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LINEAR COMBINATIONS OF ORTHOGONAL POLYNOMIALS GENERATING POSITIVE QUADRATURE FORMULAS

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ABSTRACT. Let $p_k(x) = x^k + \cdots$, $k \in \mathbf{N}_0$, be the polynomials orthogonal on $[-1, +1]$ with respect to the positive measure $d\sigma$. We give sufficient conditions on the real numbers μ_j, $j = 0, \ldots, m$, such that the linear combination of orthogonal polynomials $\sum_{j=0}^{m} \mu_j p_{n-j}$ has n simple zeros in $(-1, +1)$ and that the interpolatory quadrature formula whose nodes are the zeros of $\sum_{j=0}^{m} \mu_j p_{n-j}$ has positive weights.

1. INTRODUCTION

Let σ be a positive measure on $[-1, 1]$ such that the support of $d\sigma$ contains an infinite set of points. In this paper we consider interpolatory quadrature formulas with positive weights, i.e., quadrature formulas of the form

$$\int_{-1}^{+1} f(x)\, d\sigma(x) = \sum_{j=1}^{n} c_j f(x_j) + R_n(f), \tag{1.1}$$

where $-1 < x_1 < x_2 < \cdots < x_n < 1$, $c_j > 0$ for $j = 1, \ldots, n$, and $R_n(f) = 0$ for $f \in \mathbf{P}_{2n-1-m}$, $0 \le m \le n$ ($\mathbf{P}_n$ denotes as usual the set of polynomials of degree at most n). As in [6], such a quadrature formula is called a positive $(2n-1-m, n, d\sigma)$ quadrature formula. If σ is absolutely continuous on $[-1, 1]$, with $\sigma'(x) = w(x)$, we write also $(2n-1-m, n, w)$ instead of $(2n-1-m, n, d\sigma)$. Furthermore, we say that a polynomial $t_n \in \mathbf{P}_n$ generates a positive $(2n-1-m, n, d\sigma)$ quadrature formula if t_n has n simple zeros $x_1 < x_2 < \cdots < x_n$ in $(-1, +1)$ and the interpolatory quadrature formula based on the nodes x_j is a positive $(2n-1-m, n, d\sigma)$ quadrature formula. Since the degree of exactness is $2n-1-m$, we get with the help of (1.1) the well-known fact that such a polynomial t_n is orthogonal to $\mathbf{P}_{n-1-m}$ with respect to $d\sigma$, and hence is of the form

$$t_n(x) = \sum_{j=0}^{m} \mu_j p_{n-j}(x), \tag{1.2}$$

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where $\mu_j \in \mathbf{R}$ and $p_k(x) = x^k + \cdots$, $k \in \mathbf{N}_0$, denotes the polynomial of degree k orthogonal with respect to $d\sigma$. For that reason we are interested in conditions on the numbers μ_j such that t_n generates a positive $(2n-1-m, n, d\sigma)$ quadrature formula. For small m, $m = 1, 2, 3$, necessary and sufficient conditions on the numbers μ_j can be obtained from the general characterizations of positive quadrature formulas given by the author in [7, 8] (see in particular [8, Theorem 2(b)]), by Sottas and Wanner [10] (note that the conditions given there do not imply that the nodes are in $(-1, +1)$), and recently by H. J. Schmid [9]. But for larger m the computational work increases rapidly, and the conditions become very complex (see the examples given in [9, 10]). Thus, the problem arises to find "simple and applicable" sufficient conditions on the numbers μ_j such that $\sum_{j=0}^{m} \mu_j p_{n-j}$ generates a positive $(2n-1-m, n, d\sigma)$ quadrature formula. This problem is studied and partly solved in this paper by giving first a general sufficient condition on the μ_j's, from which simpler conditions are derived.

2. Preliminary results

In order to state our results, we need some known facts on polynomials orthogonal on $[-1, 1]$, resp. orthogonal on the circumference of the unit circle $|z| = 1$. Let us recall that the polynomials $p_n = x^n + \cdots$, $n \in \mathbf{N}$, orthogonal with respect to $d\sigma$ on $[-1, +1]$ satisfy a recurrence relation of the form

$$p_n(x) = (x - \alpha_n)p_{n-1}(x) - \lambda_n p_{n-2}(x) \quad \text{for } n \in \mathbf{N}, \tag{2.1}$$

where $p_{-1} = 0$, $p_0 = 1$, $\alpha_n \in (-1, +1)$ for $n \in \mathbf{N}$, and $\lambda_n > 0$ for $n \geq 2$. $p_n^{(1)}$, $n \in \mathbf{N}_0$, denotes the so-called associated polynomial, defined by

$$p_n^{(1)}(x) = \frac{1}{d_0} \int_{-1}^{+1} \frac{p_{n+1}(x) - p_{n+1}(t)}{x - t}\, d\sigma(t), \tag{2.2}$$

where $d_0 = \int_{-1}^{+1} d\sigma(t)$. Note that the $p_n^{(1)}$'s are polynomials of degree n with leading coefficient one, which satisfy the following recurrence relation (see e.g. [2, Chapter 3, §4])

$$p_n^{(1)}(x) = (x - \alpha_{n+1})p_{n-1}^{(1)}(x) - \lambda_{n+1} p_{n-2}^{(1)}(x) \quad \text{for } n \in \mathbf{N}, \tag{2.3}$$

where the α_n's and λ_n's are determined by (2.1).

