THE STRUCTURE OF MULTIVARIATE SUPERSPLINE SPACES OF HIGH DEGREE

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ABSTRACT. We consider splines (of global smoothness r , polynomial degree d , in a general number k of independent variables, defined on a k -dimensional triangulation $\mathcal{T}$ of a suitable domain Ω) which are $r2^{k-m-1}$ -times differentiable across every m -face ($m = 0, \dots, k-1$) of a simplex in $\mathcal{T}$. For the case $d > r2^k$ we identify a structure that allows the construction of a minimally supported basis.

1. INTRODUCTION

A κ -simplex K ($0 \leq \kappa \leq k$) is the convex hull of $\kappa + 1$ points in $\mathbb{R}^k$ called the *vertices* of K . K is *nondegenerate* if its κ -dimensional volume is nonzero, and *degenerate* otherwise. The *dimension* of a nondegenerate κ -simplex is κ . The convex hull of a subset of $\mu + 1$ vertices of K is a μ -face of K .

Let $\mathcal{V} \subset \mathbb{R}^k$ be a given set of N distinct points.

A *triangulation* $\mathcal{T}$ of the set $\mathcal{V}$ is a set of nondegenerate k -simplices satisfying the following requirements:

1. All vertices of each simplex in $\mathcal{T}$ are elements of $\mathcal{V}$.
2. The interiors of the simplices in $\mathcal{T}$ are pairwise disjoint.
3. The set

$$(1) \quad \Omega := \bigcup_{K \in \mathcal{T}} K \subset \mathbb{R}^k$$

is homeomorphic to $[0, 1]^k$.

4. Each $(k-1)$ -face of a simplex in $\mathcal{T}$ is either on the boundary of Ω , or else is a common face of exactly two simplices in $\mathcal{T}$.
5. No simplex in $\mathcal{T}$ contains any points of $\mathcal{V}$ other than its vertices.

Note that a μ -face of a simplex in $\mathcal{T}$ is itself a μ -dimensional simplex. On the triangulation $\mathcal{T}$ we define a spline space $S_d^r(\mathcal{T})$ as usual by

$$(2) \quad S_d^r(\mathcal{T}) = \{s \in C^r(\Omega) : s|_{\tau} \in \mathcal{P}_d^k \forall \tau \in \mathcal{T}\},$$

where $\mathcal{P}_d^k$ is the $\binom{k+d}{d}$ -dimensional linear space of all k -variate polynomials of total degree less than or equal to d .

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In this paper, we consider subspaces of $S_d^r(\mathcal{T})$ obtained by increasing the smoothness requirements across faces of the underlying simplices. More precisely, denoting by $\mathcal{S}_\mu$ the set of all μ -faces of the simplices in $\mathcal{T}$ ($\mu = 0, \dots, k-1$) and letting $\mathcal{S} = \bigcup_{\mu=0}^{k-1} \mathcal{S}_\mu$, we define the (superspline) space $S_d^r(\mathcal{T})$ as a subspace of $S_d^r(\mathcal{T})$ as follows:

$$(3) \quad S_d^r(\mathcal{T}) = \{s \in S_d^r(\mathcal{T}) : s \text{ is } \rho\text{-times differentiable across } \sigma \ \forall \sigma \in \mathcal{S}\},$$

where $\rho = r2^{k-\dim \sigma - 1}$.

The concept of supersplines was introduced in Chui and Lai [8], [9]. The area in between finite elements and full spline spaces was further explored by Schumaker [16] and Ibrahim and Schumaker [11].

2. THE GENERALIZED BÉZIER-BERNSTEIN FORM

Crucial to analyzing the dimension of spline spaces is the Bézier-Bernstein form of a multivariate polynomial. In the case $k \leq 2$ this form is used widely and is well known. A review of the Bézier-Bernstein form for a general number of variables is in de Boor [6]. In this paper, we use a notation that is particularly suitable for our purposes. However, generalized barycentric coordinates and global control nets have also been proposed in Alfeld [2] and de Boor [6].

