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A DETERMINISTIC ALGORITHM FOR SOLVING $n = fu^2 + gv^2$ IN COPRIME INTEGERS u AND v

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ABSTRACT. We give a deterministic algorithm for finding all primitive representations of a natural number n in the form $fu^2 + gv^2$, where f and g are given positive coprime integers, and $n \geq f+g+1$, $(n, fg) = 1$. The running time of this algorithm is at most

$$\mathcal{O}(n^{1/4}(\log n)^3(\log\log n)(\log\log\log n)),$$

uniformly in f and g.

1. INTRODUCTION

Throughout this paper, f and g denote integers such that

$$f \geq 1, \quad g \geq 1, \qquad (f, g) = 1, \tag{1.1}$$

and n denotes an integer such that

$$n \geq f + g + 1, \qquad (n, fg) = 1. \tag{1.2}$$

We are interested in the problem of determining all positive integers u and v (if any) such that

$$n = fu^2 + gv^2, \qquad (u, v) = 1. \tag{1.3}$$

If (u, v) is a solution of (1.3) in positive integers, then

$$(u, n) = (v, n) = 1, \tag{1.4}$$

and

$$u \neq v. \tag{1.5}$$

In view of (1.4), we see that $v^{-1} \pmod n$ exists, and so we can define an integer y by $y \equiv uv^{-1} \pmod n$, $0 \leq y < n$. Clearly, $(n, y) = 1$, and y is a solution of $fy^2 \equiv -g \pmod n$. In particular, we have $y \neq 0$, and $y \neq n/2$ if n is even. Replacing y by $n - y$, if necessary, we obtain a solution y of $fy^2 \equiv -g \pmod n$ satisfying $y \equiv \pm uv^{-1} \pmod n$ and $0 < y < n/2$.

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Conversely, suppose y is a solution of

$$fy^2 \equiv -g \pmod{n}, \qquad 0 < y < n/2. \tag{1.6}$$

Note that if (1.6) is insolvable, then so is (1.3). We define the (possibly empty) set $U(f, g, n, y)$ to be the set of pairs of integers (u, v) satisfying:

$$\begin{cases} n = fu^2 + gv^2, \quad (u, v) = 1, \quad uv^{-1} \equiv \pm y \pmod{n}, \\ \text{with} \begin{cases} u \geq 1, v \geq 1 & \text{if } fg \geq 2, \\ u > v \geq 1 & \text{if } fg = 1. \end{cases} \end{cases} \tag{1.7}$$

It is easily shown along the lines of the proof given in [18, pp. 332–335] that either $U(f, g, n, y)$ is empty or contains exactly one pair of integers. The main result of this paper is the following theorem which is proved in §2.

Theorem 1. *Let y be an integer satisfying (1.6) for which $U(f, g, n, y)$ is nonempty. Let*

$$r_0 \,(= y) > r_1 > \cdots > r_{s-1} \,(= 1) > r_s \,(= 0) \tag{1.8}$$

be the remainders obtained by applying the Euclidean algorithm to y and n. Let r_k $(0 \leq k \leq s)$ be the first remainder $< \sqrt{n/f}$. Then we have

$$U(f, g, n, y) = \{(u, v)\}, \tag{1.9}$$

where

$$u = r_k, \qquad v = \sqrt{(n - fr_k^2)/g}. \tag{1.10}$$

Theorem 1 enables us to give a simple algorithm for finding all the solutions (if any) of (1.3) in positive integers u and v as follows:

Algorithm. First determine all the solutions y of $fy^2 \equiv -g \pmod{n}$, $0 < y < n/2$. For each solution y, apply the Euclidean algorithm to y and n, and let $r = r(f, g, n, y)$ denote the first remainder $< \sqrt{n/f}$. Then all solutions (u, v) of (1.3) (with $u > v$ if $fg = 1$) lie among the pairs $(r, \sqrt{(n - fr^2)/g})$.

A deterministic version of this algorithm is described and analyzed in §4, see Theorem 3.

Theorem 2 below, which is proved in §3, gives an alternative expression for v to that given in (1.10). We remark that Theorem 2 is not needed in the algorithm.

Theorem 2. *With the notation of Theorem 1, we have*

$$v = \begin{cases} 1 & \text{if } k = 0, \\ r_{k+1} & \text{if } k \geq 1, \ fg = 1, \\ (r_{k-1} - cr_k)/g & \text{if } k \geq 1, \ fg \geq 2, \end{cases} \tag{1.11}$$

and c is the unique integer satisfying

$$r_k c \equiv r_{k-1} + (-1)^k fyr_k \pmod{n}, \qquad 0 \leq c < n. \tag{1.12}$$

We remark that Brillhart's modification [3] of the Hermite-Serret algorithm [8, 15] for solving $p = u^2 + v^2$, where $p \equiv 1 \pmod 4$ is prime, is a special case of Theorem 2. Our algorithm also contains those of Cornacchia [6] and Wilker [19] as special cases. Our proof is different from that of Brillhart in that the palindromic nature of the continued fraction used in [3] does not always hold.