We are now ready to state the first simple characterization of positive quadrature formulas.

Lemma 1. *Let $n, m \in \mathbf{N}_0$, $n \geq m$, and let $\mu_j \in \mathbf{R}$ for $j = 0, \ldots, m$, $\mu_0 \neq 0$. Then $\sum_{j=0}^{m} \mu_j p_{n-j}$ generates a positive $(2n-1-m, n, d\sigma)$ quadrature formula if and only if $\sum_{j=0}^{m} \mu_j p_{n-j}$ has n simple zeros in $(-1, +1)$ and the zeros of $\sum_{j=0}^{m} \mu_j p_{n-j}$ and $\sum_{j=0}^{m} \mu_j p_{n-1-j}^{(1)}$ separate each other.*

Proof. Setting

$$t_n = \sum_{j=0}^{m} \mu_j p_{n-j} \quad \text{and} \quad t_{n-1}^{(1)} = \sum_{j=0}^{m} \mu_j p_{n-1-j}^{(1)},$$

we get for the weights c_j, using relation (2.2),

$$c_j = \int_{-1}^{+1} \frac{t_n(x)}{(x - x_j)t_n'(x_j)} \, d\sigma(x) = d_0 \frac{t_{n-1}^{(1)}(x_j)}{t_n'(x_j)} \quad \text{for } j = 1, \ldots, n.$$

Hence the conditions $c_j > 0$ for $j = 1, \ldots, n$ are equivalent to the interlacing property of the zeros of t_n and $t_{n-1}^{(1)}$. $\square$

Next, denote by $P_n(z) = z^n + \cdots$, $n \in \mathbf{N}_0$, the polynomial orthogonal on $[0, 2\pi]$ with respect to the positive measure

$$\psi(\phi) = \begin{cases} -\sigma(\cos\phi) & \text{for } \phi \in [0, \pi], \\ \sigma(\cos\phi) & \text{for } \phi \in (\pi, 2\pi], \end{cases} \tag{2.4}$$

i.e.,

$$\int_0^{2\pi} e^{-ik\phi} P_n(e^{i\phi}) \, d\psi(\phi) = 0 \quad \text{for } k = 0, \ldots, n-1.$$

Note if σ is absolutely continuous on $[-1, +1]$ and $\sigma'(x) = w(x)$, then ψ is absolutely continuous with $\psi'(\phi) = w(\cos\phi)|\sin\phi|$ for $\phi \in [0, 2\pi]$. It is well known (polynomials orthogonal on the unit circle are studied extensively in [3]) that the P_n's satisfy a recurrence relation of the type

$$P_n(z) = zP_{n-1}(z) - a_{n-1}P_{n-1}^*(z) \quad \text{for } n \in \mathbf{N}, \tag{2.5}$$

where $a_n \in (-1, +1)$ for $n \in \mathbf{N}_0$, and where $P_n^*(z) = z^n P_n(z^{-1})$ denotes the reciprocal polynomial of P_n. The reason that the parameters a_n are real and have absolute value less than one consists in the facts that ψ is odd with respect to π and that ψ has an infinite set of points of increase (see [3, p. 5]). Furthermore, let $\Omega_n(z) = z^n + \cdots$ be defined by the recurrence relation

$$\Omega_n(z) = z\Omega_{n-1}(z) + a_{n-1}\Omega_{n-1}^*(z) \quad \text{for } n \in \mathbf{N}. \tag{2.6}$$

Ω_n is called the associated polynomial of P_n. It is well known that both polynomials P_n and Ω_n, $n \geq 1$, have all their zeros in the open unit disk $|z| < 1$. The following relations hold between polynomials p_n orthogonal on $[-1, 1]$ with respect to $d\sigma$ and polynomials P_n:

$$p_n(x) = 2^{-n+1} \operatorname{Re}\{z^{-n+1} P_{2n-1}(z)\}, \tag{2.7}$$

$$p_{n-1}^{(1)}(x) = 2^{-n+1} \operatorname{Im}\{z^{-n+1} \Omega_{2n-1}(z)\}/\sin\phi, \tag{2.8}$$

where $x = \frac{1}{2}(z + z^{-1})$, $z = e^{i\phi}$, $\phi \in [0, \pi]$. The parameters (a_n) are given by [3, (31.4)]

$$a_{2n-1} = 1 - (u_n + v_n) \quad \text{and} \quad a_{2n} = \frac{v_n - u_n}{v_n + u_n}, \tag{2.9}$$

where

$$u_n = \frac{p_{n+1}(1)}{p_n(1)} \quad \text{and} \quad v_n = -\frac{p_{n+1}(-1)}{p_n(-1)}.$$