We use $\mathcal{V}$ as an index set and denote by $\mathbb{N}$ the set of nonnegative integers. For vectors $\mathbf{I} = [i_v]_{v \in \mathcal{V}} \in \mathbb{N}^N$ and $\mathbf{a} = [a_v]_{v \in \mathcal{V}} \in \mathbb{R}^N$ we define

$$(4) \quad |\mathbf{I}| = \sum_{v \in \mathcal{V}} i_v,$$

$$(5) \quad \mathbf{a}^{\mathbf{I}} = |\mathbf{I}|! \prod_{v \in \mathcal{V}} a_v^{i_v} / \prod_{v \in \mathcal{V}} i_v!,$$

where

$$(6) \quad 0^0 := 1.$$

We also use the notation

$$(7) \quad \sigma(\mathbf{I}) = \text{conv}\{v : i_v > 0\}, \quad \sigma(\mathbf{a}) = \text{conv}\{v : a_v \neq 0\}$$

where $\text{conv } X$ denotes the convex hull of a point set X .

We now define *generalized barycentric coordinates* as cardinal piecewise linear functions $b_v \in S_1^0(\Omega)$ by the requirement

$$(8) \quad b_v(w) = \delta_{vw} = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{else} \end{cases} \quad \forall v, w \in \mathcal{V}.$$

Clearly, in each k -simplex $K \in \mathcal{T}$ the functions b_v , where v is a vertex of K , reduce to the ordinary barycentric coordinates. Globally, i.e., for all $x \in \Omega$, they satisfy

$$(9) \quad \sum_{v \in \mathcal{V}} b_v = 1, \quad b_v \geq 0 \quad \forall v \in \mathcal{V}, \quad \text{and} \quad x = \sum_{v \in \mathcal{V}} b_v(x)v.$$

For a given polynomial degree d , we use the *domain index set*

$$(10) \quad \mathbf{I}_d = \{\mathbf{I} \in \mathbb{N}^N : |\mathbf{I}| = d \text{ and } \sigma(\mathbf{I}) \in \mathcal{S} \cup \mathcal{T}\}.$$

Letting

$$(11) \quad \mathbf{b} = \mathbf{b}(x) = [b_v(x)]_{v \in \mathcal{V}},$$

it is clear that every function $s \in S_d^0(\mathcal{T})$ can be written as

$$(12) \quad s = \sum_{\mathbf{I} \in \mathbf{I}_d} c_{\mathbf{I}} \mathbf{b}^{\mathbf{I}}.$$

The coefficients $c_{\mathbf{I}}$ are the *Bézier ordinates* of s .

3. THE DE CASTELJAU ALGORITHM

Let $s \in S_d^0(\mathcal{T})$, $K \in \mathcal{T}$, and $s|_K = p \in \mathcal{P}_d^k$. Without loss of generality we may relabel the vertices and assume that K is the k -simplex

$$(13) \quad K = \{\mathbf{a} = (a_1, \dots, a_{k+1}) : a_1 + \dots + a_{k+1} = d, \ a_j \geq 0, \ a_j \in \mathbb{R}\}.$$

Then, with V_j denoting the vertices of K , we have

$$(14) \quad p = \sum_{\mathbf{I} \in K \cap \mathbf{I}_d} c_{\mathbf{I}} \mathbf{b}^{\mathbf{I}} \quad \text{where} \quad x = \sum_{j=1}^{k+1} b_j V_j, \quad \sum_{j=1}^{k+1} b_j = 1.$$

The Bernstein polynomials $\mathbf{b}^{\mathbf{I}}(x)$ satisfy a recurrence relation, see Farin [10]

$$(15) \quad \mathbf{b}^{\mathbf{I}}(x) = \sum_{j=1}^{k+1} b_j \mathbf{b}^{\mathbf{I} - \varepsilon^j}(x), \quad |\mathbf{I}| = d, \quad \text{and} \quad \varepsilon^j = (0, \dots, 1, \dots, 0).$$

This relation allows one to expand p in terms of Bernstein polynomials of lower degree with (polynomial) coefficients $p_{\mathbf{I}}^r(\mathbf{b})$.

Theorem 1. *We have*

$$(16) \quad p = \sum_{|\mathbf{I}|=d-r} p_{\mathbf{I}}^r(\mathbf{b}) \mathbf{b}^{\mathbf{I}}, \quad 0 \leq r \leq d,$$

where

$$(17) \quad p_{\mathbf{I}}^0(\mathbf{b}) = c_{\mathbf{I}},$$

$$(18) \quad p_{\mathbf{I}}^r(\mathbf{b}) = \sum_{j=1}^{k+1} b_j p_{\mathbf{I} + \varepsilon^j}^{r-1}(\mathbf{b}), \quad |\mathbf{I}| = d - r, \quad 0 \leq r \leq d.$$

The intermediate coefficients $p_{\mathbf{I}}^r(\mathbf{b})$ can also be written explicitly as

$$(19) \quad p_{\mathbf{I}}^r(\mathbf{b}) = \sum_{|\mathbf{M}|=r} c_{\mathbf{I} + \mathbf{M}} \mathbf{b}^{\mathbf{M}}, \quad |\mathbf{I}| = d - r.$$

The formulas in Theorem 1 may be used to evaluate p at a given point, and are referred to as the *de Casteljau Algorithm*.

