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VISCOUS SPLITTING FOR THE UNBOUNDED PROBLEM OF THE NAVIER-STOKES EQUATIONS

LUNG-AN YING

ABSTRACT. The viscous splitting for the exterior initial-boundary value problems of the Navier-Stokes equations is considered. It is proved that the approximate solutions are uniformly bounded in the space $L^\infty(0, T; H^{s+1}(\Omega))$, $s < \frac{3}{2}$, and converge with a rate of $O(k)$ in the space $L^\infty(0, T; H^1(\Omega))$, where k is the length of the time steps.

1. INTRODUCTION

Let Ω be a domain in the space $\mathbb{R}^2$. An initial-boundary value problem of the Navier-Stokes equation is given as

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \frac{1}{\rho}\nabla p = \nu \Delta u + f, \qquad x \in \Omega, \ t > 0, \tag{1.1}$$

$$\nabla \cdot u = 0, \qquad x \in \Omega, \ t > 0, \tag{1.2}$$

$$u|_{x \in \partial\Omega} = 0, \tag{1.3}$$

$$u|_{t=0} = u_0(x). \tag{1.4}$$

If $\Omega = \mathbb{R}^2$, then the boundary condition (1.3) disappears and the problem reduces to a pure initial value problem.

Beale and Majda [4] proved the convergence of a viscous splitting scheme for the initial value problem, where equation (1.1) was split in each time step into an Euler equation and a linear Stokes equation. This scheme was related to the vortex method [6], a numerical approach for high Reynold's number flow. Therefore, it is interesting to consider not only pure initial value problems, but also initial-boundary value problems. It is known that there is a boundary layer near the boundary, and that vortices are created and turbulence may develop. From the point of view of numerical analysis, the boundary condition for the Euler equation is different from that of the Navier-Stokes equation; the changing of the boundary condition in each time step creates singularities of the approximate solutions.

Alessandrini, Douglis, and Fabes considered the viscous splitting of the initial-boundary value problem in bounded domains [3], where the solutions of

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the Euler equation were replaced by polynomials. Convergence was proved, but it is not known whether this scheme is numerically realizable. Benfatto and Pulvirenti proved the convergence of a scheme for the initial-boundary value problem in the half plane [5]. A distribution vortex sheet, whose support is just the boundary, was inserted as in the vortex method, and a Neumann condition for the vorticity was introduced to replace the velocity boundary condition. The combination of those two steps generated an approximate no-slip condition at the boundary.

The author of this paper considered this problem in bounded domains [13]–[17]; a correction step was applied to maintain the no-slip condition too, but this operator was bounded in H^{s+1}, $s \geq 0$, the velocity boundary condition for the diffusion step was exact, and a nonhomogeneous term was added to the Stokes equation to neutralize the error of the above correction step. Convergence was proved. Numerical results have been obtained which will appear in a separate paper.

The purpose of this paper is to study this problem for unbounded domains. For simplicity we assume that the boundary $\partial\Omega$ of Ω is sufficiently smooth, simply closed, and Ω is its exterior. We also assume that flows tend to zero at infinity. The problem of the physically interesting case of flows having uniform velocity at infinity is still open. A simplification of the proof would suffice for the bounded case.

We now briefly summarize our main results. Denote by $x = (x_1, x_2)$ or $y = (y_1, y_2)$ a point in $\mathbb{R}^2$. The usual notations $H^s(\Omega)$ and $W^{m,p}(\Omega)$ for Sobolev spaces, and $\|\cdot\|_s$ and $\|\cdot\|_{m,p}$ for norms, are used throughout this paper. For the problems (1.1)–(1.4), we assume that $\nabla \cdot u_0 = 0$, $u_0 \in (H_0^1(\Omega))^2 \cap (H^3(\Omega))^2$, $f \in L^\infty(0, T; (H^3(\Omega))^2) \cap W^{1,\infty}(0, T; (H^1(\Omega))^2)$, and the solution $u \in L^\infty(0, T; (H^4(\Omega))^2) \cap W^{1,\infty}(0, T; (H^{5/2}(\Omega))^2)$, where T is a positive constant.

We construct a projection operator

$$\Theta\colon \{u \in (H^1(\Omega))^2;\ \nabla \cdot u = 0,\ (u \cdot n, 1)_{\partial\Omega} = 0\} \to \{u \in (H_0^1(\Omega))^2;\ \nabla \cdot u = 0\},$$

such that

$$\|\Theta u\|_{s+1} \leq C\|u\|_{s+1} \quad \forall s \geq 0, \tag{1.5}$$

where n is the unit outward normal vector, $(\cdot, \cdot)_{\partial\Omega}$ is the inner product of $L^2(\partial\Omega)$, and C is a constant depending on s. We will give an example of Θ in §2.

The following scheme is considered: We divide the interval $[0, T]$ into equal subintervals with length k. Then we construct $\tilde{u}_k(t)$, $\tilde{p}_k(t)$, $u_k(t)$, $p_k(t)$ on each interval $[ik, (i+1)k)$, $i = 0, 1, \dots$, according to the following procedure.

In the first step, we solve the following problem on the interval $[ik, (i+1)k)$:

$$\frac{\partial \tilde{u}_k}{\partial t} + (\tilde{u}_k \cdot \nabla)\tilde{u}_k + \frac{1}{\rho}\nabla \tilde{p}_k = f, \tag{1.6}$$

$$\nabla \cdot \tilde{u}_k = 0, \tag{1.7}$$

$$\tilde{u}_k \cdot n|_{x\in\partial\Omega} = 0, \tag{1.8}$$

$$\tilde{u}_k(ik) = u_k(ik-0), \tag{1.9}$$

where $u_k(-0) = u_0$.

In the second step—the projection—we construct $\Theta\tilde{u}_k((i+1)k-0)$.

In the third step, we solve the following problem on the interval $[ik, (i+1)k)$:

$$\frac{\partial u_k}{\partial t} + \frac{1}{\rho}\nabla p_k = \nu\Delta u_k + \frac{1}{k}(I-\Theta)\tilde{u}_k((i+1)k-0), \tag{1.10}$$

$$\nabla \cdot u_k = 0, \tag{1.11}$$

$$u_k|_{x\in\partial\Omega} = 0, \tag{1.12}$$

$$u_k(ik) = \Theta\tilde{u}_k((i+1)k-0), \tag{1.13}$$

where I is the identity operator. In these formulas the spacial variable x is suppressed for simplicity.

Our main result is the following:

Theorem. *If u is the solution of problem* (1.1)–(1.4), *$\tilde{u}_k$, u_k the solutions of problems* (1.6)–(1.13), *and if $0 \le s < \frac{3}{2}$, then there is a constant $k_0 > 0$ such that*

$$\sup_{0\le t\le T}(\|u_k(t)\|_{s+1}, \|\tilde{u}_k(t)\|_{s+1}) \le M, \tag{1.14}$$

$$\sup_{0\le t\le T}(\|u(t)-u_k(t)\|_1, \|u(t)-\tilde{u}_k(t)\|_1) \le M'k \tag{1.15}$$

for $0 < k \le k_0$, where the constants k_0, M, M' depend only on the domain Ω, the constants ν, s, T, the operator Θ, the functions f, u_0, and the solution u of (1.1)–(1.4).

The existence and uniqueness of the solution u_k is known [9, Chapter 4, Theorem 1 and §2], and using an argument similar to [12], we can get the existence and uniqueness of $\tilde{u}_k$; the regularity of u_k, $\tilde{u}_k$ is also obtained. Although the existence in [12] is merely local, we will show that the step length is independent of i.

2. Preliminaries

In this paper we always denote by C a generic constant which depends only on the domain Ω, the operator Θ, and the constants ν, s, T; by C_0 a generic constant which depends only on the domain Ω, the operator Θ, the constants ν, s, T, the known functions f, u_0, and the solution u of (1.1)–(1.4); by $C_1, C_2, \ldots, M_0, M_1, \ldots$ some other constants which are determined according to special requirements.

Let $E^0(\Omega)$ be a subset of $L^2(\Omega)$ such that $\omega \in E^0(\Omega)$ if and only if $\omega \in L^2(\Omega)$ and there is a $u \in (L^2(\Omega))^2$ such that $\omega = -\nabla\Lambda u$, where $\nabla\Lambda = (\partial_2, -\partial_1)$, $\partial_i = \partial/\partial x_i$. We define a norm

$$[\varphi]_1 = \left(\int_\Omega |\nabla\varphi|^2\, dx\right)^{1/2}$$

in $C_0^\infty(\overline{\Omega})$. Let $E^1(\Omega)$ be the closure of $C_0^\infty(\overline{\Omega})$ with respect to the norm $[\cdot]_1$, and let $E_0^1(\Omega)$ correspond to $C_0^\infty(\Omega)$. Letting $\omega \in E^0(\Omega)$, we consider the boundary value problem

$$\begin{cases} -\Delta\varphi = \omega, \\ \varphi|_{\partial\Omega} = 0. \end{cases} \tag{2.1}$$

The weak statement of (2.1) is: find $\varphi \in E_0^1(\Omega)$ such that

$$(\nabla\Lambda\varphi, \nabla\Lambda\psi) = (\omega, \psi) = (u, \nabla\Lambda\psi) \quad \forall\psi \in E_0^1(\Omega), \tag{2.2}$$

where $\omega = -\nabla\Lambda u$. It is easy to see that

$$(\nabla\Lambda\varphi, \nabla\Lambda\varphi) = [\varphi]_1^2.$$

By the Lax-Milgram theorem, (2.2) possesses a unique solution. Setting $\psi = \varphi$ in (2.2), we get $[\varphi]_1^2 \le \|u\|_0[\varphi]_1$, hence

$$[\varphi]_1 \le \|u\|_0. \tag{2.3}$$

If $\omega \in E^0(\Omega)$ and φ is the solution of (2.2), we define a norm

$$[\omega]_0 = (\|\omega\|_0^2 + [\varphi]_1^2)^{1/2}$$

in $E^0(\Omega)$. It is easy to see that $E^0(\Omega)$ is a Hilbert space.