Moreover,

$$a_{2n} = 0 \quad \text{for } n \in \mathbf{N}_0, \quad \text{if } \sigma(x) = -\sigma(-x) \text{ a.e. on } [-1, 1]. \tag{2.10}$$

For example, we obtain for the Jacobi polynomials $p_n^{(\alpha,\beta)}(x) = x^n + \cdots$ which are orthogonal on $[-1, 1]$ with respect to the weight function $w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$, that the corresponding parameters $a_n^{(\alpha,\beta)}$ appearing in the recurrence relation of $P_n^{(\alpha,\beta)}(z) = z^n + \cdots$ are given by

$$a_{2n+1}^{(\alpha,\beta)} = -\frac{\alpha+\beta+1}{\alpha+\beta+2n+3}, \quad a_{2n}^{(\alpha,\beta)} = \frac{\beta-\alpha}{\alpha+\beta+n+2} \quad \text{for } n \in \mathbf{N}_0. \tag{2.11}$$

Hence we get for the ultraspherical case $p_n^{(\lambda)}(x) := p_n^{(\lambda-1/2,\lambda-1/2)}(x)$ and $w^{(\lambda)}(x) = (1-x^2)^{\lambda-1/2}$ that

$$a_{2n+1}^{(\lambda)} = -\frac{\lambda}{n+1+\lambda} \quad \text{and} \quad a_{2n}^{(\lambda)} = 0 \quad \text{for } n \in \mathbf{N}_0, \tag{2.12}$$

and in particular for the Chebyshev case, i.e., for the case where $\lambda = 0$ and $w(x) = (1-x^2)^{-1/2}$, that

$$a_n = 0 \quad \text{for } n \in \mathbf{N}_0, \qquad \Omega_n(z) = P_n(z) = z^n \quad \text{for } n \in \mathbf{N}_0. \tag{2.13}$$

Finally, we shall need

Lemma 2. *Let $n \in \mathbf{N}$ and $l \in \mathbf{Z}$ with $2|l| \le n$. Assume that the real polynomial $t_n(z) = z^n + \cdots$ has all its zeros in the open unit disk $|z| < 1$. Then the cosine-polynomial $\operatorname{Re}\{z^{-l}t_n(z)\}$, resp. the sine-polynomial $\operatorname{Im}\{z^{-l}t_n(z)\}$, $z = e^{i\phi}$, $\phi \in [0, \pi]$, has $n-l$ zeros ϕ_j in $(0, \pi)$, resp. $n-l-1$ zeros ψ_j in $(0, \pi)$, and their zeros separate each other, i.e, $0 < \phi_1 < \psi_1 < \phi_2 < \cdots < \psi_{n-l-1} < \phi_{n-l} < \pi$.*

Proof. Since $\operatorname{Re}\{z^{-l}t_n(z)\}$ (respectively $\operatorname{Im}\{z^{-l}t_n(z)\}$) is zero at $z = e^{i\phi}$, $\phi \in (0, 2\pi)$, if and only if

$$z^{-2l}\frac{t_n(z)}{\overline{t_n(z)}} = -1 \quad (\text{respectively } + 1),$$

which is equivalent to

$$\arg z^{n-2l} + \arg\frac{t_n(z)}{t_n^*(z)} = (2k-1)\pi \quad (\text{respectively } 2k\pi),$$

$k \in \mathbf{N}_0$, we get, taking into consideration the fact that $\arg t_n(e^{i\phi})/t_n^*(e^{i\phi})$ increases from 0 to $2n\pi$ if ϕ varies from 0 to 2π, that both $\operatorname{Re}\{z^{-l}t_n(z)\}$ and $\operatorname{Im}\{z^{-l}t_n(z)\}$ have $2(n-l)$ zeros in $[0, 2\pi)$ and that their zeros separate each other. Observing that $\operatorname{Im}\{z^{-l}t_n(z)\}$ has a zero at $\phi = 0$ and $\phi = \pi$, the assertion follows by the symmetry of trigonometric polynomials. □

3. Main results

First, let us introduce the following polynomials, which play a crucial role in this paper.