4. SMOOTHNESS ACROSS AN INTERFACE

Given a vector $e \in \mathbb{R}^k$, the directional derivative of p in the direction of $\alpha = (\frac{\partial b_1}{\partial e}, \dots, \frac{\partial b_{k+1}}{\partial e})$, denoted D_α , is given by (Alfeld [2])

$$(20) \quad D_\alpha p = d \sum_{|\mathbf{I}|=d-1} p_{\mathbf{I}}^1(\alpha) \mathbf{b}^{\mathbf{I}},$$

where

$$(21) \quad p_{\mathbf{I}}^1(\alpha) = \sum_{|\mathbf{M}|=1} c_{\mathbf{I}+\mathbf{M}} \alpha^{\mathbf{M}}, \quad |\mathbf{I}| = d - 1.$$

In general, the r th directional derivative of p in the direction of α is given by (Farin [10])

$$(22) \quad D_\alpha^r p = \frac{d!}{(d-r)!} \sum_{|\mathbf{I}|=d-r} p_{\mathbf{I}}^r(\alpha) \mathbf{b}^{\mathbf{I}},$$

where

$$(23) \quad p_{\mathbf{I}}^r(\alpha) = \sum_{|\mathbf{M}|=r} c_{\mathbf{I}+\mathbf{M}} \alpha^{\mathbf{M}}, \quad |\mathbf{I}| = d - r, \quad \text{and} \quad p_{\mathbf{I}}^0(\alpha) = c_{\mathbf{I}}.$$

The following theorem was proved by Farin [10] in the bivariate case; here we state without proof the result in any number of variables. Let τ be an m -face of K ; without loss of generality, we will assume that

$$(24) \quad \tau = \{\mathbf{a} \in K: a_{m+2} = \dots = a_{k+1} = 0\}.$$

Theorem 2. Let $p, q \in \mathcal{P}_d^k$ be such that $p|_\tau = q|_\tau$. Then $D_\alpha^s p = D_\alpha^s q$ on τ for all directions α , $0 \leq s \leq r$, if and only if $p_{\mathbf{J}}^r = q_{\mathbf{J}}^r$ for all $\mathbf{J} = (j_1, \dots, j_{m+1}, 0, \dots, 0)$, $|\mathbf{J}| = d - r$.

5. SUBSIMPLICES AND SUBPOLYNOMIALS

Let $K \in \mathcal{T}$ be a k -simplex and τ be a face of K with $\dim \tau = n$,

$$(25) \quad K = \left\{ \mathbf{a} \in \mathbb{R}^N: \sum_{v \in K} a_v = d, a_v \geq 0 \right\},$$

$$(26) \quad \tau = \left\{ \mathbf{a} \in K: \sum_{v \in \tau} a_v = d \right\}.$$

For $\mathbf{J} \in K \cap \mathbf{I}_d$, so that $\sum_{v \in \tau} j_v = d - \rho$, define

$$(27) \quad \tau_{\mathbf{J}} = \{\mathbf{a} = [a_v]_{v \in K}: j_v \geq a_v, v \in \tau\}.$$

Clearly, $\tau_{\mathbf{J}}$ is a subsimplex of K similar to K .

Let $p \in \mathcal{P}_d^k$, and $(\mathbf{I}, c_{\mathbf{I}})_{\mathbf{I} \in K}$ be the control points of p with respect to K . The control points $(\mathbf{I}, c_{\mathbf{I}})_{\mathbf{I} \in \tau_J}$ define a polynomial p_J of degree ρ . We call τ_J the subsimplex of K associated with J , and p_J the subpolynomial of p associated with J .