Let D^m be a differential operator of mth order, $m \ge 0$, $D^m = \partial^m/\partial x_1^i \partial x_2^j$, $i + j = m$. We assume that $\omega \in E^0(\Omega) \cap H^m(\Omega)$; then by the regularity of the solutions of elliptic equations [2], we have for the solution φ of (2.2) that $\varphi \in H_{\mathrm{loc}}^{m+2}(\Omega)$. From (2.2) we get

$$(\nabla\Lambda\varphi, \nabla\Lambda D^m\partial_i\psi) = (\omega, D^m\partial_i\psi) \quad \forall\psi \in C_0^\infty(\Omega).$$

Integrating by parts, we obtain

$$(\nabla\Lambda D^m\partial_i\varphi, \nabla\Lambda\psi) = -(D^m\omega, \partial_i\psi). \tag{2.4}$$

We first assume $\omega \in C_0^\infty(\overline{\Omega})$; then φ is the solution of the Laplace equation near infinity. From the expansion of φ at infinity it is easy to see that $\nabla\Lambda D^m\partial_i\varphi \in L^2(\Omega)$, hence (2.4) also holds for $\psi \in E_0^1(\Omega)$.

Let trace $b = D^m\partial_i\varphi|_{\partial\Omega}$; then [1, Theorem 7.53]

$$\|b\|_{1/2,\partial\Omega} \le C\|\varphi\|_{m+2,\Omega'},$$

where Ω' is a neighborhood of $\partial\Omega$. By the Poincaré inequality and the local estimate of the solution φ,

$$\|\varphi\|_{m+2,\Omega'} \le C(\|\omega\|_m + [\varphi]_1).$$

Let $D^m \partial_i \varphi = \varphi_1 + \varphi_2$, where $\varphi_1 \in E_0^1(\Omega)$ is the solution of

$$(\nabla\Lambda\varphi_1, \nabla\Lambda\psi) = -(D^m\omega, \partial_i\psi) \quad \forall\psi \in E_0^1(\Omega),$$

and $\varphi_2 \in E^1(\Omega)$ is the solution of

$$(\nabla\Lambda\varphi_2, \nabla\Lambda\psi) = 0 \quad \forall\psi \in E_0^1(\Omega),$$
$$\varphi_2|_{\partial\Omega} = b.$$

φ_2 is a bounded harmonic function, hence $\|\nabla\varphi_2\|_0 \le C\|b\|_{1/2,\partial\Omega}$. By (2.3), $[\varphi_1]_1 \le \|D^m\omega\|_0$. We have

$$[D^m\partial_i\varphi]_1 \le C(\|\omega\|_m + [\varphi]_1) \le C(\|\omega\|_m + [\omega]_0). \tag{2.5}$$

$C_0^\infty(\overline{\Omega})$ is dense in $E^0(\Omega)$; therefore (2.5) holds for all $\omega \in E^0(\Omega) \cap H^m(\Omega)$. We define $[\varphi]_m = \|\nabla\varphi\|_{m-1}$ for $m \ge 1$; then

$$[\nabla\varphi]_{m+1} \le C(\|\omega\|_m + [\omega]_0).$$

By the interpolation theorem [10, Chapter 1, Theorem 5.1], we have

$$[\nabla\varphi]_{s+1} \le C(\|\omega\|_s + [\omega]_0) \quad \forall s \ge 0. \tag{2.6}$$

We denote by $E_0^m(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm $[\cdot]_m$.

Now we give an example of an operator Θ which satisfies inequality (1.5). Construct $\chi \in C_0^\infty(\overline{\Omega})$ such that $\chi \equiv 1$ near the boundary $\partial\Omega$. Let Ω' be a bounded domain whose boundary consists of simply closed curves Γ and $\partial\Omega$, where Γ is outside of $\partial\Omega$, sufficiently smooth, and $\operatorname{supp}\chi \subset \overline{\Omega}'$. Let φ be a stream function of u in Ω'. We consider the following biharmonic problem:

$$\Delta^2\Phi = 0,$$
$$\Phi|_{\partial\Omega} = -\varphi|_{\partial\Omega}, \qquad \left.\frac{\partial\Phi}{\partial n}\right|_{\partial\Omega} = -\left.\frac{\partial\varphi}{\partial n}\right|_{\partial\Omega},$$
$$\Phi|_\Gamma = 0, \qquad \Delta\Phi|_\Gamma = 0.$$

Let $u' = \nabla\Lambda(\chi\Phi)$; then $\Theta u = u + u'$ is the desired operator. In fact, if $u|_{\partial\Omega} = 0$, then $\varphi|_{\partial\Omega} = \frac{\partial\varphi}{\partial n}|_{\partial\Omega} = 0$, and $\Theta u = u$, so Θ is a projection. By the estimate of the elliptic problem and the trace theorem,

$$\|\Phi\|_{s+2} \le C\left(\|\varphi\|_{s+3/2,\partial\Omega} + \left\|\frac{\partial\varphi}{\partial n}\right\|_{s+1/2,\partial\Omega}\right) \le C\|\varphi\|_{s+2} \le C\|u\|_{s+1},$$

hence

$$\|u'\|_{s+1} \le C\|u\|_{s+1},$$

which proves (1.5).

Φ can be obtained by the Galerkin scheme. To show that, we give a weak formulation of the above biharmonic problem. Let $-\Delta\Phi = \psi$; then $-\Delta\psi = 0$.

We take a test function $v \in H^1(\Omega')$, $v|_\Gamma = 0$; then $\Phi \in H^1(\Omega')$, $\Phi|_{\partial\Omega} = -\varphi|_{\partial\Omega}$, $\Phi|_\Gamma = 0$, and

$$(\nabla\Phi, \nabla v)_{\Omega'} - \left(\frac{\partial\Phi}{\partial n}, v\right)_{\partial\Omega} = (\psi, v)_{\Omega'}.$$

Let $\omega = -\nabla\Lambda u$; then $-\Delta\varphi = \omega$, and

$$(\nabla\varphi, \nabla v)_{\Omega'} - \left(\frac{\partial\varphi}{\partial n}, v\right)_{\partial\Omega} = (\omega, v)_{\Omega'}.$$

Adding up these relations and noting the boundary condition, we have

$$(2.7)\qquad (\nabla\Phi + \nabla\varphi, \nabla v)_{\Omega'} = (\psi + \omega, v)_{\Omega'} \quad \forall v \in H^1(\Omega'),\ v|_\Gamma = 0.$$

We take another test function $v_1 \in H_0^1(\Omega')$; then we have $\psi \in H^1(\Omega')$, $\psi|_\Gamma = 0$, and

$$(2.8)\qquad (\nabla\psi, \nabla v_1) = 0 \quad \forall v_1 \in H_0^1(\Omega').$$

(2.7), (2.8) is the desired weak formulation.

Finally, we list the definitions and some properties of the Helmholtz operator P and Stokes operator A. It is known that $(L^2(\Omega))^2 = X \oplus G$, where $X =$ closure in $(L^2(\Omega))^2$ of $\{u \in (C_0^\infty(\Omega))^2;\ \nabla\cdot u = 0\}$ and $G = \{\nabla p;\ p \in E^1(\Omega)\}$. P is the orthogonal projection $P\colon (L^2(\Omega))^2 \to X$; consequently,

$$(2.9)\qquad \|Pu\|_0 \le \|u\|_0 \quad \forall u \in (L^2(\Omega))^2.$$

If $u \in (H^s(\Omega))^2$, $s \ge 1$, then [9, Chapter 1, §2.4]

$$u = \nabla\varphi + v, \qquad \nabla\cdot v = 0, \qquad v\cdot n|_{\partial\Omega} = 0,$$

and φ is the solution of

$$-\Delta\varphi = -\nabla\cdot u,$$
$$\left.\frac{\partial\varphi}{\partial n}\right|_{\partial\Omega} = u\cdot n|_{\partial\Omega}.$$

Like (2.6), we can obtain

$$(2.10)\qquad [\nabla\varphi]_s \le C\|u\|_s \quad \forall s \ge 0,$$

therefore $P\colon (H^s(\Omega))^2 \to (H^s(\Omega))^2$.

We consider the Stokes equation

$$(2.11)\qquad \frac{\partial u}{\partial t} + \frac{1}{\rho}\nabla p = \nu\Delta u + f$$

and conditions (1.2)–(1.4). Let $u = e^{\nu t}v$, $p = e^{\nu t}q$; then

$$\frac{\partial v}{\partial t} + \frac{1}{\rho}\nabla q = \nu(\Delta v - v) + e^{-\nu t}f.$$

The Stokes operator is defined as [7]: $A = -P\Delta + I$, with domain $D(A) = X \cap \{u \in (H^2(\Omega))^2;\ u|_{\partial\Omega} = 0\}$. The solution v can be expressed as

$$v = e^{-\nu tA}u_0 + \int_0^t e^{-\nu(t-\tau)A}Pe^{-\nu\tau}f(\tau)\,d\tau,$$

hence

$$u = e^{-\nu t(A-I)} u_0 + \int_0^t e^{-\nu(t-\tau)(A-I)} P f(\tau)\, d\tau. \tag{2.12}$$

We have [7]

$$\|A^\alpha e^{-tA}\| \le C t^{-\alpha}, \qquad \alpha \ge 0,\ t > 0, \tag{2.13}$$

$$C^{-1}\|u\|_{2\alpha} \le \|A^\alpha u\|_0 \le C\|u\|_{2\alpha} \quad \forall u \in D(A^\alpha),\ \alpha \ge 0. \tag{2.14}$$

And if $0 \le s < \frac{1}{2}$, $u \in X \cap (H^s(\Omega))^2$, then $u \in D(A^{s/2})$; if $1 \le s < \frac{3}{2}$, $u \in D(A) \cap (H^{s+1}(\Omega))^2$, then $u \in D(A^{(s+1)/2})$ [15].

3. Solutions of the Stokes equation

In this section we consider problems (2.11), (1.2), (1.3), (1.4) and give some estimates. It is assumed that all functions appearing below belong to $L^2(\Omega)$.