Definition. For given $n \in \mathbf{N}$ let the polynomials $Q_{\nu,2n-1}(z) = z^{\nu} + \cdots$, $\nu \in \{0, \ldots, 2n-1\}$, be defined by the recurrence relation

$$Q_{\nu,2n-1}(z) = zQ_{\nu-1,2n-1}(z) - a_{2n-1-\nu}Q^{*}_{\nu-1,2n-1} \quad \text{for } \nu = 1, \ldots, 2n-1, \tag{3.1}$$

where $Q_{0,2n-1} = 1$ and the $a_{2n-1-\nu}$'s are the parameters appearing in the recurrence relation (2.5) of the P_n's.

The polynomials $Q_{\nu,2n-1}$ have the following important properties.

Lemma 3. *Let $n \in \mathbf{N}$. The following propositions hold:*

(a) $\prod_{\kappa=0}^{\nu-1}(1 - |a_{2n-2-\kappa}|) \leq |Q^{*}_{\nu,2n-1}(z)| \leq \prod_{\kappa=0}^{\nu-1}(1 + |a_{2n-2-\kappa}|)$ *for* $|z| \leq 1$, *where* $\nu \in \{0, \ldots, 2n-1\}$. *Moreover,* $Q_{\nu,2n-1}$ *has all zeros in* $|z| < 1$.

(b) *Let* $\nu \in \{0, \ldots, n-1\}$; *then* ($z = e^{i\phi}$, $x = \cos\phi$, $\phi \in [0, \pi]$)

$$p_n(x) = 2^{-n+1}\,\mathrm{Re}\{z^{-n+1}Q_{2\nu,2n-1}(z)P_{2(n-\nu)-1}(z)\}$$

and

$$p^{(1)}_{n-1}(x) = 2^{-n+1}\,\mathrm{Im}\{z^{-n+1}Q_{2\nu,2n-1}(z)\Omega_{2(n-\nu)-1}(z)\}/\sin\phi.$$

Proof. (a) follows immediately from (3.1) and [3, (26.6)].

(b) We first note that the recurrence relations (2.5), resp. (3.1), imply (see [3, (3.6)]) that

$$P^{*}_{n}(z) = P^{*}_{n-1}(z) - a_{n-1}zP_{n-1}(z) \quad \text{for } n \in \mathbf{N}, \tag{2.5$'$}$$

and

$$Q^{*}_{\nu,2n-1}(z) = Q^{*}_{\nu-1,2n-1}(z) - a_{2n-1-\nu}zQ_{\nu-1,2n-1}(z) \quad \text{for } \nu = 1, \ldots, 2n-1. \tag{3.1$'$}$$

With the help of all these recurrence relations it follows by induction arguments that

$$zP_{2n-1}(z) + P^{*}_{2n-1}(z) = zQ_{\nu,2n-1}(z)P_{2n-1-\nu}(z) + Q^{*}_{\nu,2n-1}(z)P^{*}_{2n-1-\nu}(z),$$

which, in view of (2.7) and taking into consideration the fact that for $z = e^{i\phi}$

$$2\,\mathrm{Re}\{z^{-n+1}P_{2n-1}(z)\} = z^{-n}(zP_{2n-1}(z) + P^{*}_{2n-1}(z)),$$

gives the first relation.

Analogously as above, one demonstrates that

$$z\Omega_{2n-1}(z) - \Omega^{*}_{2n-1}(z) = zQ_{\nu,2n-1}(z)\Omega_{2n-1-\nu}(z) - Q^{*}_{\nu,2n-1}(z)\Omega^{*}_{2n-1-\nu}(z),$$

which in conjunction with (2.8) gives the second relation. □

The main result is now the following

Theorem 1. *Let* $n, m \in \mathbf{N}_0$, $m \le n$, $\mu_0, \dots, \mu_m \in \mathbf{R}$ *and* $\mu_0 \neq 0$. *Then* $\sum_{j=0}^m \mu_j p_{n-j}$ *generates a positive* $(2n-1-m, n, d\sigma)$ *quadrature formula if* $\sum_{j=0}^m \tilde{\mu}_j z^j Q_{2m-2j, 2(n-j)-1}(z)$, *where* $\tilde{\mu}_j = 2^j \mu_j$, *has all its zeros in the open unit disk* $|z| < 1$.