De Boor [6] introduces the concepts of subsimplices and subpolynomials and proves most of the results of this section. Since our definitions are slightly different, we restate two key facts involving subpolynomials. Theorem 2 can be restated as follows:

Theorem 3. Let $p, q \in \mathcal{P}_d^k$, and τ be a face of K with $\dim \tau = m$ such that $p|_{\tau} = q|_{\tau}$. Then

$$(28) \quad D_{\alpha}^s p = D_{\alpha}^s q \text{ on } \tau, \text{ for all directions } \alpha \text{ and } 0 \leq s \leq \rho,$$

if and only if

$$(29) \quad p_{\mathbf{I}} = q_{\mathbf{I}} \quad \forall \mathbf{I} \in K \text{ with } \text{dist}(\mathbf{I}, \tau) = \rho.$$

The above theorem immediately yields the following:

Theorem 4. Let $K, K' \in \mathcal{T}$, $\tau \subset K \cap K'$, $p, q \in \mathcal{P}_d^k$, and let $s \in S_d^0(\Omega)$ with $s|_K = p$, $s|_{K'} = q$. Then $s \in C^{\rho}(\tau)$ if and only if $p_{\mathbf{I}} = q_{\mathbf{I}} \quad \forall \mathbf{I} \in K'$ with $\text{dist}(\mathbf{I}, \tau) = \rho$.

6. DETERMINING SETS

We now generalize the concept of a *determining set* known from the bivariate case (see Alfeld and Schumaker [4]).

Definition 5. A set $D \subset \mathbf{I}_d$ is a determining set of $\mathcal{S}_d'(\mathcal{T})$ if, for all $s \in \mathcal{S}_d'(\mathcal{T})$,

$$(30) \quad c_{\mathbf{I}} = 0 \quad \forall \mathbf{I} \in D \implies s \equiv 0.$$

D is a minimal determining set if there is no determining set which has fewer elements than D .

It is clear from elementary linear algebra that the number of elements in a determining set of $\mathcal{S}_d'(\mathcal{T})$ provides an upper bound on the dimension of $\mathcal{S}_d'(\mathcal{T})$, and that the number of elements in a minimal determining set is unique and equals the dimension of $\mathcal{S}_d'(\mathcal{T})$.

7. A MINIMAL DETERMINING SET OF $\mathcal{S}_d'(\mathcal{T})$

For each simplex $\sigma \in \mathcal{S} \cup \mathcal{T}$ let $\rho = r2^{k-\dim \sigma - 1}$ as before; we define two sets of domain indices recursively by

$$(31) \quad \overline{\mathcal{D}}(\sigma) = \left\{ \mathbf{I} \in \mathbf{I}_d : \sum_{v \in \sigma} i_v \geq d - \rho \right\}$$

and

$$(32) \quad \mathcal{D}(\sigma) = \overline{\mathcal{D}}(\sigma) \setminus \bigcup_{\tau \prec \sigma} \mathcal{D}(\tau),$$

where τ is a proper face of σ .

Definition 6. Let $\sigma \in \mathcal{S} \cup \mathcal{T}$. A set $\mathcal{A}(\sigma) \subset D(\sigma)$ is a determining set of $\overline{\mathcal{D}}(\sigma)$ if, for all $s \in \mathbb{S}'_d(\mathcal{T})$,

$$(33) \quad c_{\mathbf{I}} = 0 \quad \forall \mathbf{I} \in \mathcal{A}(\sigma) \cup (\overline{\mathcal{D}}(\sigma) \setminus \mathcal{D}(\sigma)) \implies c_{\mathbf{I}} = 0 \quad \forall \mathbf{I} \in \overline{\mathcal{D}}(\sigma).$$

The set $\mathcal{A}$ is a minimal determining set of $\overline{\mathcal{D}}(\sigma)$ if there is no determining set of $\overline{\mathcal{D}}(\sigma)$ with fewer elements.

Lemma 7. Let $d > r2^k$. Then, for all $\mathbf{I} \in I_d$ there exists a unique $\sigma \in \mathcal{S} \cup \mathcal{T}$ such that $\mathbf{I} \in \mathcal{D}(\sigma)$.