Lemma 1. *If u is the solution of* (2.11), (1.2), (1.3), (1.4), *then*

$$\|u(t)\|_0^2 \le e^t \left(\|u_0\|_0^2 + \int_0^t \|f(\tau)\|_0^2\, d\tau \right). \tag{3.1}$$

Proof. Taking the inner product of (2.11) with u, we get $(u, \frac{\partial u}{\partial t}) = \nu(\Delta u, u) + (f, u)$. Integrating by parts, we obtain $\frac{1}{2}\frac{d}{dt}\|u\|_0^2 + \nu(\nabla u, \nabla u) = (f, u)$. Thus, $\frac{d}{dt}\|u\|_0^2 \le \|u\|_0^2 + \|f\|_0^2$. By the Gronwall lemma, this gives (3.1). □

Lemma 2. *Let $\omega = -\nabla\Lambda u$, and let u be the solution of* (2.11), (1.2), (1.3), (1.4); *then*

$$\frac{d}{dt}\|\omega(t)\|_0^2 \le \frac{1}{2\nu}\|f\|_0^2. \tag{3.2}$$

Proof. We apply the operator $-\nabla\Lambda$ to equation (2.11) and obtain

$$\frac{\partial \omega}{\partial t} = \nu\Delta\omega - \nabla\Lambda f. \tag{3.3}$$

The stream function ψ is the solution of (2.1). Thus, the weak formulation is: find $\omega \in E^1(\Omega)$ and $\psi \in E_0^1(\Omega)$ such that

$$\frac{d}{dt}(\omega, v) + \nu(\nabla\omega, \nabla v) = -(\nabla\Lambda f, v) = (f, \nabla\Lambda v) \quad \forall v \in E_0^1(\Omega), \tag{3.4}$$

$$(\nabla\psi, \nabla\chi) = (\omega, \chi) \quad \forall \chi \in E^1(\Omega). \tag{3.5}$$

We take $\chi \in E_0^1(\Omega)$; then by (3.4) and (3.5),

$$\left(\nabla\frac{\partial\psi}{\partial t}, \nabla\chi\right) = \frac{d}{dt}(\nabla\psi, \nabla\chi) = \frac{d}{dt}(\omega, \chi) = (f, \nabla\Lambda\chi) - \nu(\nabla\omega, \nabla\chi).$$

Let $\chi = \frac{\partial\psi}{\partial t}$; then

$$\left(\nabla\frac{\partial\psi}{\partial t}, \nabla\frac{\partial\psi}{\partial t}\right) = \left(f, \nabla\Lambda\frac{\partial\psi}{\partial t}\right) - \nu\left(\nabla\omega, \nabla\frac{\partial\psi}{\partial t}\right). \tag{3.6}$$

(3.5) also yields

$$\left(\nabla\frac{\partial\psi}{\partial t}, \nabla\chi\right) = \left(\frac{\partial\omega}{\partial t}, \chi\right) \quad \forall\chi \in E^1(\Omega).$$

Set $\chi = \omega$; then

$$\left(\nabla\frac{\partial\psi}{\partial t}, \nabla\omega\right) = \left(\frac{\partial\omega}{\partial t}, \omega\right).$$

Substitute this into (3.6) to obtain

$$\left\|\nabla\frac{\partial\psi}{\partial t}\right\|_0^2 + \nu\left(\frac{\partial\omega}{\partial t}, \omega\right) = \left(f, \nabla\Lambda\frac{\partial\psi}{\partial t}\right) \le \frac{1}{4}\|f\|_0^2 + \left\|\nabla\frac{\partial\psi}{\partial t}\right\|_0^2.$$

Therefore,

$$\frac{\nu}{2}\frac{d}{dt}\|\omega\|_0^2 \le \frac{1}{4}\|f\|_0^2,$$

which is (3.2). □

Lemma 3. *If* $u_0 \in D(A) \cap (H^{s+1}(\Omega))^2$, $0 \le s < \frac{3}{2}$, $f \in L^\infty(0, T; (H^1(\Omega))^2)$, *and* u *is the solution of* (2.11), (1.2), (1.3), (1.4), *then*

$$\|u(t)\|_{s+1} \le C\left(\|u_0\|_{s+1} + \sup_{0\le\tau\le T}\|f(\tau)\|_1\right). \tag{3.7}$$

Proof. We estimate the terms of (2.12). According to the statement at the end of §2, $u_0 \in D(A^{(s+1)/2})$. By (2.13), (2.14),

$$\begin{aligned}\|e^{-\nu t(A-I)}u_0\|_{s+1} &\le C\|A^{(s+1)/2}e^{-\nu tA}e^{\nu t}u_0\|_0 = C\|e^{-\nu tA}A^{(s+1)/2}e^{\nu t}u_0\|_0\\ &\le C\|A^{(s+1)/2}e^{\nu t}u_0\|_0 \le C\|e^{\nu t}u_0\|_{s+1} \le C\|u_0\|_{s+1}.\end{aligned}$$

Take a positive constant r such that $s - 1 < r < \frac{1}{2}$; then $Pf(\tau) \in D(A^{r/2})$ $\forall\tau \in [0, T]$, and

$$\begin{aligned}\left\|\int_0^t e^{-\nu(t-\tau)(A-I)}Pf(\tau)\,d\tau\right\|_{s+1} &\le C\int_0^t \|A^{(s+1)/2}e^{-\nu(t-\tau)(A-I)}Pf(\tau)\|_0\,d\tau\\ &= C\int_0^t \|A^{(s+1-r)/2}e^{-\nu(t-\tau)(A-I)}A^{r/2}Pf(\tau)\|_0\,d\tau\\ &\le C\int_0^t (\nu(t-\tau))^{-(s+1-r)/2}\|e^{\nu(t-\tau)}A^{r/2}Pf(\tau)\|_0\,d\tau\\ &\le C\int_0^t (\nu(t-\tau))^{-(s+1-r)/2}\|f(\tau)\|_r\,d\tau\\ &\le C\sup_{0\le\tau\le T}\|f(\tau)\|_1. \quad \square\end{aligned}$$

Now we apply scheme (1.6)–(1.13) to problem (2.11), (1.2), (1.3), (1.4). Equation (1.6) reduces to

$$\frac{\partial\tilde{u}_k}{\partial t} + \frac{1}{\rho}\nabla\tilde{p}_k = f. \tag{3.8}$$

Applying the operator P to (3.8), we obtain $\partial \tilde{u}_k/\partial t = Pf$, thus

$$\tilde{u}_k(t) = u_k(ik-0) + \int_{ik}^{t} Pf(\tau)\,d\tau, \qquad ik \le t < (i+1)k. \tag{3.9}$$

By induction and (2.12), it can be proved that

$$\begin{aligned} u_k(t) = {} & e^{-\nu t(A-I)}u_0 + \sum_{i=0}^{[t/k]} e^{-\nu(t-ik)(A-I)} \int_{ik}^{(i+1)k} \Theta P f(\tau)\,d\tau \\ & + \sum_{i=0}^{[t/k]-1} \int_{ik}^{(i+1)k} e^{-\nu(t-\tau)(A-I)} \frac{1}{k}\int_{ik}^{(i+1)k} (I-\Theta)Pf(\zeta)\,d\zeta\,d\tau \\ & + \int_{[t/k]k}^{t} e^{-\nu(t-\tau)(A-I)} \frac{1}{k}\int_{[t/k]k}^{([t/k]+1)k} (I-\Theta)Pf(\zeta)\,d\zeta\,d\tau, \end{aligned} \tag{3.10}$$

where [ ] denotes the integral part of a number.

Lemma 4. *If* $u_0 \in D(A)\cap (H^{s+1}(\Omega))^2$, $0 \le s < \frac{3}{2}$, $f \in L^\infty(0, T; (H^1(\Omega))^2)$, *then*

$$\|u_k(jk-0)\|_{s+1} \le C\left(\|u_0\|_{s+1} + \sup_{0\le\tau<jk}\|f(\tau)\|_1\right).$$

Proof. We estimate the second term of (3.10); the estimate of other terms is similar. Let r be a positive constant such that $s-1 < r < \frac{1}{2}$; then

$$\begin{aligned} & \left\| \sum_{i=0}^{j-1} e^{-\nu(j-i)k(A-I)} \int_{ik}^{(i+1)k} \Theta P f(\tau)\,d\tau \right\|_{s+1} \\ & \le C \left\| \sum_{i=0}^{j-1} A^{(s+1)/2} e^{-\nu(j-i)k(A-I)} \int_{ik}^{(i+1)k} \Theta P f(\tau)\,d\tau \right\|_0 \\ & = C \left\| \sum_{i=0}^{j-1} A^{(s+1-r)/2} e^{-\nu(j-i)k(A-I)} A^{r/2} \int_{ik}^{(i+1)k} \Theta P f(\tau)\,d\tau \right\|_0 \\ & \le C \sum_{i=0}^{j-1} (\nu(j-i)k)^{-(s+1-r)/2} \int_{ik}^{(i+1)k} \|e^{\nu(j-i)k} A^{r/2} \Theta P f(\tau)\|_0\,d\tau \\ & \le C \sup_{0\le\tau<jk}\|f(\tau)\|_1 \sum_{i=0}^{j-1} (\nu(j-i)k)^{-(s+1-r)/2} k \\ & \le C \sup_{0\le\tau<jk}\|f(\tau)\|_1 \int_0^{jk} (\nu(jk-\tau))^{-(s+1-r)/2}\,d\tau \\ & \le C \sup_{0\le\tau<jk}\|f(\tau)\|_1. \qquad \square \end{aligned}$$

Lemma 5. *If* $u_0 \in D(A)\cap (H^{s+1}(\Omega))^2$, $0 \le s < \frac{3}{2}$, $f \in L^\infty(0, T; (H^3(\Omega))^2) \cap W^{1,\infty}(0, T; (H^1(\Omega))^2)$, u *is the solution of problem* (2.11), (1.2), (1.3), (1.4),

$\frac{\partial u}{\partial t} \in L^\infty(0, T; (H^{s+1}(\Omega))^2)$, *and* u_k, $\tilde{u}_k$ *are the solutions of problems* (3.8), (1.7)–(1.13), *then*

$$\sup_{0\le t\le T}(\|u(t)-u_k(t)\|_{s+1}, \|u(t)-\tilde{u}_k(t)\|_{s+1}) \le C_0 k. \tag{3.11}$$