2. The integers c_i and d_i and the proof of Theorem 1

Let y be an integer satisfying (1.6) for which

$$U(f, g, n, y) \neq \varnothing. \tag{2.1}$$

By the remarks following (1.7), there is a *unique* pair of integers (u, v) such that

$$\begin{cases} n = fu^2 + gv^2, \quad (u, v) = 1, \quad uv^{-1} \equiv \pm y \pmod n, \\ \text{with} \begin{cases} u \geq 1, \ v \geq 1 & \text{if } fg \geq 2, \\ u > v \geq 1 & \text{if } fg = 1. \end{cases} \end{cases} \tag{2.2}$$

We define ε $(= \pm 1)$ to be the *unique* integer satisfying

$$y \equiv \varepsilon uv^{-1} \pmod n. \tag{2.3}$$

Applying the Euclidean algorithm to y and n, we obtain

$$\begin{cases} y = q_0 n + r_0, \\ n = q_1 r_0 + r_1, \\ r_{i-2} = q_i r_{i-1} + r_i \qquad (i = 2, \ldots, s), \end{cases} \tag{2.4}$$

where

$$s \geq 1, \tag{2.5}$$

$$r_0 \ (= y) > r_1 > r_2 > \cdots > r_{s-1} \ (= 1) > r_s \ (= 0), \tag{2.6}$$

and

$$\begin{aligned} q_0 = [y/n] = 0, \qquad q_1 = [n/r_0] = [n/y] \geq 2, \\ q_i = [r_{i-2}/r_{i-1}] \geq 1 \qquad (i = 2, \ldots, s). \end{aligned} \tag{2.7}$$

The continued fraction for y/n is

$$\frac{y}{n} = [q_0, q_1, q_2, \ldots, q_s]. \tag{2.8}$$

The ith convergent to y/n is

$$\frac{A_i}{B_i} = [q_0, q_1, q_2, \ldots, q_i] \qquad (i = 0, 1, \ldots, s), \tag{2.9}$$

so that, in particular, we have

$$\begin{aligned} A_0 &= 0, & B_0 &= 1, \\ A_1 &= 1, & B_1 &= q_1, \\ A_2 &= q_2, & B_2 &= q_1 q_2 + 1, \\ &\vdots & &\vdots \\ A_s &= y, & B_s &= n. \end{aligned} \tag{2.10}$$

Moreover, we have

$$\begin{aligned} A_i &= q_i A_{i-1} + A_{i-2} \qquad (i = 2, \dots, s), \\ B_i &= q_i B_{i-1} + B_{i-2} \qquad (i = 2, \dots, s). \end{aligned} \tag{2.11}$$

From (2.4) and (2.11), we obtain, for $i = 1, \dots, s-1$,

$$(B_{i+1} - B_{i-1})r_i = q_{i+1}B_i r_i = (r_{i-1} - r_{i+1})B_i,$$

so that

$$r_i B_{i+1} + r_{i+1}B_i = r_{i-1}B_i + r_i B_{i-1} \qquad (i = 1, \dots, s-1),$$

and so, for $i = 1, \dots, s-1$, we have

$$r_i B_{i+1} + r_{i+1}B_i = r_0 B_1 + r_1 B_0 = r_0 q_1 + r_1 = n,$$

and thus

$$r_i B_{i+1} + r_{i+1}B_i = n \qquad (i = 0, 1, \dots, s-1). \tag{2.12}$$

An easy induction argument on i, using (2.4), (2.6), and (2.11), shows that

$$r_i = (-1)^i (B_i y - A_i n) \qquad (i = 0, 1, \dots, s), \tag{2.13}$$

so that

$$r_i \equiv (-1)^i B_i y \pmod{n} \qquad (i = 0, 1, \dots, s). \tag{2.14}$$

From (2.2), (2.3), and (2.14), we see that for $i = 0, 1, \dots, s$,

$$\begin{cases} f r_i u + \varepsilon(-1)^i g B_i v \equiv 0 \pmod{n}, \\ r_i v - \varepsilon(-1)^i B_i u \equiv 0 \pmod{n}. \end{cases} \tag{2.15}$$

Hence, we may define integers c_i and d_i $(i = 0, 1, \dots, s)$ by

$$\begin{cases} c_i = (f r_i u + \varepsilon(-1)^i g B_i v)/n, \\ d_i = (r_i v - \varepsilon(-1)^i B_i u)/n. \end{cases} \tag{2.16}$$

Using (2.2), (2.4), (2.11), and (2.16), it is easy to show that

$$\begin{cases} c_i = -q_i c_{i-1} + c_{i-2} \qquad (i = 2, 3, \dots, s), \\ d_i = -q_i d_{i-1} + d_{i-2} \qquad (i = 2, 3, \dots, s), \end{cases} \tag{2.17}$$

$$c_i^2 + f g d_i^2 = (f r_i^2 + g B_i^2)/n \qquad (i = 0, 1, \dots, s), \tag{2.18}$$

and

$$c_i d_{i+1} - c_{i+1} d_i = (-1)^i \varepsilon \qquad (i = 0, 1, \dots, s-1). \tag{2.19}$$

We note that

$$c_0 = (fyu + \varepsilon g v)/n, \qquad d_0 = (yv - \varepsilon u)/n, \tag{2.20}$$

$$c_1 = fu - q_1 c_0, \qquad d_1 = v - q_1 d_0, \tag{2.21}$$

$$c_s = \varepsilon(-1)^s g v, \qquad d_s = \varepsilon(-1)^{s+1} u. \tag{2.22}$$

We emphasize that c_s and d_s are nonzero.

Lemma 1. *Suppose that* $c_i = 0$ *for some integer* i *with* $0 \le i \le s-1$. *Set*

$$a = \begin{cases} 0, & i \text{ even}, \\ 1, & i \text{ odd}, \end{cases} \qquad b = \begin{cases} 0, & s-i \text{ even}, \\ 1, & s-i \text{ odd}. \end{cases}$$

Then we have $\varepsilon = (-1)^{i+1}$ *and*

(a) $c_a > \cdots > c_{i-2} > c_i = 0 > c_{i+2} > \cdots > c_{s-b}$,

(b) $c_{1-a} > \cdots > c_{i-3} > c_{i-1} = 1 = c_{i+1} < c_{i+3} < \cdots < c_{s-(1-b)}$,

(c) $d_{1-a} > \cdots > d_{i-1} \ge 0 \ge d_{i+1} > d_{i+3} > \cdots > d_{s-(1-b)}$, *where at most one of the equality signs holds, and*

(d) $d_a > \cdots > d_{i-2} \ge d_i = 1 \le d_{i+2} < \cdots < d_{s-b}$.