Proof. To prove the lemma, we have to show first that for all $\sigma, \tau \in \mathcal{S} \cup \mathcal{T}$:

$$(34) \quad \sigma \neq \tau \implies \mathcal{D}(\sigma) \cap \mathcal{D}(\tau) = \emptyset.$$

To establish this, suppose there is a domain index $\mathbf{I} \in \mathcal{D}(\sigma) \cap \mathcal{D}(\tau)$, for two simplices $\sigma, \tau \in \mathcal{S} \cup \mathcal{T}$, such that $\dim \sigma \geq \dim \tau$, and τ is not a face of σ . Thus,

$$(35) \quad \sum_{v \in \sigma} i_v \geq d - r2^{k-\dim \sigma-1} \quad \text{and} \quad \sum_{v \in \tau} i_v \geq d - r2^{k-\dim \tau-1},$$

which implies

$$(36) \quad \sum_{v \in \sigma} i_v + \sum_{v \in \tau} i_v \geq 2d - r2^{k-\dim \sigma-1} - r2^{k-\dim \tau-1}.$$

Moreover,

$$(37) \quad \sum_{v \in \sigma} i_v + \sum_{v \in \tau} i_v = \sum_{v \in \sigma \cap \tau} i_v + \sum_{v \in \sigma \cup \tau} i_v.$$

Now, rearranging (37), substituting (36) into (37), and using $\sum_{v \in \sigma \cup \tau} i_v \leq d$, we obtain

$$(38) \quad \sum_{v \in \sigma \cap \tau} i_v \geq d - r2^{k-\dim \tau}.$$

So there exists a face $\tilde{\tau}$ of τ such that $\mathbf{I} \in \mathcal{D}(\tilde{\tau})$, which is a contradiction.

Finally, we need to prove that $\mathbf{I}_d = \bigcup_{\sigma \in \mathcal{S} \cup \mathcal{T}} \mathcal{D}(\sigma)$.

To see this, we only need to show that for each domain index $\mathbf{I} \in \mathbf{I}_d$ there exists a simplex $\sigma \in \mathcal{S} \cup \mathcal{T}$ such that $\mathbf{I} \in \mathcal{D}(\sigma)$. However, this is trivial, since for each domain index $\mathbf{I}$ there exists at least one k -simplex K such that $\sum_{v \in K} i_v = d$. Thus, $\mathbf{I} \in \mathcal{D}(K)$, and there must be a simplex $\sigma \prec K$ such that $\mathbf{I} \in \mathcal{D}(\sigma)$. $\square$

Lemma 8. Let $d > r2^k$, $\rho = r2^{k-\dim \sigma-1}$, and $\mathcal{A}(\sigma) := \mathcal{D}(\sigma) \cap K$, where $K \in \mathcal{T}$ and $\sigma \prec K$. Then $\mathcal{A}(\sigma)$ is a minimal determining set of $\overline{\mathcal{D}}(\sigma)$.

Proof. First we establish that $\mathcal{A}(\sigma)$ is determining.

Let $s \in \mathbb{S}'_d(\mathcal{T})$, and let $s|_K = p$, with $p \in \mathcal{P}_d^k$. If

$$c_{\mathbf{I}} = 0 \quad \forall \mathbf{I} \in \mathcal{A}(\sigma) \cup (\overline{\mathcal{D}}(\sigma) \setminus \mathcal{D}(\sigma)),$$

then

$$p_{\mathbf{I}} = 0 \quad \forall \mathbf{I} \in \mathcal{A}(\sigma) \cup (\overline{\mathcal{D}}(\sigma) \setminus \mathcal{D}(\sigma)),$$

where p_I is the subpolynomial of p of degree $\leq \rho$ associated with I (notice that $\text{dist}(I, \sigma) \leq \rho \quad \forall I \in \mathcal{A}(\sigma)$). Let K' be any other k -simplex of $\mathcal{T}$ such that $\sigma \prec K'$ and $K \neq K'$, and let $q = s|_{K'}$, $q \in \mathcal{P}_d^k$. Now, $s \in C^\rho(\sigma)$; then $\forall I \in (\mathcal{D}(\sigma) \cap K') \cup (\overline{\mathcal{D}}(\sigma) \setminus \mathcal{D}(\sigma)) p_I = q_I$, where q_I is the subpolynomial of q of degree $\leq \rho$ associated with I . But,

$$(39) \quad p_I = 0 \quad \forall I \in (\mathcal{D}(\sigma) \cap K) \cup (\overline{\mathcal{D}}(\sigma) \setminus \mathcal{D}(\sigma))$$

$$(40) \quad \Rightarrow \quad q_I = 0 \quad \forall I \in (\mathcal{D}(\sigma) \cap K') \cup (\overline{\mathcal{D}}(\sigma) \setminus \mathcal{D}(\sigma))$$