Proof. By (2.12), (3.10) we have

$$\begin{aligned}
u(t)-u_k(t) = {} & \sum_{i=0}^{[t/k]-1}\int_{ik}^{(i+1)k}(e^{-\nu(t-\tau)(A-I)} - e^{-\nu(t-ik)(A-I)})\Theta P f(\tau)\,d\tau \\
& + \int_{[t/k]k}^{t}(e^{-\nu(t-\tau)(A-I)} - e^{-\nu(t-[t/k]k)(A-I)})\Theta P f(\tau)\,d\tau \\
& - \int_{t}^{([t/k]+1)k} e^{-\nu(t-[t/k]k)(A-I)}\Theta P f(\tau)\,d\tau \\
& + \sum_{i=0}^{[t/k]-1}\int_{ik}^{(i+1)k} e^{-\nu(t-\tau)(A-I)}\frac{1}{k} \\
& \qquad \cdot \int_{ik}^{(i+1)k}(I-\Theta)P(f(\tau)-f(\zeta))\,d\zeta\,d\tau \\
& + \int_{[t/k]k}^{t} e^{-\nu(t-\tau)(A-I)}\frac{1}{k}\int_{[t/k]k}^{([t/k]+1)k}(I-\Theta)P(f(\tau)-f(\zeta))\,d\zeta\,d\tau.
\end{aligned}\tag{3.12}$$

We estimate the terms in (3.12). With regard to the first term,

$$\begin{aligned}
I_1 &= \left\|\sum_i \int_{ik}^{(i+1)k}(e^{-\nu(t-\tau)(A-I)} - e^{-\nu(t-ik)(A-I)})\Theta P f(\tau)\,d\tau\right\|_{s+1} \\
&\le C\left\|\sum_i \int_{ik}^{(i+1)k} A^{(s+1)/2} e^{-\nu(t-\tau)(A-I)}(I - e^{-\nu(\tau-ik)(A-I)})\Theta P f(\tau)\,d\tau\right\|_0 \\
&= C\left\|\sum_i \int_{ik}^{(i+1)k} A^{(s+1)/2}\nu(A-I)e^{-\nu(t-\tau)(A-I)}\right. \\
&\qquad\qquad \left.\cdot\int_0^{\tau-ik} e^{-\nu\zeta(A-I)}\,d\zeta\cdot\Theta P f(\tau)\,d\tau\right\|_0 .
\end{aligned}$$

Taking a constant s_1, $s < s_1 < \frac{3}{2}$, we get

$$\begin{aligned}
I_1 &\le C\left\|\sum_i \int_{ik}^{(i+1)k} A^{(s-s_1)/2}\,(A-I)e^{-\nu(t-\tau)A}A^{(s_1+1)/2}e^{\nu(t-\tau)}\right. \\
&\qquad\qquad \left.\cdot\int_0^{\tau-ik} e^{-\nu\zeta(A-I)}\,d\zeta\cdot\Theta P f(\tau)\,d\tau\right\|_0 \\
&\le C\sum_i \int_{ik}^{(i+1)k}(t-\tau)^{-1+(s_1-s)/2}\int_0^{\tau-ik}\|A^{(s_1+1)/2}\Theta P f(\tau)\|_0\,d\zeta\,d\tau \\
&\le C\sup_{0\le\tau\le T}\|\Theta P f(\tau)\|_{s_1+1}k\int_0^t (t-\tau)^{-1+(s_1-s)/2}\,d\tau \\
&\le Ck\sup_{0\le\tau\le T}\|f(\tau)\|_{s_1+1}.
\end{aligned}$$

With regard to the fourth term, we take a positive constant r, $s-1<r<\frac{1}{2}$; then applying (2.13), (2.14), we get

$$\begin{aligned}
I_4 &= \left\| \sum_i \int_{ik}^{(i+1)k} e^{-\nu(t-\tau)(A-I)} \frac{1}{k} \int_{ik}^{(i+1)k} (I-\Theta)P(f(\tau)-f(\zeta))\,d\zeta\,d\tau \right\|_{s+1} \\
&= \left\| \sum_i \int_{ik}^{(i+1)k} e^{-\nu(t-\tau)(A-I)} \frac{1}{k} \int_{ik}^{(i+1)k} \int_\zeta^\tau (I-\Theta)Pf'(\xi)\,d\xi\,d\zeta\,d\tau \right\|_{s+1} \\
&\le C \left\| \sum_i \int_{ik}^{(i+1)k} A^{(s+1)/2} e^{-\nu(t-\tau)(A-I)} \frac{1}{k} \int_{ik}^{(i+1)k} \int_\zeta^\tau (I-\Theta)Pf'(\xi)\,d\xi\,d\zeta\,d\tau \right\|_0 \\
&\le C \sum_i \int_{ik}^{(i+1)k} (\nu(t-\tau))^{-(s+1-r)/2} \frac{1}{k} \\
&\qquad \cdot \int_{ik}^{(i+1)k} \int_{ik}^{(i+1)k} \|(I-\Theta)Pf'(\xi)\|_r\,d\xi\,d\zeta\,d\tau \\
&\le C \int_0^t (t-\tau)^{-(s+1-r)/2}\,d\tau \cdot \max_i \int_{ik}^{(i+1)k} \|(I-\Theta)Pf'(\xi)\|_r\,d\xi \\
&\le Ck \sup_{0\le\xi\le T} \|f'(\xi)\|_1 .
\end{aligned}$$

The rest of the terms can be estimated in a similar way, and therefore we get the desired estimate of $\|u(t)-u_k(t)\|_{s+1}$.

Now we estimate $\|u(t)-\tilde{u}_k(t)\|_{s+1}$. Since $\|\frac{\partial u}{\partial t}\|_{s+1}$ is bounded, we have

$$\|u(t)-u(ik)\|_{s+1} \le C_0 k, \qquad t\in[ik,(i+1)k).$$

By (3.9),

$$\|\tilde{u}_k(t)-u_k(ik-0)\|_{s+1} \le Ck \sup_{ik\le\tau\le t} \|f(\tau)\|_{s+1}.$$

Therefore,

$$\|u(t)-\tilde{u}_k(t)\|_{s+1} \le C_0 k + \|u(ik)-u_k(ik-0)\|_{s+1} \le C_0 k. \quad \square$$

4. Solutions of the Euler equation

We consider

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u + \frac{1}{\rho}\nabla p = f, \tag{4.1}$$

$$\nabla\cdot u = 0, \tag{4.2}$$

$$u\cdot n|_{x\in\partial\Omega} = 0, \tag{4.3}$$

$$u|_{t=0} = u_0(x). \tag{4.4}$$

The existence and uniqueness theorem has been proved by several authors. We apply the result of [12] here. Although only bounded domains were considered in [12], a slight modification of the proof will yield the result for unbounded domains. In brief, if $u_0\in(H^m(\Omega))^2$, $f\in L^1(0,T;(H^m(\Omega))^2)$, $m\ge 3$, then the local solution $u\in L^\infty(0,T;(H^m(\Omega))^2)$.

Lemma 6. *If the integer* $m \geq \max(3, s+1)$, $s \geq -1$, $\|u_0\|_m \leq M_1$, $u_0 \in X$, *then there exists a constant* $C > 0$ *such that if*

$$k_0 = \frac{1}{C(M_1 + \sup_{0\leq t\leq T} \|f(t)\|_m + 1)} \tag{4.5}$$

and $0 \leq t \leq k_0$, *then the solution* u *of* (4.1)–(4.4) *satisfies*

$$\|u\|_{s+1} \leq C_1(\|u_0\|_{s+1} + 1), \tag{4.6}$$

where the constant C_1 *depends only on the constant* T *and* $\sup_{0\leq t\leq T} \|f(t)\|_m$.

Proof. From (4.1), (4.2) we get [12]

$$\frac{1}{\rho}\Delta p = \nabla \cdot f - \nabla \cdot ((u \cdot \nabla)u),$$
$$\left.\frac{\partial p}{\partial n}\right|_{x\in\partial\Omega} = \rho f \cdot n + \sum_{i,j} \phi_{ij} u_i u_j,$$

where ϕ_{ij} are bounded functions. Analogously to the proof in [12], it can be proved that $p \in E^{m+1}(\Omega)$, and

$$\|\nabla p\|_m \leq C\{\|f(t)\|_m + \|u(t)\|_m^2\}. \tag{4.7}$$

By (4.1), (4.7) we get

$$\frac{d}{dt}\|u\|_m \leq C(\|u\|_m^2 + \|f\|_m).$$

Therefore, $\|u(t)\|_m \leq y(t)$, where $y(t)$ is the solution of the initial value problem $y' = Cy^2 + C\|f(t)\|_m$, $y(0) = \|u_0\|_m$. We take

$$M = 3\left(\|u_0\|_m + \sup_{0\leq t\leq T} \|f(t)\|_m\right)$$

and impose the restriction $|y| \leq M$. Then

$$0 \leq y(t) \leq \|u_0\|_m + C\int_0^t \|f(\tau)\|_m \, d\tau + CM\int_0^t y(\tau)\, d\tau.$$

By the Gronwall lemma,

$$y(t) \leq e^{CMt}\left(\|u_0\|_m + C\int_0^t \|f(\tau)\|_m \, d\tau\right). \tag{4.8}$$

We take $t > 0$ such that $t \leq 1/(CM + C)$; then $CMt \leq 1$ and $Ct \leq 1$. (4.8) yields $y(t) \leq M$. Comparing the upper bound of t with (4.5), it will suffice that $t \leq k_0$ for a suitable constant C.