Proof. As $c_i = 0$, appealing to (2.16), we obtain $fr_i u = \varepsilon(-1)^{i+1} gB_i v$. Now $r_i \ge 1$, as $0 \le i \le s-1$, and so, as f, g, u, v, B_i are all positive, we see that $\varepsilon = (-1)^{i+1}$.

For $0 \le i \le s-1$, from (2.19), we obtain $c_{i+1} d_i = (-1)^{i+1}\varepsilon = 1$. As c_{i+1} and d_i are both integers, we must have $c_{i+1} = d_i = \pm 1$ But, from (2.16), we obtain $d_i = (r_i v + B_i u)/n > 0$, so we must have

$$c_{i+1} = d_i = 1 \qquad (0 \le i \le s-1).$$

For $1 \le i \le s-1$, from (2.19), we obtain (as $c_i = 0$, $d_i = 1$)

$$c_{i-1} = 1 \qquad (1 \le i \le s-1).$$

For $0 \le i \le s-2$, from (2.16), we have

$$c_i - c_{i+2} = (fu(r_i - r_{i+2}) + gv(B_{i+2} - B_i))/n > 0,$$

so that $c_{i+2} < 0$ $(0 \le i \le s-2)$.

For $2 \le i \le s-1$, from (2.16), we obtain

$$c_{i-2} - c_i = (fu(r_{i-2} - r_i) + gv(B_i - B_{i-2}))/n > 0,$$

so that $c_{i-2} > 0$ $(2 \le i \le s-1)$.

For $2 \le i \le s-1$, from (2.16), we obtain

$$d_{i-2} = (r_{i-2} v + B_{i-2} u)/n > 0,$$

so that $d_{i-2} \ge 1$ $(2 \le i \le s-1)$. Further, for $2 \le i \le s-1$, appealing to (2.19), we obtain

$$c_{i-2} d_{i-1} - c_{i-1} d_{i-2} = \varepsilon(-1)^{i-2} = -1,$$

so that $c_{i-2} d_{i-1} = d_{i-2} - 1 \ge 0$, and thus $d_{i-1} \ge 0$ $(2 \le i \le s-1)$. This inequality is also true for $i = 1$ as we now show. When $i = 1$ we have $\varepsilon = 1$, $c_0 = 1$, and so, by (2.20), $n = nc_0 = fyu + gv$ and

$$d_0 = \frac{yv - u}{n} = \frac{fuyv - fu^2}{fun} = \frac{fuyv - (n - gv^2)}{fun}$$

$$= \frac{v(fuy + gv) - n}{fun} = \frac{v-1}{fu} \ge 0.$$

Hence we have $d_{i-1} \ge 0$ $(1 \le i \le s-1)$.

For $0 \le i \le s-2$, from (2.16), we obtain $d_{i+2} = (r_{i+2}v + B_{i+2}u)/n > 0$, so that $d_{i+2} \ge 1$ $(0 \le i \le s-2)$. Further, for $0 \le i \le s-2$, from (2.19), we have $c_{i+1}d_{i+2} - c_{i+2}d_{i+1} = 1$, so that $c_{i+2}d_{i+1} = d_{i+2} - 1 \ge 0$, and thus $d_{i+1} \le 0$ $(0 \le i \le s-2)$. But the last inequality also holds for $i = s-1$, as $d_{i+1} = d_s = \varepsilon(-1)^{s+1}u = -u < 0$. Hence we have $d_{i+1} \le 0$ $(0 \le i \le s-1)$.

For $a \le t \le s-b-2$ and $t \equiv i \pmod 2$, we have, by (2.16),

$$c_t - c_{t+2} = (fu(r_t - r_{t+2}) + gv(B_{t+2} - B_t))/n > 0,$$

so that $c_t > c_{t+2}$ $(a \le t \le s-b-2,\ t \equiv i \pmod 2)$. This completes the proof of (a).

For $1-a \le t \le i-3$ and $t \equiv i+1 \pmod 2$, we have, by (2.17) and (a),

$$c_t - c_{t+2} = q_{t+2}c_{t+1} \ge q_{t+2}c_{i-2} > 0,$$

so that $c_t > c_{t+2}$ $(1-a \le t \le i-3,\ t \equiv i+1 \pmod 2)$. For $i+1 \le t \le s-3+b$ and $t \equiv i+1 \pmod 2$, we have, by (2.17) and (a),

$$c_t - c_{t+2} = q_{t+2}c_{t+1} \le q_{t+2}c_{i+2} < 0,$$

so that $c_t < c_{t+2}$ $(i+1 \le t \le s-3+b,\ t \equiv i+1 \pmod 2)$. This completes the proof of (b).

For $1-a \le t \le s-3+b$ and $t \equiv i+1 \pmod 2$, we have, by (2.16),

$$d_t - d_{t+2} = ((r_t - r_{t+2})v + (B_{t+2} - B_t)u)/n > 0,$$

so that $d_t > d_{t+2}$ $(1-a \le t \le s-3+b,\ t \equiv i+1 \pmod 2)$. This completes the proof of (c).

For $a \le t \le i-4$ and $t \equiv i \pmod 2$, we have, by (2.17) and (c),

$$d_t - d_{t+2} = q_{t+2}d_{t+1} \ge q_{t+2}d_{i-3} > q_{t+2}d_{i-1} \ge 0,$$

so that $d_t > d_{t+2}$ $(a \le t \le i-4,\ t \equiv i \pmod 2)$.