Proof. Putting

$$t_n(x) = \sum_{j=0}^m \mu_j p_{n-j}(x) \quad \text{and} \quad t_{n-1}^{(1)}(x) = \sum_{j=0}^m \mu_j p_{n-1-j}^{(1)}(x),$$

we get with the help of Lemma 3(b) that $(z = e^{i\phi},\ x = \cos\phi,\ \phi \in [0, \pi])$

$$t_n(x) = 2^{-n+1} \operatorname{Re}\{z^{-m} q_{2m}(z) z^{-(n-m)+1} P_{2(n-m)-1}(z)\} \tag{3.2}$$

and

$$t_{n-1}^{(1)}(x) = 2^{-n+1} \operatorname{Im}\{z^{-m} q_{2m}(z) z^{-(n-m)+1} \Omega_{2(n-m)-1}(z)\} / \sin\phi,$$

where

$$q_{2m}(z) = \sum_{j=0}^m \tilde{\mu}_j z^j Q_{2m-2j, 2(n-j)-1}(z). \tag{3.3}$$

Assume now that q_{2m} has all its zeros in $|z| < 1$. Since the same is true for $P_{2(n-m)-1}$, it follows from Lemma 2 that t_n has n simple zeros in $(-1, +1)$. Thus, by Lemma 1, it remains to demonstrate that the zeros of t_n and $t_{n-1}^{(1)}$ separate each other.

Using the relation

$$\operatorname{Re} a \operatorname{Re} b + \operatorname{Im} a \operatorname{Im} b = \operatorname{Re}\{a\bar{b}\},$$

where $a, b \in \mathbf{C}$, we get for $z = e^{i\phi}$

$$\begin{aligned}
&\operatorname{Re}\{z^{-(n-1)} q_{2m}(z) P_{2(n-m)-1}(z)\} \operatorname{Re}\{z^{-(n-1)} q_{2m}(z) \Omega_{2(n-m)-1}(z)\} \\
&\quad + \operatorname{Im}\{z^{-(n-1)} q_{2m}(z) P_{2(n-m)-1}(z)\} \operatorname{Im}\{z^{-(n-1)} q_{2m}(z) \Omega_{2(n-m)-1}(z)\} \\
&= |q_{2m}(z)|^2 \operatorname{Re}\{P_{2(n-m)-1}(z) \overline{\Omega_{2(n-m)-1}(z)}\} \\
&= c|q_{2m}(z)|^2, \qquad c \in \mathbf{R}^+,
\end{aligned} \tag{3.4}$$

where the last equality follows from the known relation [3, (5.6)]

$$\begin{aligned}
&P_{2(n-m)-1}(z) \Omega_{2(n-m)-1}^*(z) + \Omega_{2(n-m)-1}(z) P_{2(n-m)-1}^*(z) \\
&\quad = \tilde{c} z^{2n-2m-1}, \quad \text{where } \tilde{c} \in \mathbf{R}^+.
\end{aligned}$$

Considering relation (3.4) at the zeros x_j, $-1 < x_1 < x_2 < \cdots < x_n < 1$, of $t_n(x)$ and taking into account that by Lemma 2 the zeros of $t_n(x)$ and $r_{n-1}(x) := \operatorname{Im}\{z^{-(n-1)} q_{2m}(z) P_{2(n-m)-1}(z)\} / \sin\phi$, $x = \frac{1}{2}(z + 1/z)$, $z = e^{i\phi}$,

$\phi \in [0, \pi]$, separate each other, we obtain

$$(-1)^{n-j} t_{n-1}^{(1)}(x_j) > 0 \quad \text{for } j = 1, \ldots, n,$$

which proves the interlacing property of t_n and $t_{n-1}^{(1)}$ and thus the theorem. □

Remark 1. From the general characterization of positive quadrature formulas given by the author in [7, Theorem 2] it follows with the help of relation (3.2) that the sufficient condition of Theorem 1 is also necessary if $2m \le n$.

From Theorem 1 we obtain, using some ideas of Cauchy and Kojima on the location of the zeros of polynomials (see [4, §30, in particular Exercise 6]), the following sufficient conditions which are easy to verify.

Corollary 1. *Let* $n, m \in \mathbf{N}_0$, $m \le n$, $\mu_0, \ldots, \mu_m \in \mathbf{R}$ *and* $\mu_0 \ne 0$. *Put* $A_0 = |\mu_0|$,

$$A_j = 2^j |\mu_j| \frac{\prod_{\kappa=0}^{2m-1-2j}(1 + |a_{2(n-j-1)-\kappa}|)}{\prod_{\kappa=0}^{2m-1}(1 - |a_{2(n-1)-\kappa}|)} \quad \textit{for } j = 1, \ldots, m, \tag{3.5}$$

and let $j_\nu \in \{0, 1, \ldots, m\}$, $j_0 := 0 < j_1 < \cdots < j_{m^*}$ *be those indices for which* $A_{j_\nu} \ne 0$ *for* $\nu = 1, \ldots, m^*$ *and* $A_j = 0$ *for* $j \in \{1, \ldots, m\} \backslash \{j_0, j_1, \ldots, j_{m^*}\}$. *Then each of the following two conditions is sufficient that* $\sum_{j=0}^m \mu_j p_{n-j}$ *generates a positive* $(2n-1-m, n, d\sigma)$ *quadrature formula*:

(1) $\sum_{\nu=1}^{m^*} A_{j_\nu} < A_0$.