We consider the auxiliary linear problem

$$\begin{aligned} &\frac{\partial \tilde{u}}{\partial t} + (u \cdot \nabla)\tilde{u} + \nabla\tilde{\pi} = \hat{f}, \qquad \nabla \cdot \tilde{u} = 0, \\ &\tilde{u} \cdot n|_{x\in\partial\Omega} = 0, \qquad \tilde{u}|_{t=0} = \tilde{u}_0(x). \end{aligned} \tag{4.9}$$

When $\tilde{u}_0 = u_0$, $\tilde{f} = f$, then by uniqueness $\tilde{u} = u$. In a manner similar to [12], we get

$$\|\nabla \tilde{\pi}(t)\|_m \leq C(\|\tilde{f}(t)\|_m + \|u(t)\|_m \|\tilde{u}(t)\|_m);$$

then we can prove

$$\frac{1}{2}\frac{d}{dt}\|\tilde{u}(t)\|_m^2 \leq C(\|\tilde{f}(t)\|_m + \|u(t)\|_m \|\tilde{u}(t)\|_m)\|\tilde{u}(t)\|_m .$$

But $\|u(t)\|_m \leq M$, and by Gronwall's lemma

$$\|\tilde{u}(t)\|_m \leq e^{CMt}\left(\|\tilde{u}_0\|_m + Ck_0 \sup_{0\leq\tau\leq t} \|\tilde{f}(\tau)\|_m\right).$$

By (4.5),

$$\|\tilde{u}(t)\|_m \leq e\left(\|\tilde{u}_0\|_m + \sup_{0\leq\tau\leq t} \|\tilde{f}(\tau)\|_m\right) \tag{4.10}$$

under the restriction $t \leq k_0$. Taking the inner product of $\tilde{u}$ with equation (4.9), we get $(\frac{\partial \tilde{u}}{\partial t}, \tilde{u}) = (\tilde{f}, \tilde{u})$, hence

$$\|\tilde{u}(t)\|_0 \leq \|\tilde{u}_0\|_0 + \int_0^t \|\tilde{f}(\tau)\|_0 \, d\tau .$$

The mapping $(\tilde{u}_0, \tilde{f}) \to \tilde{u}$ is linear, and by the interpolation theorem and (4.10),

$$\|\tilde{u}(t)\|_{s+1} \leq C\left(\|\tilde{u}_0\|_{s+1} + \sup_{0\leq\tau\leq t} \|\tilde{f}(\tau)\|_{s+1}\right).$$

Letting $\tilde{u}_0 = u_0$, $\tilde{f} = f$, one obtains (4.6). $\square$

Now, u is assumed to be an arbitrary vector function which belongs to $L^\infty(0, T; (W^{2,\infty}(\Omega))^2)$, and with $u(\cdot, t) \in X$, $u_0 \in (H^1(\Omega))^2 \cap X$ we let ω be the solution of

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = -\nabla \Lambda f \equiv F, \qquad \omega|_{t=0} = -\nabla \Lambda u_0 \equiv \omega_0 . \tag{4.11}$$

We denote by $\xi(y, t; \tau)$ the characteristic which satisfies

$$\frac{\partial}{\partial t}\xi(y, t; \tau) = u(\xi(y, t; \tau), t), \qquad \xi(y, \tau; \tau) = y .$$

Let $\psi \in E_0^2(\Omega)$ be the stream function corresponding to u_0, and

$$\Psi(y) = \psi(\xi(y, 0; t)), \qquad \theta = -\Delta\Psi .$$

Then we have the following lemma.

Lemma 7. *If* $u_0 \in D(A)$, *then*

$$\|\theta(t) - \omega(t)\|_0 \leq C_2 t \|u_0\|_1 + \int_0^t \|F(\tau)\|_0 \, d\tau , \tag{4.12}$$

where the constant C_2 *depends only on the domain* Ω *and the function* u.

Proof. We have

$$\begin{aligned}\Delta\Psi &= \partial_1^2\psi\cdot|\nabla\xi_1|^2 + 2\partial_1\partial_2\psi\cdot(\partial_1\xi_1\cdot\partial_1\xi_2 + \partial_2\xi_1\cdot\partial_2\xi_2) + \partial_2^2\psi\cdot|\nabla\xi_2|^2\\ &\quad + \partial_1\psi\cdot\Delta\xi_1 + \partial_2\psi\cdot\Delta\xi_2,\\ \partial_i\xi_j &= \delta_{ij} + O(t), \qquad \Delta\xi = O(t),\end{aligned}$$

hence

$$-\Delta\Psi = \omega_0(\xi(y, 0; t)) + R_1, \tag{4.13}$$

where

$$\|R_1\|_0 = O(t)[\psi]_2. \tag{4.14}$$

Integrating equation (4.11) along characteristics, we obtain

$$\omega(x, t) = \omega_0(\xi(x, 0; t)) + \int_0^t F(\xi(x, \zeta; t), \zeta)\, d\zeta.$$

Since the mapping $x \to \xi(x, \tau; t)$ is measure-preserving, we get

$$\begin{aligned}\|\omega(t) - \omega_0(\xi(\cdot, 0; t))\|_0 &= \left\|\int_0^t F(\xi(\cdot, \zeta; t), \zeta)\, d\zeta\right\|_0\\ &\le \int_0^t \|F(\xi(\cdot, \zeta; t), \zeta)\|_0\, d\zeta = \int_0^t \|F(\zeta)\|_0\, d\zeta.\end{aligned}$$

By (4.13), $\theta(t) = \omega(t) + R_1 + R_2$, and

$$\|R_2\|_0 \le \int_0^t \|F(\zeta)\|_0\, d\zeta. \tag{4.15}$$

Then (4.14), (4.15) give (4.12). □

5. Some estimates for the viscous splitting scheme

In this section we give some estimates for the solutions of the scheme (1.6)–(1.13). We always denote by u, ω the solution of problem (1.1)–(1.4), and by ω_k, $\tilde{\omega}_k$ the vorticity corresponding to u_k, $\tilde{u}_k$. We recall that we assume $u_0 \in D(A)\cap(H^3(\Omega))^2$, $f \in L^\infty(0, T; (H^3(\Omega))^2)\cap W^{1,\infty}(0, T; (H^1(\Omega))^2)$ and $u \in L^\infty(0, T; (H^4(\Omega))^2) \cap W^{1,\infty}(0, T; (H^{5/2}(\Omega))^2)$.

Lemma 8. *If* $1 < s < \frac{3}{2}$, *and if there is a constant* M_0 *such that*

$$\|\tilde{u}_k(t)\|_1 \le M_0, \qquad 0 \le t \le T, \tag{5.1}$$

and constants C_1, $k_0 > 0$ *such that*

$$\|\tilde{u}_k(t)\|_{s+1} \le C_1(\|\tilde{u}_k(ik)\|_{s+1} + 1), \qquad ik \le t < (i+1)k, \tag{5.2}$$

for $0 < k \le k_0$, *then*

$$\sup_{0\le t\le T} \|\tilde{u}_k(t)\|_{s+1} \le M_2 \tag{5.3}$$

for $0 < k \le k_0$, where the constant M_2 depends only on the domain Ω, the operator Θ, the constants C_1, M_0, T, s, ν, and the functions f, u_0.

Proof. We denote by C_3 a generic constant depending only on the domain Ω, the operator Θ, the constants C_1, T, s, ν, and the functions f, u_0. Set $f_1(\tau) = f(\tau) - (\tilde{u}_k \cdot \nabla)\tilde{u}_k$; then by Lemma 4,

$$\|u_k(jk-0)\|_{s+1} \le C\left(\|u_0\|_{s+1} + \sup_{0\le\tau<jk} \|f_1(\tau)\|_1\right).$$

The norm of the nonlinear term has an upper bound

$$\|(\tilde{u}_k \cdot \nabla)\tilde{u}_k\|_1 \le C(\|\tilde{u}_k\|_{1,4}^2 + \|\tilde{u}_k\|_{0,\infty}\|\tilde{u}_k\|_2).$$

We take a constant q, $1 < q < s$. Then using the imbedding theorem [1, Theorem 7.57],

$$\|f_1(\tau)\|_1 \le \|f(\tau)\|_1 + C(\|\tilde{u}_k\|_{3/2}^2 + \|\tilde{u}_k\|_q\|\tilde{u}_k\|_2),$$

and by the interpolation inequality [10, Chapter 1, Remark 9.1],

$$\begin{aligned}\|f_1(\tau)\|_1 \le \|f(\tau)\|_1 &+ C(\|\tilde{u}_k\|_1^{2-1/s}\|\tilde{u}_k\|_{s+1}^{1/s} \\ &+ \|\tilde{u}_k\|_1^{1-(q-1)/s}\|\tilde{u}_k\|_{s+1}^{(q-1)/2}\|\tilde{u}_k\|_1^{1-1/s}\|\tilde{u}_k\|^{1/s}) \\ &= \|f(\tau)\|_1 + C(\|\tilde{u}_k\|_1^{2-1/s}\|\tilde{u}_k\|_{s+1}^{1/s} + \|\tilde{u}_k\|_1^{2-q/s}\|\tilde{u}_k\|_{s+1}^{q/s}.\end{aligned}$$

Hence,

$$\|u_k(jk-0)\|_{s+1} \le C_3 + C\sup_{0\le\tau<jk}(\|\tilde{u}_k\|_1^{2-1/s}\|\tilde{u}_k\|_{s+1}^{1/s} + \|\tilde{u}_k\|_1^{2-q/s}\|\tilde{u}_k\|_{s+1}^{q/s}). \tag{5.4}$$

By (5.2) and initial condition (1.9) we obtain

$$\|\tilde{u}_k(t)\|_{s+1} \le C_3 + C_3\sup_{0\le\tau\le T}(\|\tilde{u}_k\|_1^{2-1/s}\|\tilde{u}_k\|_{s+1}^{1/s} + \|\tilde{u}_k\|_1^{2-q/s}\|\tilde{u}_k\|_{s+1}^{q/s}) + C_1.$$

Taking the supremum of the left-hand side and applying (5.1), we get

$$\sup_{0\le t\le T}\|\tilde{u}_k\|_{s+1} \le C_3 + C_3\left(M_0^{2-1/s}\sup_{0\le t\le T}\|\tilde{u}_k\|_{s+1}^{1/s} + M_0^{2-q/s}\sup_{0\le t\le T}\|\tilde{u}_k\|_{s+1}^{q/s}\right) + C_1.$$

Then (5.3) follows. □

If we replace $(\tilde{u}_k \cdot \nabla)\tilde{u}_k$ in equation (1.6) by $(u \cdot \nabla)u$, then it becomes a linear equation

$$\frac{\partial \tilde{u}_k}{\partial t} + \frac{1}{\rho}\nabla\tilde{p}_k = f - (u\cdot\nabla)u. \tag{5.5}$$

The solutions of problem (5.5), (1.7)–(1.13) are denoted by $\tilde{u}^*$, $\tilde{p}^*$, u^*, p^*. Let $\tilde{\omega}^*$, ω^* be the associated vorticities. By Lemma 5, for any $0 \le s' < \frac{3}{2}$,

$$\sup_{0\le t\le T}(\|u(t) - u^*(t)\|_{s'+1}, \|u(t) - \tilde{u}^*(t)\|_{s'+1}) \le C_0 k. \tag{5.6}$$

Lemma 9. *If* $1 < s < \frac{3}{2}$, $\|\tilde{u}_k\|_{s+1} \le M_3$, *then*

$$(5.7)\quad \|(I-\Theta)(\tilde{u}^* - \tilde{u}_k)((i+1)k-0)\|_1 \le C_4 k\left(\sup_{ik\le\tau<(i+1)k} \|(\tilde{u}^* - \tilde{u}_k)(\tau)\|_1 + k\right),$$

where the constant C_4 *depends only on the domain* Ω, *the operator* Θ, *the constants* s, ν, T, M_3, *the functions* f, u_0, *and the solution* u *of* (1.1)–(1.4).