For $i+2 \le t \le s-2-b$ and $t \equiv i \pmod 2$, we have, by (2.17) and (c),

$$d_t - d_{t+2} = q_{t+2}d_{t+1} \le q_{t+2}d_{i+3} < q_{t+2}d_{i+1} \le 0,$$

so that $d_t < d_{t+2}$ $(i+2 \le t \le s-2-b,\ t \equiv i \pmod 2)$. This completes the proof of (d). The proof of Lemma 1 is now complete. □

Lemma 2. *Suppose that* $d_i = 0$ *for some integer* i *with* $0 \le i \le s-1$. *Set*

$$a = \begin{cases} 0, & i \text{ even}, \\ 1, & i \text{ odd}, \end{cases} \qquad b = \begin{cases} 0, & s-i \text{ even}, \\ 1, & s-i \text{ odd}. \end{cases}$$

Then we have $\varepsilon = (-1)^i$ *and*

(a) $c_{1-a} > \cdots > c_{i-1} \ge 0 \ge c_{i+1} > \cdots > c_{s-(1-b)}$, *where at most one of the equality signs holds,*

(b) $c_a > \cdots > c_{i-4} > c_{i-2} \ge c_i = 1 \le c_{i+2} < c_{i+4} < \cdots < c_{s-b}$,

(c) $d_a > \cdots > d_{i-2} > d_i = 0 > d_{i+2} > \cdots > d_{s-b}$, *and*

(d) $d_{1-a} > \cdots > d_{i-3} > d_{i-1} = 1 = d_{i+1} < d_{i+3} < \cdots < d_{s-(1-b)}$.

Proof. The proof of Lemma 2 is similar to that of Lemma 1 and will be omitted. □

Lemma 3. (a) *If* $c_p = c_q = 0$ *with* $0 \le p \le q \le s-1$, *then* $p = q$.

(b) *If* $d_p = d_q = 0$ *with* $0 \le p \le q \le s-1$, *then* $p = q$.

(c) *If* $c_p = d_q = 0$ *with* $0 \le p \le s-1$, $0 \le q \le s-1$, *then either* $p = q+1$ *or* $p = q-1$.

Proof. (a) Immediate from Lemma 1(a), (b).

(b) Immediate from Lemma 2(c), (d).

(c) Immediate from Lemma 1 (or from Lemma 2). □

We now define the nonnegative integers k and j which are central to the proofs of Theorems 1 and 2. We let r_k $(0 \le k \le s)$ be the largest remainder which is less than $\sqrt{n/f}$, and B_j $(0 \le j \le s)$ the largest denominator of the convergents to y/n which is less than $\sqrt{n/g}$. Clearly, $r_{s-1} = 1 < \sqrt{n/f}$, showing that $0 \le k \le s-1$. Also we have $\sqrt{n/g} \le \sqrt{n} < n = B_s$, so that $0 \le j \le s-1$.

If $k = 0$, then $y = r_0 < \sqrt{n/f}$, $fy^2 + g \equiv 0 \pmod{n}$, $fy^2 + g < 2n$, so that $n = fy^2 + g1^2$, showing that $(y, 1) \in U(f, g, n, y)$. Hence we have $(y, 1) = (u, v)$, and so in the case $k = 0$ we have

$$u = y = r_0, \qquad v = 1 = B_0, \tag{2.23}$$

as asserted in Theorems 1 and 2. When $k = 0$ we also show that

$$j = \begin{cases} 0 & \text{if } fg \ge 2, \\ 1 & \text{if } fg = 1, \end{cases}$$

as follows. From $n = fy^2 + g1^2$ and $n > f+g+1$, we obtain $y^2 \ge 1 + 1/f > 1$, so that (as $0 < y < n/2$) we have $y \ge 2$.

We first treat the case $fg \ge 2$. We suppose that $j \ge 1$ and obtain a contradiction. We have

$$fy \le fy + \left[\frac{g}{y}\right] = \left[fy + \frac{g}{y}\right] = \left[\frac{fy^2 + g}{y}\right] = \left[\frac{n}{y}\right] = q_1$$

$$= B_1 \le B_j < \sqrt{\frac{n}{g}} = \sqrt{\frac{fy^2 + g}{g}} = \sqrt{1 + \frac{fy^2}{g}},$$

so that

$$f^2y^2 < 1 + \frac{fy^2}{g}.$$

If $f = 1$, we have $g \ge 2$, and the inequality becomes

$$y^2 < 1 + \frac{y^2}{g} \le 1 + \frac{y^2}{2},$$

which is impossible, as $y \ge 2$. On the other hand, if $f \ge 2$, the inequality gives

$$2y^2 \le f(f-1)y^2 < 1 - fy^2(g-1)/g < 1,$$

which is again impossible. Hence we must have $j = 0$ as claimed.

Next we treat the case $fg = 1$, that is, $f = g = 1$. In this case we have $n = fy^2 + g1^2 = y^2 + 1$, where $y \geq 2$, and the Euclidean algorithm applied to y and n just consists of three lines, namely

$$\begin{cases} y = 0n + y, \\ n = yy + 1, \\ y = y1 + 0, \end{cases}$$

so that $s = 2$, $r_0 = y$, $r_1 = 1$, $r_2 = 0$, $q_0 = 0$, and $q_1 = q_2 = y$. Thus we have

$$B_1 = q_1 = y < \sqrt{n} < n = B_2,$$

and $j = 1$ as claimed.

This completes the treatment of the case $k = 0$, and so from here until the end of §3, we may assume $k \geq 1$. Thus we have

$$r_k < \sqrt{n/f} \leq r_{k-1} \qquad (1 \leq k \leq s-1), \tag{2.24}$$

and

$$B_j < \sqrt{n/g} \leq B_{j+1} \qquad (0 \leq j \leq s-1). \tag{2.25}$$

We are now ready to prove Theorem 1.

