)$.)

*Remark.* An application of Stirling's formula shows that

$$\varepsilon_n^{(2)} \sim \frac{\Gamma(2\delta)}{\sqrt{2\delta-1}} \frac{1}{n^{2\delta-1}} \quad \text{as } n \to \infty.$$

Next, we determine the corresponding sequence-to-sequence summability method $B = (b_{n,k})_{k=0,\dots,n;\,n=0,1,2,\dots}$. Recall that $b_{n,k}$ is given by $\beta_n(t) = \sum_{k=0}^n b_{n,k} t^k$, where $t\beta_n(t) = \gamma_n(0) - (1-t)\gamma_n(t)$.

**Theorem 2.** *The sequence-to-sequence method $B = (b_{n,k})$ is given by*

$$b_{n,k} = \frac{(-1)^{n+k}}{2\delta}(n+1)\frac{\binom{n+2\delta+k+1}{k+1}\binom{n+1}{k+1}}{\binom{n+2\delta}{n}}.$$

*Proof.* Using (7) and (5), we first have

$$\gamma_n(0) = 1 - \frac{1}{2\delta}\frac{n+1}{\binom{n+2\delta}{n}} P_{n+1}^{(2\delta-1,0)}(-1) = 1 + \frac{(-1)^n}{2\delta} \cdot \frac{n+1}{\binom{n+2\delta}{n}}.$$

Hence,

$$\begin{aligned} t\beta_n(t) &= \gamma_n(0) - (1-t)\gamma_n(t) \\ &= \frac{(-1)^n}{2\delta} \frac{n+1}{\binom{n+2\delta}{n}} + \frac{1}{2\delta} \frac{n+1}{\binom{n+2\delta}{n}} P_{n+1}^{(2\delta-1,0)}(2t-1) \end{aligned}$$

by Theorem 1. Since $t\beta_n(t) = \sum_{k=0}^n b_{n,k} t^{k+1}$, we have

$$\begin{aligned} b_{n,k} &= \left.\frac{(d/dt)^{k+1}(t\beta_n(t))}{(k+1)!}\right|_{t=0} \\ &= \frac{1}{2\delta}\frac{n+1}{\binom{n+2\delta}{n}}\frac{1}{(k+1)!}2^{k+1}\left.\left(\frac{d}{dx}\right)^{k+1} P_{n+1}^{(2\delta-1,0)}(x)\right|_{x=-1} \\ &= \frac{1}{2\delta}\frac{n+1}{\binom{n+2\delta}{n}}\frac{1}{(k+1)!}\frac{\Gamma(n+2\delta+k+2)}{\Gamma(n+2\delta+1)} P_{n-k}^{(2\delta+k,k+1)}(-1) \end{aligned}$$

by using (8) with $m = k+1$, $\alpha = 2\delta - 1$, $\beta = 0$, and $n$ replaced by $n+1$. Applying (7) and (5) again, we finally get the result for $b_{n,k}$. □

*Remarks.* (i) An application of Stirling's formula shows that $|b_{n,k}| = O(n^{2k+3-2\delta})$ ($k$ fixed). Thus, the $B$-method is *not regular*. In this context, it is interesting that Wimp [7] has noted that

> "nonregular methods often work so well—i.e. accelerate convergence so dramatically—on the sequences for which they do preserve convergence, that one is willing to forego the convenience of using a method which works for every convergent sequence".

Compared with Wimp's work, our approach has the advantage that we do not need the row sum condition $\sum_{k=0}^{n} b_{n,k} = 1$. In fact, for the "best" sequence-to-sequence method $B$ constructed above, this condition holds *asymptotically*, since

$$\sum_{k=0}^{n} b_{n,k} = \beta_n(1) = \gamma_n(0) = 1 + \frac{(-1)^n}{2\delta}\frac{n+1}{\binom{n+2\delta}{n}} \to 1 \quad \text{as } n \to \infty .$$

Moreover, our assumptions on the class of series $\sum_{k=0}^{\infty} a_n$ are very "mild", whereas Wimp requires a representation of the form

$$a_n = \xi + \int_0^{\infty} e^{-nt} f(t)\,dt$$

and makes assumptions on the differentiability of the function $f$.

(ii) In the case $\delta = 1$, we have

$$b_{n,k} = \frac{(-1)^{n+k}}{n+2}\binom{n+1}{k+1}\binom{n+k+3}{k+1} \quad \text{and} \quad \varepsilon_n^{(2)} = \frac{1}{n+2}.$$

This case is of particular interest, since it corresponds to the series $\sum_{k=0}^{\infty} a_k$, where $a_k$ can be written as the *difference* of two moments.

## 4. Some remarks on the case $q = 1$, $p = \infty$

We have

$$\varepsilon_n^{(1)} = \min_{\substack{\gamma_n \text{ polynomial} \\ \deg(\gamma_n) \le n}} \int_0^1 |(1-t)^{\delta}\gamma_n(t) - (1-t)^{\delta-1}|\,dt \qquad (\delta > 0).$$

We proceed as Achieser suggests in [1, pp. 82–88]. Then we first have to determine polynomials $Q_n(t) = \prod_{j=1}^{n}(t - c_{j,n})$ that satisfy

$$0 < c_{1,n} < \cdots < c_{n,n} < 1 \tag{10}$$

and

$$\int_0^1 (1-t)^{\delta} t^k \operatorname{sign} Q_n(t)\,dt = 0 \quad \text{for } k = 0, 1, \ldots, n-1. \tag{11}$$

Note that Achieser shows that there exists a unique polynomial $Q_n(t)$ satisfying (10) and (11).