Proof. We denote by C_4 a generic constant which possesses the above property. By (5.5) and (1.6),

$$\frac{\partial\tilde{\omega}^*}{\partial t} + u\cdot\nabla\omega = F, \qquad \frac{\partial\tilde{\omega}_k}{\partial t} + \tilde{u}_k\cdot\nabla\tilde{\omega}_k = F.$$

On subtracting the two equations, we obtain

$$(5.8)\qquad \frac{\partial(\tilde{\omega}^* - \tilde{\omega}_k)}{\partial t} + u\cdot\nabla(\tilde{\omega}^* - \tilde{\omega}_k) = u\cdot\nabla(\tilde{\omega}^* - \omega) - (u - \tilde{u}_k)\cdot\nabla\tilde{\omega}_k.$$

By Lemma 7,

$$(5.9)\qquad \begin{aligned} &\|\theta - (\tilde{\omega}^* - \tilde{\omega}_k)((i+1)k-0)\|_0 \\ &\quad \le C_4 k\|(\tilde{u}^* - \tilde{u}_k)(ik)\|_1 \\ &\quad + \int_{ik}^{(i+1)k} \|u\cdot\nabla(\tilde{\omega}^* - \omega) - (u-\tilde{u}_k)\cdot\nabla\tilde{\omega}_k\|_0\, d\tau, \end{aligned}$$

where $\theta = -\Delta\Psi$, $\Psi(y) = \psi(\xi(y, ik; (i+1)k))$, and ψ is the stream function corresponding to $(\tilde{u}^* - \tilde{u}_k)(ik)$.

We estimate the integrand. By (5.6),

$$\|u\cdot\nabla(\tilde{\omega}^* - \omega)\|_0 \le C_4\|\tilde{u}^* - u\|_2 \le C_4 k.$$

Let $p = 2/(2-s)$ and $q = 2/(s-1)$; then

$$\begin{aligned} \|(u-\tilde{u}_k)\cdot\nabla\tilde{\omega}_k\|_0 &= \left(\int_\Omega |(u-\tilde{u}_k)\cdot\nabla\tilde{\omega}_k|^2\,dx\right)^{1/2} \\ &\le \left(\int_\Omega |\nabla\tilde{\omega}_k|^p\,dx\right)^{1/p}\left(\int_\Omega |u-\tilde{u}_k|^q\,dx\right)^{1/q} \\ &\le \|\tilde{\omega}_k\|_{1,p}\|u-\tilde{u}_k\|_{0,q}. \end{aligned}$$

Using the imbedding theorem and (5.6),

$$\|\tilde{\omega}_k\|_{1,p} \le C\|\tilde{\omega}_k\|_s,$$

$$\|u-\tilde{u}_k\|_{0,q} \le C\|u-\tilde{u}_k\|_1 \le C_0(\|\tilde{u}^* - \tilde{u}_k\|_1 + k).$$

Therefore,

$$(5.10)\qquad \|u\cdot\nabla(\tilde{\omega}^* - \omega) - (u-\tilde{u}_k)\cdot\nabla\tilde{\omega}_k\|_0 \le C_4(\|\tilde{u}^* - \tilde{u}_k\|_1 + k).$$

Substituting (5.10) into (5.9), we obtain

$$(5.11)\quad \|\theta - (\tilde{\omega}^* - \tilde{\omega}_k)((i+1)k-0)\|_0 \le C_4 k\left(\sup_{ik\le\tau<(i+1)k} \|\tilde{u}^*(\tau) - \tilde{u}_k(\tau)\|_1 + k\right).$$

On subtracting equations (5.5) and (1.6), we have

$$\frac{\partial(\tilde{u}^* - \tilde{u}_k)}{\partial t} + \frac{1}{\rho}\nabla(\tilde{p}^* - \tilde{p}_k) = (\tilde{u}_k \cdot \nabla)\tilde{u}_k - (u \cdot \nabla)u \equiv \mathscr{F}, \tag{5.12}$$

hence

$$\begin{aligned} &(\tilde{u}^* - \tilde{u}_k)(t) - (\tilde{u}^* - \tilde{u}_k)(ik) \\ &\qquad = -\frac{1}{\rho}\int_{ik}^{t} \nabla(\tilde{p}^* - \tilde{p}_k)\,d\tau + \int_{ik}^{t} \mathscr{F}\,d\tau, \\ &\|(\tilde{u}^* - \tilde{u}_k)(t) - (\tilde{u}^* - \tilde{u}_k)(ik)\|_0 \\ &\qquad \le \frac{1}{\rho}\int_{ik}^{t} \|\nabla(\tilde{p}^* - \tilde{p}_k)\|_0\,d\tau + \int_{ik}^{t} \|\mathscr{F}\|_0\,d\tau. \end{aligned} \tag{5.13}$$

Similarly as in the proof of Lemma 6, it can be shown that $\tilde{p}^* - \tilde{p}_k$ is the solution of

$$\begin{aligned} &\frac{1}{\rho}\Delta(\tilde{p}^* - \tilde{p}_k) = \nabla \cdot \mathscr{F}, \\ &\left.\frac{\partial(\tilde{p}^* - \tilde{p}_k)}{\partial n}\right|_{x\in\partial\Omega} = \sum_{i,j} \phi_{ij}((\tilde{u}_k)_i(\tilde{u}_k)_j - u_i u_j) \\ &\qquad\qquad = \sum_{i,j} \phi_{ij}((\tilde{u}_k)_i(\tilde{u}_k - u)_j - (u - \tilde{u}_k)_i u_j). \end{aligned}$$

In weak formulation, $\tilde{p}^* - \tilde{p}_k \in E^1(\Omega)$, and

$$\begin{aligned} &(\nabla(\tilde{p}^* - \tilde{p}_k), \nabla v)_\Omega + \left(\sum_{i,j} \phi_{ij}((\tilde{u}_k)_i(\tilde{u}_k - u)_j - (u - \tilde{u}_k)_i u_j), v\right)_{\partial\Omega} \\ &\quad = (\rho\mathscr{F}, \nabla v)_\Omega \quad \forall v \in E^1(\Omega). \end{aligned}$$

We may assume that $(\tilde{p}^* - \tilde{p}_k, 1)_{\partial\Omega} = 0$; then

$$\|\tilde{p}^* - \tilde{p}_k\|_{0,\partial\Omega} \le C[\tilde{p}^* - \tilde{p}_k]_1.$$

Taking $v = \tilde{p}^* - \tilde{p}_k$ we get

$$\begin{aligned} [\tilde{p}^* - \tilde{p}_k]_1^2 \le &\left\|\sum_{i,j} \phi_{ij}((\tilde{u}_k)_i(\tilde{u}_k - u)_j - (u - \tilde{u}_k)_i u_j)\right\|_{0,\partial\Omega} \|\tilde{p}^* - \tilde{p}_k\|_{0,\partial\Omega} \\ &+ \|\rho\mathscr{F}\|_0[\tilde{p}^* - \tilde{p}_k]_1, \end{aligned}$$

thus

$$[\tilde{p}^* - \tilde{p}_k]_1 \le C\left\|\sum_{i,j} \phi_{ij}((\tilde{u}_k)_i(\tilde{u}_k - u)_j - (u - \tilde{u}_k)_i u_j)\right\|_{0,\partial\Omega} + \|\rho\mathscr{F}\|_0.$$

By (5.6),

$$\begin{aligned} \|\mathscr{F}\|_0 &\le \|(u \cdot \nabla)(\tilde{u}_k - u)\|_0 + \|((\tilde{u}_k - u) \cdot \nabla)\tilde{u}_k\|_0 \\ &\le C_4\|\tilde{u}_k - u\|_1 \le C_4(\|\tilde{u}^* - \tilde{u}_k\|_1 + k), \end{aligned} \tag{5.14}$$

and

$$\left\| \sum_{i,j} \phi_{ij}((\tilde{u}_k)_i(\tilde{u}_k - u)_j - (u - \tilde{u}_k)_i u_j) \right\|_{0,\partial\Omega}$$
$$\le C_4 \|\tilde{u}_k - u\|_{0,\partial\Omega} \le C_4 \|\tilde{u}_k - u\|_{1,\Omega} \le C_4(\|\tilde{u}^* - \tilde{u}_k\|_1 + k).$$

Therefore,

$$[\tilde{p}^* - \tilde{p}_k]_1 \le C_4(\|\tilde{u}^* - \tilde{u}_k\|_1 + k). \tag{5.15}$$

Substituting (5.14), (5.15) into (5.13), we obtain

$$\|(\tilde{u}^* - \tilde{u}_k)(t) - (\tilde{u}^* - \tilde{u}_k)(ik)\|_0 \le C_4 k \left(\sup_{[ik,(i+1)k)} \|\tilde{u}^* - \tilde{u}_k\|_1 + k \right). \tag{5.16}$$

Let $U = \nabla \Lambda \Psi$. By definition of the function Ψ we have

$$\Psi(y) - \psi(y) = -\int_{ik}^{(i+1)k} \sum_j \partial_j \psi(\xi(y, t; (i+1)k)) \frac{d}{dt} \xi_j(y, t; (i+1)k)\, dt$$
$$= -\int_{ik}^{(i+1)k} \sum_j \partial_j \psi(\xi(y, t; (i+1)k)) u_j(\xi(y, t; (i+1)k))\, dt.$$