(2) $A_{j_\nu} \ge 2A_{j_{\nu+1}}$ *for* $\nu = 0, \ldots, m^*-2$ *and* $A_{j_{m^*-1}} > A_{j_{m^*}}$.

Proof. First let us note that condition (2) implies condition (1). In fact, applying successively the inequalities given in (2), we obtain

$$A_{j_0} \ge A_{j_1} + A_{j_1} \ge A_{j_1} + A_{j_2} + A_{j_2} \ge \cdots > \sum_{\nu=1}^{m^*-1} A_{j_\nu} + A_{j_{m^*}},$$

which is condition (1).

Next we show that condition (1) implies that

$$q_{2m}^*(z) := \sum_{j=0}^m \tilde{\mu}_j z^j Q_{2m-2j,2(n-j)-1}^*(z), \qquad \tilde{\mu}_j = 2^j \mu_j,$$

has all zeros in $|z| > 1$, which is equivalent to the fact that

$$\sum_{j=0}^m \tilde{\mu}_j z^j Q_{2m-2j,2(n-j)-1}(z)$$

has all zeros in $|z| < 1$ and proves the corollary. Assume, to the contrary, that q_{2m}^* has a zero ζ in $|z| \le 1$. Then it follows, using from Lemma 3 the fact that $Q_{2m,2n-1}^*$ has no zero in $|z| \le 1$, that

$$|\mu_0| = \left| \sum_{j=1}^m \tilde{\mu}_j \zeta^j \frac{Q_{2m-2j,2(n-j)-1}^*(\zeta)}{Q_{2m,2n-1}^*(\zeta)} \right| \le \sum_{j=1}^m A_j |\zeta|^j \le \sum_{j=1}^m A_j, \tag{3.6}$$

where the first inequality follows with the help of Lemma 3, which is a contradiction to (1). □

Let us give an illustrative

Example. Let $n, m \in \mathbf{N}_0$, $n > m$, and suppose that the parameters a_ν satisfy

$$0 < 1/\gamma \leq 1 - |a_\nu| \quad \text{for } \nu = 2(n-m)-1, \dots, 2n-2.$$

Then we get by Corollary 1 that

$$p_n - \mu_m p_{n-m}, \qquad |\mu_m| < (2\gamma^2)^{-m},$$

generates a positive $(2n-1-m, n, d\sigma)$ quadrature formula, where because of (2.10) the condition on $|\mu_m|$ can be replaced by $|\mu_m| < (2\gamma)^{-m}$ if σ is odd. In particular, we obtain for the Jacobi weight by a rough estimate of the parameters $a_n^{(\alpha,\beta)}$ from (2.11) that

$$p_n^{(\alpha,\beta)} - \mu_m p_{n-m}^{(\alpha,\beta)}, \qquad |\mu_m| < 2^{-3m},$$

generates a positive $(2n-1-m, n, (1-x)^\alpha(1+x)^\beta)$ quadrature formula for each $n \geq m + \max\{2, \alpha+\beta+1+2|\beta-\alpha|\}$. In the ultraspherical case $\alpha = \beta = \lambda - 1/2$, $\lambda \in (-1/2, \infty)$, the conditions on $|\mu_m|$, resp. n, can be replaced by $|\mu_m| < 2^{-2m}$ and $n \geq m + \max\{\lambda, -3\lambda\}$.

Let us note in this connection that the conditions of Corollary 1 are in general too rough to get the known results (see [1]) on the positivity of $(n-1, n, (1-x)^\alpha(1+x)^\beta)$ quadrature formulas generated by $p_n^{(a,b)}$, $a, b > -1$. But this is not surprising because the proof of such results requires very special properties of Jacobi polynomials.

In order to weaken the sufficient conditions of Corollary 1, a better estimate for $\max_{0\leq\phi\leq 2\pi} |Q^*_{2m-2j,2(n-j)-1}(e^{i\phi})/Q^*_{2m,2n-1}(e^{i\phi})|$ than that one used in (3.6) would be needed.

In the following, let T_n, resp. U_n, denote the Chebyshev polynomial of the first, resp. second, kind of degree n and $\widehat{T}_n(x) = 2^{-n+1}T_n(x) = x^n + \cdots$, resp. $\widehat{U}_n(x) = 2^{-n}U_n(x) = x^n + \cdots$. For the case of the Chebyshev distribution $d\sigma(x) = (1-x^2)^{-1/2}dx$ we get in view of (2.13) particularly simple conditions, which hold also for the distribution $d\sigma(x) = (1-x^2)^{1/2}dx$.