Proof of Theorem 1. From (2.12) and (2.24), we obtain

$$\sqrt{n/f}B_k \leq r_{k-1}B_k \leq r_{k-1}B_k + r_kB_{k-1} = n,$$

so that

$$B_k \leq \sqrt{fn}. \tag{2.26}$$

From (2.12) and (2.25), we obtain

$$\sqrt{n/g}r_j \leq r_jB_{j+1} \leq r_jB_{j+1} + r_{j+1}B_j = n,$$

so that

$$r_j \leq \sqrt{gn}. \tag{2.27}$$

Then, by (2.18), (2.24), and (2.26), we have

$$c_k^2 + fgd_k^2 = (fr_k^2 + gB_k^2)/n < 1 + fg, \tag{2.28}$$

and, by (2.18), (2.25), and (2.27), we have

$$c_j^2 + fgd_j^2 = (fr_j^2 + gB_j^2)/n < fg + 1. \tag{2.29}$$

Hence, from (2.28), we deduce

$$d_k = 0 \quad \text{or} \quad d_k = \pm 1, \quad c_k = 0, \tag{2.30}$$

and, from (2.29), we deduce

$$d_j = 0 \quad \text{or} \quad d_j = \pm 1, \quad c_j = 0. \tag{2.31}$$

We first show that neither of the possibilities

(a) $d_k = \pm 1$, $c_k = 0$, $d_j = 0$,
(b) $d_k = \pm 1$, $c_k = 0$, $d_j = \pm 1$, $c_j = 0$,

can occur.

(a) $d_k = \pm 1$, $c_k = 0$, $d_j = 0$. By Lemma 3(c) we have $j = k + 1$ or $j = k - 1$. First, suppose that $j = k + 1$. By Lemma 1(b), (d), we have

$$c_k = 0, \quad d_k = 1, \quad c_{k+1} = 1, \quad d_{k+1} = 0, \quad \varepsilon = (-1)^{k+1}.$$

Appealing to (2.16), we obtain

$$fur_k - gvB_k = 0, \qquad vr_k + uB_k = n,$$

and

$$fur_{k+1} + gvB_{k+1} = n, \qquad vr_{k+1} - uB_{k+1} = 0.$$

Solving these linear equations for r_k, B_k and r_{k+1}, B_{k+1}, we obtain

$$r_k = gv, \quad B_k = fu, \qquad r_{k+1} = u, \quad B_{k+1} = v.$$

As $r_{k+1} < r_k < \sqrt{n/f}$ and $B_k < B_{k+1} < \sqrt{n/g}$, we deduce that

$$u < gv < \sqrt{n/f}, \qquad fu < v < \sqrt{n/g}.$$

Further, as $u > v$ for $fg = 1$, we see that $fg \geq 2$. Then we have

$$1 + fg < (fg)^2 < \frac{fn}{v^2} = \frac{f^2u^2}{v^2} + fg < 1 + fg,$$

which is impossible.

Next, suppose that $j = k - 1$. By Lemma 1(b), (d), we have

$$c_k = 0, \quad d_k = 1, \quad c_{k-1} = 1, \quad d_{k-1} = 0, \quad \varepsilon = (-1)^{k+1}.$$

Appealing to (2.16), we obtain

$$fur_k - gvB_k = 0, \qquad vr_k + uB_k = n,$$

and

$$fur_{k-1} + gvB_{k-1} = n, \qquad vr_{k-1} - uB_{k-1} = 0.$$

Solving these linear equations for r_k, B_k and r_{k-1}, B_{k-1}, we obtain

$$r_k = gv, \quad B_k = fu, \qquad r_{k-1} = u, \quad B_{k-1} = v.$$

Appealing to (2.24), we obtain $\sqrt{n/f} \leq u$, and so

$$n = fu^2 + gv^2 > fu^2 \geq n,$$

which is impossible. Thus case (a) cannot occur.

(b) $d_k = \pm 1$, $c_k = 0$, $d_j = \pm 1$, $c_j = 0$. By Lemma 3(a) we have $j = k$, and by Lemma 1(d) we have $c_k = 0$, $d_k = 1$, and $\varepsilon = (-1)^{k+1}$. Appealing to (2.16), we obtain

$$fur_k - gvB_k = 0, \qquad vr_k + uB_k = n.$$

Solving these linear equations for r_k and B_k, we obtain $r_k = gv$, $B_k = fu$. Then, from (2.24) and (2.25), we deduce

$$gv < \sqrt{n/f}, \qquad fu < \sqrt{n/g}.$$

If $fg \geq 2$, then we have

$$n = fu^2 + gv^2 < \frac{n}{fg} + \frac{n}{fg} = \frac{2n}{fg} \leq n,$$

which is impossible. Thus we must have $fg = 1$, so that

$$f = g = 1, \quad r_k = v, \quad B_k = u.$$

Then, as $c_{k-1} = 1$ by Lemma 1(b), we obtain from (2.12) (with $i = k - 1$) $n = ur_{k-1} + vB_{k-1}$. Thus, as $(u, v) = 1$, there is an integer t such that

$$r_{k-1} = u + vt, \qquad B_{k-1} = v - ut.$$

As $0 < B_{k-1} \leq B_k$ and $u > v > 0$, we have $-1 < v/u - 1 \leq t < v/u < 1$, so that $t = 0$, and thus $r_{k-1} = u$, $B_{k-1} = v$. Then, from (2.24) and (2.25), we obtain the contradiction $\sqrt{n} \leq u$, $u < \sqrt{n}$. Thus case (b) cannot occur.