Define $\psi_k(t) = \int_0^t (1-\tau)^\delta \tau^k \, d\tau$, $k = 0, 1, 2, \ldots$. Then (11) is equivalent to

$$(11') \qquad 2\sum_{j=1}^{n}(-1)^{j+1}\psi_k(c_{j,n}) + (-1)^n \cdot \frac{\Gamma(\delta+1)\cdot k!}{\Gamma(\delta+k+2)} = 0 \qquad \text{for } k = 0, 1, \ldots, n-1.$$

Now a short calculation shows that

$$\psi_k(t) = \frac{k}{k+\delta+1}\psi_{k-1}(t) - \frac{1}{k+\delta+1}(1-t)^{\delta+1}t^k$$

whenever $k \geq 1$. Using this and $\psi_0(t) = -(1-t)^{\delta+1}/(\delta+1) + 1/(\delta+1)$, we can replace $(11')$ by the following equivalent set of conditions:

$$(11'') \qquad \sum_{j=1}^{n}(-1)^{j+1}(1-c_{j,n})^{\delta+1}c_{j,n}^k = \begin{cases} 1/2 & \text{if } k = 0, \\ 0 & \text{if } k \in \{1, \ldots, n-1\}. \end{cases}$$

If we introduce new variables $d_{j,n} = 1 - c_{j,n}$, $j = 1, \ldots, n$, then from $(11'')$ we have the following equations:

$$(12) \qquad \sum_{j=1}^{n}(-1)^{j+1} d_{j,n}^{k+\delta+1} = \frac{1}{2} \quad \text{for } k = 0, \ldots, n-1 \text{ and}$$

$$1 > d_{1,n} > \cdots > d_{n,n} > 0.$$

Since $f(t) = (1-t)^{\delta-1} \in L_1[0, 1]$, and since $f^{(n)}(t)$ has no zero in (0,1), we conclude that

$$(13) \qquad \begin{aligned} \varepsilon_n^{(1)} &= \left| \int_0^1 (1-t)^{\delta-1} \operatorname{sign} Q_{n+1}(t)\, dt \right| \\ &= \left| \frac{2}{\delta}\sum_{j=1}^{n+1}(-1)^{j+1}(1-c_{j,n+1})^\delta - \frac{1}{\delta} \right| \\ &= \frac{2}{\delta}\left| \sum_{j=1}^{n+1}(-1)^{j+1} d_{j,n+1}^\delta - \frac{1}{2} \right|, \end{aligned}$$

and the optimal polynomial $\gamma_n(t)$ is given by the interpolation conditions

$$(14) \qquad \frac{1}{1-c_{j,n+1}} = \gamma_n(c_{j,n+1}), \qquad j = 1, 2, \ldots, n+1.$$

*Remark.* The equations (12) also make sense when $\delta = 0$. In this case it is known that

$$d_{k,n} = \frac{1-\cos(k\pi/(n+1))}{2} = \sin^2\left(\frac{k\pi}{2(n+1)}\right)$$

(in fact, $1 - d_{k,n}$ are the zeros of $U_{n+1}(2x-1)$, where $U_n(x)$ denotes the $n$th degree Chebyshev polynomial of the second kind). In general, the numbers $d_{1,n+1}, \ldots, d_{n+1,n+1}$, and $\varepsilon_n^{(1)}$ can be calculated by standard numerical techniques. For example, we have for $\delta = 1$

(15)

| $n$ | $\varepsilon_n^{(1)}$ | |
|---|---|---|
| 0 | 0.414214 | $(= \sqrt{2} - 1)$ |
| 1 | 0.227774 | |
| 2 | 0.144298 | |
| 3 | 0.099662 | |
| 4 | 0.072982 | |
| 5 | 0.055759 | |

## Acknowledgment

The authors want to thank Dr. Hermann Brunner for providing the numerical results in table (15).

## Bibliography

1. N. I. Achieser, *Theroy of approximation*, Ungar, New York, 1956.
2. A. Erdélyi (Editor), *Higher transcendental functions*, Vol. II, McGraw-Hill, New York, 1955.
3. H. Fiedler and W. B. Jurkat, *On best approximation of alternating series*, J. Approx. Theory **34** (1982), 423–424.
4. W. B. Jurkat and B. L. R. Shawyer, *Best approximation of alternating series*, J. Approx. Theory **34** (1982), 397–422.
5. B. L. R. Shawyer, *On best approximation of alternating power series*, J. Math. Anal. Appl. **114** (1986), 360–375.
6. ——, *On best approximation of positive power series*, Math. Comp. **43** (1984), 529–534.
7. J. Wimp, *New methods for accelerating the convergence of sequences arising in Laplace transform theory*, SIAM J. Numer. Anal. **14** (1977), 194–204.

Abteilung Mathematik I, Universität Ulm, Oberer Eselsberg, D-7900 Ulm, Federal Republic of Germany

Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland A1C 5S7, Canada. *E-mail*: bshawyer@mun.bitnet