Corollary 2. *Let* $n, m \in \mathbf{N}_0$, $m \leq n$, $\mu_0, \dots, \mu_m \in \mathbf{R}$, $\mu_0 \neq 0$, *and put* $\tilde{\mu}_j = 2^j \mu_j$ *for* $j = 0, \dots, m$. *Then the following propositions hold*:

(a) $\sum_{j=0}^m \mu_j \widehat{T}_{n-j}$ *generates a positive* $(2n-1-m, n, (1-x^2)^{-1/2})$ *quadrature formula if* $\sum_{j=0}^m \tilde{\mu}_j z^{m-j}$ *has all its zeros in the open unit disk* $|z| < 1$. *In particular* (*besides conditions* (1) *and* (2) *of Corollary* 1), *the condition*

(3) $\tilde{\mu}_0 > \tilde{\mu}_1 > \cdots > \tilde{\mu}_m > 0$

is sufficient that $\sum_{j=0}^m \mu_j \widehat{T}_{n-j}$ *generates a positive* $(2n-1-m, n, (1-x^2)^{-1/2})$ *quadrature formula.*

(b) *The sufficient conditions given in* (a) (*including conditions* (1) *and* (2) *of Corollary* 1 *with* $a_n = 0$ *for* $n \in \mathbf{N}_0$) *are also sufficient for* $\sum_{j=0}^{m} \mu_j \widehat{U}_{n-j}$ *to generate a positive* $(2n-1-m, n, (1-x^2)^{1/2})$ *quadrature formula.*

Proof. (a) The first statement follows immediately from Theorem 1. Since by the Kakeya-Eneström Theorem (see, e.g., [4]) condition (3) implies that $\sum_{j=0}^{m} \tilde{\mu}_j z^{m-j}$ has all zeros in $|z| < 1$, part (a) is proved.

(b) We shall demonstrate, independently from Theorem 1, that $\sum_{j=0}^{m} \mu_j \widehat{U}_{n-j}$ generates a positive $(2n-1-m, n, (1-x^2)^{1/2})$ quadrature formula if $\sum_{j=0}^{m} \tilde{\mu}_j \cdot z^{m-j}$ has all zeros in $|z| < 1$, which also implies all other statements of (b). Setting

$$r_n(z) = z^{n-m} \sum_{j=0}^{m} \tilde{\mu}_j z^j$$

and

$$2^n t_n(x) = \sum_{j=0}^{m} \tilde{\mu}_j U_{n-j}(x) = \operatorname{Im}\{z r_n(z)\} / \sin\phi,$$

we obtain, since, as is well known, the associated polynomial of $\widehat{U}_k$ is $\widehat{U}_{k-1}$, $k \in \mathbf{N}_0$, that the associated polynomial $t_{n-1}^{(1)}$ of t_n with respect to $(1-x^2)^{1/2}$ is of the form

$$2^{n-1} t_{n-1}^{(1)}(x) = \sum_{j=0}^{m} \tilde{\mu}_j U_{n-1-j}(x) = \operatorname{Im}\{r_n(z)\} / \sin\phi.$$

Observing that

$$\operatorname{Re}\{r_n(z)\} \frac{\operatorname{Im}\{z r_n(z)\}}{\sin\phi} - \operatorname{Re}\{z r_n(z)\} \frac{\operatorname{Im}\{r_n(z)\}}{\sin\phi} = |r_n(z)|^2, \tag{3.7}$$

we deduce with the help of Lemma 2, by considering relation (3.7) at the n zeros of t_n, that t_n and $t_{n-1}^{(1)}$ have interlacing zeros. In view of Lemma 1 the assertion is proved. □

The sufficiency of condition (3) for the Chebyshev weight $(1-x^2)^{-1/2}$ is due to C. A. Micchelli [5], who derived this result in order to demonstrate that the ultraspherical polynomials $p_n^{(\lambda)}$, $0 \le \lambda < 1$, generate a positive $(n-1, n, (1-x^2)^{-1/2})$ quadrature formula. Let us mention in this connection (for a different approach see [5]) that for $-1/2 < \lambda \le 0$ the positivity can be demonstrated with the help of condition (1), using the simple fact that $T_k(1) = 1$ for $k \in \mathbf{N}_0$. Proceeding similarly as in the proof of Corollary 2(b), it could also be demonstrated that Corollary 2(b) holds for the more general weight $(1-x)^\alpha (1+x)^\beta$, $\alpha, \beta \in \{-1/2, 1/2\}$, a result which has been given by the author in [8, Corollary 2], using different methods.