Hence, from (2.30) and (2.31), we see that we must have $d_k = 0$. By Lemma 2(b) we have $\varepsilon = (-1)^k$ and $c_k = 1$. Appealing to (2.16) (with $i = k$), we obtain

$$fur_k + gvB_k = n, \qquad vr_k - uB_k = 0.$$

Solving these linear equations for r_k and B_k, we obtain $r_k = u$, $B_k = v$. This completes the proof of Theorem 1. □

We finish this section by showing that

$$j = \begin{cases} k & \text{if } fg \geq 2, \\ k+1 & \text{if } fg = 1. \end{cases} \tag{2.32}$$

Equation (2.32) has already been proved when $k = 0$, so we may assume that $k \geq 1$. We have shown that

$$c_k = 1, \quad d_k = 0, \quad \varepsilon = (-1)^k, \quad r_k = u, \quad B_k = v. \tag{2.33}$$

Hence we have $B_k = v < \sqrt{n/g}$, so that

$$j \geq k. \tag{2.34}$$

From (2.31) we see that either

$$d_j = 0 \tag{2.35}$$

or

$$c_j = 0, \qquad d_j = \pm 1. \tag{2.36}$$

If (2.36) holds, we have $c_j = d_k = 0$, and so by Lemma 2(c) we have $j = k + 1$ or $j = k - 1$. The second possibility is excluded by (2.34). Thus, $j = k + 1$

and $c_{k+1} = 0$, $d_{k+1} = \pm 1$. By Lemma 1(d) we deduce $d_{k+1} = 1$. Appealing to (2.16), we obtain

$$fur_{k+1} - gvB_{k+1} = 0, \qquad vr_{k+1} + uB_{k+1} = n.$$

Solving for r_{k+1}, B_{k+1}, we deduce $r_{k+1} = gv$, $B_{k+1} = fu$. Since $r_{k+1} < r_k < \sqrt{n/f}$ and $B_k < B_{k+1} < \sqrt{n/g}$, we have $gv < u < \sqrt{n/f}$ and $v < fu < \sqrt{n/g}$, so that

$$n = fu^2 + gv^2 < f\left(\frac{1}{f}\sqrt{\frac{n}{g}}\right)^2 + g\left(\frac{1}{g}\sqrt{\frac{n}{f}}\right)^2 = \frac{n}{fg} + \frac{n}{fg} = \frac{2n}{fg}.$$

But this is a contradiction when $fg \geq 2$. Hence (2.35) holds when $fg \geq 2$ and so, by Lemma 3(b), we have $j = k$ as asserted.

Finally, we treat the case $fg = 1$, that is, $f = g = 1$. If (2.35) holds, we have $d_k = d_j = 0$, and so, by Lemma 3(b), we have $j = k$. By Lemma 2(a), (d), we obtain $0 \geq c_{k+1}$, $d_{k+1} = 1$. By (2.16) we have

$$ur_{k+1} - vB_{k+1} = c_{k+1}n, \qquad vr_{k+1} + uB_{k+1} = n.$$

Solving for r_{k+1}, B_{k+1}, we deduce

$$r_{k+1} = v + c_{k+1}u, \qquad B_{k+1} = u - c_{k+1}v.$$

Since $0 \leq r_{k+1} < r_k$ and $B_k < \sqrt{n} \leq B_{k+1}$, we have

$$0 \leq v + c_{k+1}u < u \tag{2.37}$$

and

$$v < \sqrt{n} \leq u - c_{k+1}v. \tag{2.38}$$

But, by (1.7), we have $u > v$, so that by (2.37)

$$-1 < -v/u \leq c_{k+1} < 1 - v/u < 1,$$

and so $c_{k+1} = 0$. But then from (2.38) we have $u \geq \sqrt{n}$, contradicting $u < \sqrt{n}$. Hence, (2.36) must hold, and so $j = k + 1$ as before.

This completes the proof of (2.32). □

3. Proof of Theorem 2

In §2 we showed that $c_k = 1$, $d_k = 0$, $\varepsilon = (-1)^k$, $u = r_k$, and $v = B_k$. Thus, to prove Theorem 2, we must show that

$$B_k = \begin{cases} r_{k+1} & \text{if } k \geq 1,\ fg = 1, \\ (r_{k-1} - cr_k)/g & \text{if } k \geq 1,\ fg \geq 2. \end{cases}$$

We first suppose that $k \geq 1$ and $fg = 1$. From the analysis at the end of §2 we have $c_{k+1} = 0$, and $r_{k+1} = v$, $B_{k+1} = u$. Hence, $B_k = v = r_{k+1}$ as required.

Next we suppose that $k \geq 1$ and $fg \geq 2$. As $(u, n) = 1$, we have $(r_k, n) = 1$, and so the congruence

$$r_k c \equiv r_{k-1} + (-1)^k fyr_k \pmod{n}, \qquad 0 \leq c < n,$$

has a *unique* solution c. From (2.16) (with $i = k-1$), recalling that $\varepsilon = (-1)^k$ and noting that $d_{k-1} = 1$ by Lemma 2(d), we obtain

$$fur_{k-1} - gvB_{k-1} = c_{k-1}n, \qquad vr_{k-1} + uB_{k-1} = n,$$

so that $r_{k-1} = c_{k-1}u + gv$, $B_{k-1} = fu - c_{k-1}v$, and hence

$$v = (r_{k-1} - c_{k-1}r_k)/g.$$

Next, appealing to Lemma 2(a), we note that

$$0 \le c_{k-1} \le c_{k-1}v = fu - B_{k-1} < fu \le fu^2 < n,$$

and modulo n we observe that

$$\begin{aligned} r_k c_{k-1} &\equiv uv^{-1}(vc_{k-1}) \equiv uv^{-1}(fr_k - B_{k-1}) \equiv \varepsilon y(fr_k - B_{k-1}) \\ &\equiv (-1)^k yfr_k - (-1)^k yB_{k-1} \equiv r_{k-1} + (-1)^k fyr_k. \end{aligned}$$

This shows that

$$r_k c_{k-1} \equiv r_{k-1} + (-1)^k fyr_k \pmod{n}, \qquad 0 \le c_{k-1} < n,$$

proving that $c_{k-1} = c$. The proof of Theorem 2 is now complete. □

4. The algorithm

Step 1. Use the Adleman-Pomerance-Rumely primality test [1] on the integer n. This is a deterministic algorithm with a worst case running time of $\mathscr{O}((\log n)^{\alpha \log\log\log n})$, where $\alpha > 0$ is an absolute constant and the constant implied by the $\mathscr{O}$-symbol is also absolute. If n is composite, go to Step 2; else set $r = 1$, $n = p_1$, $a_1 = 1$ and go to Step 3.