Since u, $\partial_1 u$, $\partial_2 u$, $\partial_1 \xi$, $\partial_2 \xi$ are bounded, we get

$$\|\nabla \Lambda(\Psi - \psi)\|_0 \le C_0 k [\psi]_2,$$

that is

$$\|U - (\tilde{u}^* - \tilde{u}_k)(ik)\|_0 \le C_0 k \|(\tilde{u}^* - \tilde{u}_k)(ik)\|_1. \tag{5.17}$$

By (5.16), (5.17) we have

$$\|U - (\tilde{u}^* - \tilde{u}_k)((i+1)k - 0)\|_0 \le C_4 k \left(\sup_{[ik,(i+1)k)} \|\tilde{u}^* - \tilde{u}_k\|_1 + k \right). \tag{5.18}$$

Since $\theta = -\nabla \Lambda U$, by (5.11), (5.18), and (2.6) we have

$$\|U - (\tilde{u}^* - \tilde{u}_k)((i+1)k - 0)\|_1 \le C_4 k \left(\sup_{ik \le \tau < (i+1)k} \|\tilde{u}^*(\tau) - \tilde{u}_k(\tau)\|_1 + k \right).$$

But we know that $U \in (H_0^1(\Omega))^2$, $\nabla \cdot U = 0$, hence $(I - \Theta)U = 0$. Θ is a bounded operator, thus

$$\|(I - \Theta)(\tilde{u}^* - \tilde{u}_k)((i+1)k - 0)\|_1$$
$$= \|(I - \Theta)(U - (\tilde{u}^* - \tilde{u}_k)((i+1)k - 0))\|_1$$
$$\le C_4 k \left(\sup_{ik \le \tau < (i+1)k} \|\tilde{u}^*(\tau) - \tilde{u}_k(\tau)\|_1 + k \right). \quad \square$$

Lemma 10. *If* $1 \le s < \frac{3}{2}$, $k \le 1$, $\|\tilde{u}_k(t)\|_{s+1} \le M_2$ *for* $ik \le t < (i+1)k$, *then*

$$\|u_k(t)\|_3 \le C_5 (t - ik)^{s/2 - 1} \tag{5.19}$$

on the same interval, where the constant C_5 depends only on the domain Ω, the operator Θ, the constants s, ν, T, M_2, the functions f, u_0, and the solution u of (1.1)–(1.4).

Proof. Let $w = \partial u_k/\partial t$, $\pi = \partial p_k/\partial t$. Differentiating equations (1.10)–(1.12) formally with respect to t, we obtain

$$\frac{\partial w}{\partial t} + \frac{1}{\rho}\nabla\pi = \nu\Delta w, \quad \nabla\cdot w = 0, \quad w|_{x\in\partial\Omega} = 0,$$

$$\begin{aligned} w(ik) = \left.\frac{\partial u_k}{\partial t}\right|_{t=ik} &= -\nu(A-I)\Theta\tilde{u}_k((i+1)k-0) \\ &\quad + \frac{1}{k}P(I-\Theta)\tilde{u}_k((i+1)k-0). \end{aligned}$$

It was proved in [9, Chapter 4, §2, Corollary 1] that $\partial u_k/\partial t$ is the weak solution of it. But the above problem possesses a strong solution

$$w(t) = e^{-\nu(t-ik)(A-I)}w(ik);$$

therefore,

$$\frac{\partial u_k}{\partial t} = e^{-\nu(t-ik)(A-I)}w(ik).$$

By (2.13), (2.14),

$$\begin{aligned} \left\|\frac{\partial u_k}{\partial t}\right\|_1 &\le C\|A^{1/2}e^{-\nu(t-ik)(A-I)}w(ik)\|_0 \\ &= C\|A^{1-s/2}e^{-\nu(t-ik)(A-I)}A^{(s-1)/2}w(ik)\|_0 \\ &\le C(t-ik)^{s/2-1}\|A^{(s-1)/2}e^{\nu(t-ik)}w(ik)\|_0 \\ &\le C(t-ik)^{s/2-1}\|w(ik)\|_{s-1}. \end{aligned} \tag{5.20}$$

Applying the operator P to equation (1.10), we obtain

$$\frac{\partial u_k}{\partial t} = -\nu(A-I)u_k + \frac{1}{k}P(I-\Theta)\tilde{u}_k((i+1)k-0).$$

Consequently,

$$\begin{aligned} \|u_k\|_3 &\le C\|Au_k\|_1 \\ &= C\left\|u_k + \frac{1}{k\nu}P(I-\Theta)\tilde{u}_k((i+1)k-0) - \frac{1}{\nu}\frac{\partial u_k}{\partial t}\right\|_1 \\ &= C\left\|u_k(ik) + \int_{ik}^{t}\frac{\partial u_k}{\partial \tau}\,d\tau + \frac{1}{k\nu}P(I-\Theta)\tilde{u}_k((i+1)k-0) - \frac{1}{\nu}\frac{\partial u_k}{\partial t}\right\|_1. \end{aligned}$$

Then by (5.20),

$$\begin{aligned}\|u_k\|_3 &\le C\left\|u_k(ik)+\frac{1}{k\nu}P(I-\Theta)\tilde{u}_k((i+1)k-0)\right\|_1\\&\quad+C(t-ik)^{s/2-1}\|w(ik)\|_{s-1}\\&\le C\|u_k(ik)\|_1+\frac{1}{k\nu}\|P(I-\Theta)\tilde{u}_k((i+1)k-0)\|_1\\&\quad+C(t-ik)^{s/2-1}\Big\|-\nu(A-I)\Theta\tilde{u}_k((i+1)k-0)\\&\qquad\qquad+\frac{1}{k}P(I-\Theta)\tilde{u}_k((i+1)k-0)\Big\|_{s-1}.\end{aligned}\tag{5.21}$$

By Lemma 9 and (5.6),

$$\begin{aligned}&\frac{1}{k}\|P(I-\Theta)\tilde{u}_k((i+1)k-0)\|_1\\&\quad\le\frac{1}{k}\|P(I-\Theta)(\tilde{u}^*-\tilde{u}_k)((i+1)k-0)\|_1\\&\qquad+\frac{1}{k}\|P(I-\Theta)(u-\tilde{u}^*)((i+1)k-0)\|_1\\&\quad\le C_5\left(\sup_{ik\le\tau<(i+1)k}\|\tilde{u}^*(\tau)\|_1+\sup_{ik\le\tau<(i+1)k}\|\tilde{u}_k(\tau)\|_1+C_0\right)\le C_5.\end{aligned}$$

Substituting this into (5.21), we obtain (5.19). □

Lemma 11. *If* $1<s<\frac{3}{2}$, $\|\tilde{u}_k\|_{s+1}\le M_3$, *then*

$$\sup_{0\le t\le T}(\|u(t)-u_k(t)\|_1,\ \|u(t)-\tilde{u}_k(t)\|_1)\le C_6k,\tag{5.22}$$

where the constant C_6 *depends only on the domain* Ω, *the operator* Θ, *the constants* s, ν, T, M_3, *the functions* f, u_0, *and the solution* u *of* (1.1)–(1.4).

Proof. We denote by C_6 a generic constant which possesses the above property. Taking the inner product of (5.8) with $\tilde{\omega}^*-\tilde{\omega}_k$ and noting that

$$(u\cdot\nabla(\tilde{\omega}^*-\tilde{\omega}_k),\ \tilde{\omega}^*-\tilde{\omega}_k)=0,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\omega}^*-\tilde{\omega}_k\|_0^2\le\|u\cdot\nabla(\tilde{\omega}^*-\omega)-(u-\tilde{u}_k)\cdot\nabla\tilde{\omega}_k\|_0\|\tilde{\omega}^*-\tilde{\omega}_k\|_0.$$

By (5.10), the right-hand side is bounded by

$$\frac{C_6}{2}(\|\tilde{u}^*-\tilde{u}_k\|_1^2+k^2)+\frac{1}{2}\|\tilde{\omega}^*-\tilde{\omega}_k\|_0^2.$$

By (5.12), (5.14) we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\tilde{u}^*-\tilde{u}_k\|_0^2&\le\frac{1}{2}(\|\mathscr{F}\|_0^2+\|\tilde{u}^*-\tilde{u}_k\|_0^2)\\&\le C_6(\|\tilde{u}^*-\tilde{u}_k\|_1^2+k^2)+\frac{1}{2}\|\tilde{u}^*-\tilde{u}_k\|_0^2.\end{aligned}$$

Thus we have

$$\frac{d}{dt}[\tilde{\omega}^* - \tilde{\omega}_k]_0^2 \leq C_6([\tilde{\omega}^* - \tilde{\omega}_k]_0^2 + k^2).$$

By the Gronwall lemma,

$$[(\tilde{\omega}^* - \tilde{\omega}_k)(t)]_0^2 \leq e^{C_6 k}([(\tilde{\omega}^* - \tilde{\omega}_k)(ik)]_0^2 + C_6 k^3). \tag{5.23}$$

Using the triangle inequality,

$$\begin{aligned}
[(\omega^* - \omega_k)(ik)]_0 &\leq [(\tilde{\omega}^* - \tilde{\omega}_k)((i+1)k-0)]_0 \\
&\quad + [(\tilde{\omega}^* - \tilde{\omega}_k)((i+1)k-0) - (\omega^* - \omega_k)(ik)]_0 \\
&\leq [(\tilde{\omega}^* - \tilde{\omega}_k)((i+1)k-0)]_0 \\
&\quad + \|(I-\Theta)(\tilde{u}^* - \tilde{u}_k)((i+1)k-0)\|_1,
\end{aligned}$$

hence, by Lemma 9,

$$\begin{aligned}
[(\omega^* - \omega_k)(ik)]_0 &\leq [(\tilde{\omega}^* - \tilde{\omega}_k)((i+1)k-0)]_0 \\
&\quad + C_6 k\left(\sup_{ik \leq \tau < (i+1)k} \|(\tilde{u}^* - \tilde{u}_k)(\tau)\|_1 + k\right).
\end{aligned}$$