Using the fact that the sufficient condition of Theorem 1 is also necessary if $2m \leq n$ (see Remark 1), we get

Corollary 3. *Let* $n, m \in \mathbf{N}_0$, $2m \leq n$, $\mu_0, \ldots, \mu_m \in \mathbf{R}$ *and* $\mu_0, \mu_m \in \mathbf{R}\backslash\{0\}$. *For* $k \in \{0, \ldots, m\}$ *put* $A_k^{(k)} = 2^k|\mu_k|$ *and*

$$A_j^{(k)} = 2^j|\mu_j| \frac{\prod_{\kappa=0}^{2m-1-2j}(1 + |a_{2(n-j-1)-\kappa}|)}{\prod_{\kappa=0}^{2m-1-2k}(1 - |a_{2(n-k-1)-\kappa}|)} \quad \text{for } j = 0, \ldots, m, \ j \neq k.$$

If there is a $k \in \{1, \ldots, m\}$ *such that* $A_k^{(k)} > \sum_{j=0, j\neq k}^{m} A_j^{(k)}$, *then* $\sum_{j=0}^{m} \mu_j \cdot p_{n-j}$ *does not generate a positive* $(2n-1-m, n, d\sigma)$ *quadrature formula.*

Proof. In view of Remark 1 it is sufficient to demonstrate that

$$q_{2m}^*(z) := \sum_{j=0}^{m} \tilde{\mu}_j z^j Q_{2m-2j, 2(n-j)-1}^*(z), \qquad \tilde{\mu}_j = 2^j \mu_j,$$

has at least one zero in $|z| < 1$. With the help of Lemma 3 we get on the circumference $|z| = 1$

$$\begin{aligned} |\tilde{\mu}_k z^k Q_{2m-2k, 2(n-k)-1}^*(z)| &\geq |\tilde{\mu}_k| \prod_{\kappa=0}^{2m-1-2k} (1 - |a_{2(n-k-1)-\kappa}|) \\ &> \sum_{\substack{j=0 \\ j\neq k}}^{m} |\tilde{\mu}_j| \prod_{\kappa=0}^{2m-1-2j} (1 + |a_{2(n-j-1)-\kappa}|) \\ &\geq \left| \sum_{\substack{j=0 \\ j\neq k}}^{m} \tilde{\mu}_j z^j Q_{2m-2j, 2(n-j)-1}^*(z) \right|. \end{aligned}$$

Using the fact that $Q_{2m-2k, 2(n-k)-1}^*$ has no zero in $|z| < 1$, this implies by Rouché's Theorem that q_{2m}^* has k zeros in $|z| < 1$, which proves the assertion. □

If one is interested only in such linear combinations of orthogonal polynomials whose zeros are simple and are in $(-1, +1)$, conditions (1) and (2) can be weakened in the following way.

Theorem 2. *Let* $n, m \in \mathbf{N}_0$, $m \leq n$, $\mu_0, \ldots, \mu_m \in \mathbf{R}$ *and* $\mu_0 \neq 0$. *Put* $|B_0| = \mu_0$,

$$B_j = 2^j|\mu_j| / \prod_{\kappa=2(n-j)-1}^{2n-2} (1 - |a_\kappa|) \quad \text{for } j = 1, \ldots, m,$$

and let $j_\nu \in \{0, 1, \ldots, m\}$, $j_0 := 0 < j_1 < \cdots < j_{m^*}$ *be those indices for which* $B_{j_\nu} \neq 0$ *for* $\nu = 1, \ldots, m^*$ *and* $B_j = 0$ *for* $j \in \{1, \ldots, m\}\backslash\{j_0, j_1, \ldots, j_{m^*}\}$. *Then each of the following two conditions is sufficient that* $\sum_{j=0}^{m} \mu_j p_{n-j}$ *has* n

simple zeros in $(-1, 1)$:

(1′) $\sum_{\nu=1}^{m^*} B_{j_\nu} < B_0$.

(2′) $B_{j_\nu} \geq 2B_{j_{\nu+1}}$ *for* $\nu = 0, \ldots, m^* - 1$ *and* $B_{j_{m^*-1}} > B_{j_{m^*}}$.

Proof. Since by (2.7)

$$\sum_{j=0}^{m} \mu_j p_{n-j}(x) = 2^{-n+1} \operatorname{Re}\left\{ \sum_{j=0}^{m} \tilde{\mu}_j z^j P_{2(n-j)-1}(z) \right\},$$

where $\tilde{\mu}_j = 2^j \mu_j$, we deduce with the help of Lemma 2 that $\sum_{j=0}^{m} \mu_j p_{n-j}(x)$ has n simple zeros in $(-1, +1)$ if $\sum_{j=0}^{m} \mu_j z^j P^*_{2(n-j)-1}$ has all zeros in $|z| > 1$. Observing that by relation (26.5) of [3]

$$\max_{|z|\leq 1} \left| \frac{P^*_{2(n-j)-1}(z)}{P^*_{2n-1}(z)} \right| \leq \frac{1}{\prod_{\kappa=2(n-j)-1}^{2n-2} (1 - |a_\kappa|)} \quad \text{for } j = 1, \ldots, m,$$

the assertion can be proved in the same way as Corollary 1. □

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