Step 2. Factor n into primes. The fastest known, fully proven, deterministic factoring algorithm is the Pollard-Strassen method discussed by Pomerance in [11, §4]. This algorithm has a running time of

$$\mathscr{O}(n^{1/4}(\log n)^3 \log\log n \log\log\log n),$$

where the constant implied by the $\mathscr{O}$-symbol is absolute. This step in the algorithm is the dominant one. In practice, one would use one of the following methods: ρ method, $p-1$ method, elliptic curve method, quadratic sieve, etc. (see [10, 11, 13]). Set

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}, \tag{4.1}$$

where $a_1, a_2, \ldots, a_r$ are r (≥ 1) positive integers and $p_1, p_2, \ldots, p_r$ are distinct primes with $p_1 = 2$ if n is even. Continue with Step 3.

Step 3. Calculate $f^{-1} \pmod{n}$ and determine $m \equiv f^{-1}g \pmod{n}$, $0 < m < n$. Note $(m, n) = 1$. The congruence (1.6) is equivalent to $y^2 \equiv -m \pmod{n}$.

(a) If n is odd go to (b). Else n is even, m is odd, $p_1 = 2$, $a_1 \ge 1$. If $a_1 = 2$ and $m \equiv 1 \pmod 4$, or $a_1 \ge 3$ and $m \not\equiv 7 \pmod 8$, the congruence

$$y_1^2 \equiv -m \pmod{2^{a_1}} \tag{4.2}$$

is insolvable and the algorithm terminates at this point. Otherwise the congruence (4.2) is solvable and we continue with (b).

(b) For each odd prime p_i, compute the Legendre symbol $(-m/p_i)$, using the Euclidean algorithm. If any of these symbols has the value -1, the congruence

$$y_i^2 \equiv -m \pmod{p_i^{a_i}} \tag{4.3}$$

is insolvable and the algorithm terminates, else all the Legendre symbols have the value $+1$ and we go to Step 4.

Step 3(a),(b) has a running time of at most

$$\mathcal{O}(\log n) + \sum_{i=1}^{r} \mathcal{O}(\log^2 p_i) = \mathcal{O}(\log n) + \mathcal{O}(\log^2 p_1 \cdots p_r) = \mathcal{O}(\log^2 n),$$

where the constant implied by the $\mathcal{O}$-symbol is absolute.

Step 4. For each odd prime p_i a solution z_i of the congruence

$$z_i^2 \equiv -m \pmod{p_i} \tag{4.4}$$

is found. The congruence (4.4) is solvable as $(-m/p_i) = +1$, and a solution z_i can be found in

$$\mathcal{O}(p_i^{(4\sqrt{e})^{-1}+\varepsilon})$$

steps, where $\varepsilon > 0$ and the constant implied by the $\mathcal{O}$-symbol depends only on ε. This is clear, as (4.4) can be solved in time $\mathcal{O}(\log p_i)$, once some quadratic nonresidue $(\bmod p_i)$ has been found (see, for example, [2, 5, 9, 16, 17]), and it is known [4] that the least quadratic nonresidue $(\bmod p_i)$ can be found in

$$\mathcal{O}(p_i^{(4\sqrt{e})^{-1}+\varepsilon})$$

steps, where the implied constant depends only on ε. The two solutions $\pm y_i$ of the congruence (4.3) are then found by means of the recurrence relation

$$\begin{cases} y_{i,1} = z_i, \\ y_{i,k+1} \equiv y_{i,k} - (2y_{i,k})^{-1}(y_{i,k}^2 + m) \pmod{p_i^{k+1}} \\ \qquad\qquad (k = 1, \ldots, a_i - 1), \\ y_i = y_{i,a_i}, \end{cases} \tag{4.5}$$

where the inverse of $2y_{i,k}$ is taken modulo p_i. Knowing z_i, y_i can be determined in $\mathcal{O}(a_i \log p_i)$ steps.

When n is even, it is also necessary to solve the congruence (4.2), which is known to be solvable from Step 3. For $a_1 = 1$ or 2, the solution y_1 of (4.2) is given by

$$y_1 \equiv \begin{cases} 1 \pmod 2 & \text{if } a_1 = 1, \\ \pm 1 \pmod 4 & \text{if } a_1 = 2. \end{cases} \tag{4.6}$$

For $a_1 \geq 3$ the solutions of (4.2) are computed by means of the recurrence relation

$$\text{(4.7)} \qquad \begin{cases} y_{1,3} = 1, \\ y_{1,k+1} \equiv (y_{1,k}^3 + (m+2)y_{1,k})/2 \pmod{2^k} \\ \qquad\qquad (k = 3, 4, \ldots, a_i - 1), \\ y_1 = y_{1,a_1}. \end{cases}$$

The four solutions of (4.2) are given by

$$\text{(4.8)} \qquad y \equiv \pm y_1, \pm(y_1 + 2^{a_1-1}) \pmod{2^{a_1}},$$

and can be calculated in $\mathcal{O}(a_1)$ steps.

Step 4 has a running time of

$$\sum_{i=1}^{r} \mathcal{O}(p_i^{(4\sqrt{e})^{-1}+\varepsilon})\mathcal{O}(a_i \log p_i) = \mathcal{O}\left(\sum_{i=1}^{r} a_i p_i^{(4\sqrt{e})^{-1}+2\varepsilon}\right)$$
$$= \mathcal{O}\left(n^{(4\sqrt{e})^{-1}+2\varepsilon} \sum_{i=1}^{r} a_i\right) = \mathcal{O}\left(n^{(4\sqrt{e})^{-1}+2\varepsilon} \log n\right) = \mathcal{O}(n^{(4\sqrt{e})^{-1}+3\varepsilon}),$$

where the constant implied by the $\mathcal{O}$-symbol depends only upon ε. Thus the time for Step 4 is $\mathcal{O}(n^{(4\sqrt{e})^{-1}+\varepsilon})$ uniformly in f and g.