By (5.23),

$$[(\omega^* - \omega_k)(ik)]_0 \leq [(\tilde{\omega}^* - \tilde{\omega}_k)((i+1)k-0)]_0 + C_6 k([(\tilde{\omega}^* - \tilde{\omega}_k)(ik)]_0 + k).$$

Taking the square of both sides of the above inequality and applying (5.23) again, we get

$$\begin{aligned}
&[(\omega^* - \omega_k)(ik)]_0^2 \\
&\quad \leq [(\tilde{\omega}^* - \tilde{\omega}_k)((i+1)k-0)]_0^2 + 2[(\tilde{\omega}^* - \tilde{\omega}_k)((i+1)k-0)]_0 \\
&\qquad \cdot C_6 k([(\tilde{\omega}^* - \tilde{\omega}_k)(ik)]_0 + k) + C_6 k^2([(\tilde{\omega}^* - \tilde{\omega}_k)(ik)]_0^2 + k^2) \\
&\quad \leq [(\tilde{\omega}^* - \tilde{\omega}_k)((i+1)k-0)]_0^2 + C_6 k([(\tilde{\omega}^* - \tilde{\omega}_k)(ik)]_0^2 + k^2).
\end{aligned} \tag{5.24}$$

By (1.10)–(1.13), $u^* - u_k$, $p^* - p_k$ is the solution of

$$\begin{gathered}
\frac{\partial(u^* - u_k)}{\partial t} + \frac{1}{\rho}\nabla(p^* - p_k) \\
= \nu\Delta(u^* - u_k) + \frac{1}{k}(I-\Theta)(\tilde{u}^* - \tilde{u}_k)((i+1)k-0), \\
\nabla\cdot(u^* - u_k) = 0, \\
(u^* - u_k)|_{x\in\partial\Omega} = 0, \\
(u^* - u_k)(ik) = \Theta(\tilde{u}^* - \tilde{u}_k)((i+1)k-0).
\end{gathered} \tag{5.25}$$

By Lemmas 1 and 2,

$$\begin{aligned}
\|(u^* - u_k)(t)\|_0^2 \leq e^k \Bigg(&\|(u^* - u_k)(ik)\|_0^2 \\
&+ \int_{ik}^t \left\|\frac{1}{k}(I-\Theta)(\tilde{u}^* - \tilde{u}_k)((i+1)k-0)\right\|_0^2 d\tau\Bigg),
\end{aligned}$$

$$\|(\omega^* - \omega_k)(t)\|_0^2 \le \|(\omega^* - \omega_k)(ik)\|_0^2 + \frac{1}{2\nu}\int_{ik}^{t}\left\|\frac{1}{k}(I-\Theta)(\tilde{u}^* - \tilde{u}_k)((i+1)k-0)\right\|_0^2 d\tau .$$

Using Lemma 9,

$$[(\omega^* - \omega_k)(t)]_0^2 \le e^k\left([(\omega^* - \omega_k)(ik)]_0^2 + C_6 k\left(\sup_{ik\le\tau<(i+1)k}\|(\tilde{u}^* - \tilde{u}_k)(\tau)\|_1^2 + k^2\right)\right).$$

By (5.23),

$$(5.26)\quad [(\omega^* - \omega_k)(t)]_0^2 \le e^k([(\omega^* - \omega_k)(ik)]_0^2 + C_6 k([(\tilde{\omega}^* - \tilde{\omega}_k)(ik)]_0^2 + k^2)).$$

By (5.26), (5.24), (5.23) we obtain

$$[(\omega^* - \omega_k)((i+1)k-0)]_0^2 \le (1 + C_6 k)[(\tilde{\omega}^* - \tilde{\omega}_k)(ik)]_0^2 + C_6 k^3 .$$

Using the initial condition (1.9),

$$[(\omega^* - \omega_k)((i+1)k-0)]_0^2 \le (1 + C_6 k)[(\omega^* - \omega_k)(ik-0)]_0^2 + C_6 k^3 .$$

Therefore, by induction,

$$[(\omega^* - \omega_k)((i+1)k-0)]_0^2 \le C_6 k^2 e^{C_6 T},$$

hence

$$\|(u^* - u_k)((i+1)k-0)\|_1^2 \le C_6 k^2 .$$

Applying (5.23), (5.24), (5.26), and (5.6), we obtain (5.22). □

Lemma 12. *If* $i \ge 0$, $0 \le s < \frac{3}{2}$, *and* $\|\tilde{u}_k(t)\|_{s+1} \le M_2$ *for* $ik \le t < (i+1)k$, *then* $\|u_k(t)\|_{s+1} \le M_4$ *on the same interval, where the constant* M_4 *depends only on the domain* Ω, *the operator* Θ, *the constants* ν, s, T, M_2, *the functions* f, u_0, *and the solution* u *of problem* (1.1)–(1.4).

Proof. We apply Lemma 3 to the initial-boundary value problem (5.25) and obtain

$$\|(u^* - u_k)(t)\|_{s+1} \le C\left(\|\Theta(\tilde{u}^* - \tilde{u}_k)((i+1)k-0)\|_{s+1} + \frac{1}{k}\|(I-\Theta)(\tilde{u}^* - \tilde{u}_k)((i+1)k-0)\|_1\right).$$

It is known that $\|\tilde{u}^*\|_{s+1}$ is bounded, and by Lemma 9 we can estimate the right-hand side. The upper bound of $\|(u^* - u_k)(t)\|_{s+1}$ is given, and $\|u^*\|_{s+1}$ is also bounded; thus the desired upper bound of $\|u_k(t)\|_{s+1}$ follows. □

6. Proof of the Theorem

We assume that $1 < s < \frac{3}{2}$. Let $M_0 = 2\max_{0\le t\le T}\|(u(t)\|_1$. We take $m = 3$ and determine the constant C_1 in Lemma 6; then we determine the constant

M_2 in Lemma 8 and the constant C_5 in Lemma 10. By Lemma 6, we take $k_0 \in (0, 1]$ such that

$$k_0 \leq \frac{1}{C(C_5 k_0^{s/2-1} + \sup_{0\leq t\leq T} \|f(t)\|_3 + 1)}, \tag{6.1}$$

that is,

$$C\left(C_5 k_0^{s/2} + k_0 \sup_{0\leq t\leq T} \|f(t)\|_3 + k_0\right) \leq 1,$$

which always holds if k_0 is small enough. By (5.4) we set

$$M_5 = C_3 + C M_0^{2-1/s} M_2^{1/s} + M_0^{2-q/s} M_2^{q/s}, \tag{6.2}$$

where $1 < q < s$. By Lemma 6 we set

$$M_3 = \max(C_1 M_5 + C_1, M_2). \tag{6.3}$$

We determine the constant C_6 according to Lemma 11, and the constant M_4 according to Lemma 12, and reduce k_0, if necessary, such that

$$\|u_0\|_3 \leq C_5 k_0^{s/2-1}, \tag{6.4}$$

$$C_6 k_0 \leq M_0/2. \tag{6.5}$$

With the constants so determined, we prove by induction that if $0 < k \leq k_0$, then

$$\|\tilde{u}_k(t)\|_1 \leq M_0, \qquad \|u_k(t)\|_1 \leq M_0, \qquad \|\tilde{u}_k(t)\|_{s+1} \leq M_2,$$
$$\|u(t) - u_k(t)\|_1 \leq C_6 k, \qquad \|u(t) - \tilde{u}_k(t)\|_1 \leq C_6 k.$$

Two cases are considered simultaneously: (a) $j = 0$; (b) $j > 0$ and the above assertion is valid for $0 \leq t < jk$. If $j > 0$, then by (5.4) and (6.2),

$$\|u_k(jk - 0)\|_{s+1} \leq M_5. \tag{6.6}$$

(6.6) also holds for $j = 0$. If $j > 0$, then by Lemma 10,

$$\|u_k(jk-0)\|_3 \leq C_5 k^{s/2-1};$$

by (6.4) this also holds for $j = 0$. Using Lemma 6 and (6.1), (6.3), $\|\tilde{u}_k(t)\|_{s+1} \leq M_3$ for $jk \leq t < (j+1)k$. By Lemma 11,

$$\|u(t) - u_k(t)\|_1, \ \|u(t) - \tilde{u}_k(t)\|_1 \leq C_6 k$$

always holds for $0 \leq t < (j+1)k$; by (6.5), $\|\tilde{u}_k(t)\|_1 \leq M_0$, $\|u_k(t)\|_1 \leq M_0$ on the same interval. By Lemmas 6 and 8, $\|\tilde{u}_k(t)\|_{s+1} \leq M_2$ for $0 \leq t < (j+1)k$. Thus the induction is complete.

Applying Lemma 12, we obtain the upper bound of $\|u_k(t)\|_{s+1}$. □

7. Remark

If the Euler equation possesses global solutions, then the conclusion of the theorem is also true for $k \geq k_0$; since there are at most $1 + [T/k_0]$ steps, the upper bounds in (1.14) and (1.15) are easily obtained.

A sufficient condition for global existence was given in [8, 11], namely the initial value u_0 and body force f should satisfy, in addition, $\nabla\Lambda u_0 \in L^1(\Omega)$, $\nabla\Lambda f \in L^1(\Omega\times(0, T))$. Under that restriction we can prove by induction global existence for problem (1.6)–(1.9), for any i. In fact, if $\omega_k(ik-0) \in L^1$, then $\tilde{\omega}_k(t) \in L^1(\Omega)$ for $t \in [ik, (i+1)k)$ (see [11]). For the operator Θ given in §2, $(I-\Theta)\tilde{u}_k((i+1)k-0)$ has compact support, so $\omega_k(ik) = -\nabla\Lambda\Theta\tilde{u}_k((i+1)k-0) \in L^1(\Omega)$, and ω_k satisfies

$$\frac{\partial\omega_k}{\partial t} = \nu\Delta\omega_k - \nabla\Lambda(I-\Theta)\tilde{u}_k((i+1)k-0),$$

$$\omega|_{t=ik} = \omega_k(ik).$$

Using the fundamental solution of the heat equation, it is easy to prove that $\omega_k(t) \in L^1(\Omega)$ for $t \in [ik, (i+1)k)$.

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Department of Mathematics, Peking University, Beijing, 100871, China