We remark that if Schoof's algorithm [14] is used for solving (4.4), Step 4 has a running time of $\mathcal{O}(\log^{10} n)$ but the implied constant depends (strongly) on m.

Step 5. The Chinese remainder theorem is used to find the 2^r solutions $y \pmod n$ of

$$\text{(4.9)} \qquad y \equiv \pm y_i \pmod{p_i^{a_i}} \qquad (i = 1, 2, \ldots, r), \text{ if } n \equiv 1 \pmod 2;$$

the 2^{r-1} solutions $y \pmod n$ of

$$\text{(4.10)} \qquad y \equiv \begin{cases} 1 \pmod 2, \\ \pm y_i \pmod{p_i^{a_i}} \qquad (i = 2, \ldots, r), \end{cases} \text{ if } n \equiv 2 \pmod 4;$$

the 2^r solutions $y \pmod n$ of

$$\text{(4.11)} \qquad y \equiv \begin{cases} \pm 1 \pmod 4, \\ \pm y_i \pmod{p_i^{a_i}} \qquad (i = 2, \ldots, r), \end{cases} \text{ if } n \equiv 4 \pmod 8;$$

and the 2^{r+1} solutions $y \pmod n$ of

$$\text{(4.12)} \qquad y \equiv \begin{cases} \pm y_1, \pm(y_1 + 2^{a_1-1}) \pmod{2^{a_1}}, \\ \pm y_i \pmod{p_i^{a_i}} \qquad (i = 2, \ldots, r), \end{cases} \text{ if } n \equiv 0 \pmod 8.$$

The values y obtained are the solutions $\pmod n$ of $y^2 \equiv -m \pmod n$. Step 5 can be accomplished in $\mathcal{O}(2^r \log^2 n) = \mathcal{O}(2^{\beta \log n / \log\log n})$ steps, where β is a positive absolute constant, and the constant implied by the $\mathcal{O}$-symbol is absolute.

Step 6. For each of the solutions y of $y^2 \equiv -m \pmod{n}$ with $0 < y < n/2$ found in Step 5, we apply the Euclidean algorithm to y and n, and determine the first remainder r less than $\sqrt{n/f}$. By Theorem 1 all the solutions (u, v) of $n = fu^2 + gv^2$, $(u, v) = 1$, in positive integers (with $u > v$ if $fg = 1$) lie among the pairs $(r, \sqrt{(n - fr^2)/g})$. They are easily found by checking whether $\sqrt{(n - fr^2)/g}$ is an integer. Step 6 takes $\mathcal{O}(2^r \log n) = \mathcal{O}(2^{\beta \log n / \log\log n})$ steps, where the implied constant is absolute.

We have thus proved the following theorem.

Theorem 3. *Let n, f, g be integers satisfying* (1.1) *and* (1.2). *Then there is a deterministic algorithm which gives all the solutions of $n = fu^2 + gv^2$ in positive coprime integers u and v, with a worst case running time of*

$$\mathcal{O}(n^{1/4}(\log n)^3(\log\log n)(\log\log\log n)),$$

where the constant implied by the $\mathcal{O}$-symbol is independent of f and g.

We remark that the worst case running time for a rigorous random version of our algorithm is

$$\mathcal{O}(2^{\beta \log n / \log\log n}) \qquad (\beta > 0),$$

with the dominant steps being Step 5 and Step 6.

5. Numerical example

This algorithm was implemented in ALGEB on a RAVEN 286/10 IBM AT clone at Carleton University. The following example illustrates the calculation of solutions (u, v) to (1.3) in the case

$$n = 9,198,968,367,101, \quad f = 4, \quad g = 61,$$

using Steps 1–6 as described in §4.

Steps 1, 2. n is composite and the parameters in (4.1) are:

$$p_1 = 12613, \quad a_1 = 1; \qquad p_2 = 20333, \quad a_2 = 1; \qquad p_3 = 35869, \quad a_3 = 1.$$

Step 3. We have

$$f^{-1} \pmod{n} = 6,899,226,275,326,$$
$$m = 6,899,226,275,341,$$

$$\left(\frac{-m}{12613}\right) = \left(\frac{-m}{20333}\right) = \left(\frac{-m}{35869}\right) = +1.$$

Step 4. The solutions y_i of (4.3) corresponding to the primes p_i, $i = 1, 2, 3$, are

$$y_1 = 4853, \qquad y_2 = 9570, \qquad y_3 = 14037.$$

Step 5. The four solutions y of $y^2 \equiv -m \pmod{n}$ with $0 < y < n/2$ are

$$382,072,735,980,$$
$$1,154,613,726,359,$$
$$1,579,334,330,612,$$
$$3,116,020,792,951.$$

Step 6. Applying the Euclidean algorithm to n and each of the solutions y from Step 5 above, we find the corresponding remainders r_k with $r_k < \sqrt{n/f} \leq r_{k-1}$. All four of the values of y give rise to solutions of (1.3) as follows:

y	r_k	$\sqrt{(n - fr_k^2)/g}$
382,072,735,980	717,088	342,175
1,154,613,726,359	577,520	359,071
1,579,334,330,612	1,376,188	163,135
3,116,020,792,951	381,100	375,871

The values of r_{k-1} and c corresponding to the four solutions above are:

k	r_{k-1}	c	$(r_{k-1} - cr_k)/g$
10	26,609,379	8	342,175
10	25,368,451	6	359,071
13	55,365,439	33	163,135
9	24,452,531	4	375,871

in accordance with Theorem 2.

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