$$(41) \quad \Rightarrow \quad c_I = 0 \quad \forall I \in (\mathcal{D}(\sigma) \cap K') \cup (\overline{\mathcal{D}}(\sigma) \setminus \mathcal{D}(\sigma))$$

$$(42) \quad \Rightarrow \quad c_I = 0 \quad \forall I \in \left(\bigcup_{\substack{K \in \mathcal{T} \\ \sigma \prec K}} \mathcal{D}(\sigma) \cap K \right) \cup (\overline{\mathcal{D}}(\sigma) \setminus \mathcal{D}(\sigma))$$

$$(43) \quad \Rightarrow \quad c_I = 0 \quad \forall I \in \mathcal{D}(\sigma) \cup (\overline{\mathcal{D}}(\sigma) \setminus \mathcal{D}(\sigma)).$$

Therefore,

$$(44) \quad c_I = 0 \quad \forall I \in \overline{\mathcal{D}}(\sigma).$$

To see that $\mathcal{A}(\sigma)$ is indeed minimal, take $I \in \mathcal{A}(\sigma)$ and consider the set $\widetilde{\mathcal{A}}(\sigma) = \mathcal{A}(\sigma) \setminus \{I\}$, define a polynomial on K whose Bézier coefficients are equal to zero except at the domain point I , where $c_I = 1$, and extend this polynomial globally on the rest of the triangulation.

The coefficients c_J are equal to zero on $\overline{\mathcal{D}}(\sigma) \setminus \mathcal{D}(\sigma)$, since the smoothness conditions there only involve domain indices in $\mathcal{D}(\tau)$ for $\tau \prec \sigma$.

Hence, $c_J = 0 \quad \forall J \in \widetilde{\mathcal{A}}(\sigma) \cup (\overline{\mathcal{D}}(\sigma) \setminus \mathcal{D}(\sigma))$, but this does not imply that $c_J = 0 \quad \forall J \in \overline{\mathcal{D}}(\sigma)$ since in particular $I \in \overline{\mathcal{D}}(\sigma)$ and $c_I = 1$.

Therefore, $\widetilde{\mathcal{A}}(\sigma)$ cannot be determining and $\mathcal{A}(\sigma)$ is a minimal determining set of $\overline{\mathcal{D}}(\sigma)$. $\square$

The following theorem is the central result of this paper.

Theorem 9. Let $r \geq 0$, $d > r2^k$ and let $\mathcal{A}(\sigma) = \mathcal{D}(\sigma) \cap K$, where $K \in \mathcal{T}$ is a k -simplex so that σ is a face of K . Then

$$(45) \quad \mathcal{A} := \bigcup_{\sigma \in \mathcal{S} \cup \mathcal{T}} \mathcal{A}(\sigma)$$

is a minimal determining set of $\mathbb{S}_d^r(\mathcal{T})$.

Proof. Let $s \in \mathbb{S}_d^r(\mathcal{T})$ and assume that $c_I = 0 \quad \forall I \in \mathcal{A}$. Using induction on $\dim \sigma$, we first establish that $\mathcal{A}$ is a determining set.

(1) If $\dim \sigma = 0$ then $c_I = 0 \quad \forall I \in \mathcal{A}(\sigma)$ implies $c_I = 0 \quad \forall I \in \overline{\mathcal{D}}(\sigma)$, since $\mathcal{D}(\sigma) = \overline{\mathcal{D}}(\sigma)$.

(2) We assume now that $c_I = 0 \quad \forall I \in \overline{\mathcal{D}}(\sigma)$ and $\forall \sigma$ with $\dim \sigma < n$.

(3) Let $\dim \sigma = n$, $c_I = 0 \quad \forall I \in \mathcal{A}(\sigma)$. Let $\mathcal{B}(\sigma) = \mathcal{A}(\sigma) \cup (\overline{\mathcal{D}}(\sigma) \setminus \mathcal{D}(\sigma))$; notice that $B(\sigma) \subset \mathcal{A}(\sigma) \cup (\bigcup_{\tau \prec \sigma} \overline{\mathcal{D}}(\tau))$.

By the induction hypothesis, $c_I = 0 \quad \forall I \in \bigcup_{\tau \prec \sigma} \overline{\mathcal{D}}(\tau)$ implies that $c_I = 0 \quad \forall I \in \mathcal{B}(\sigma)$. But $\mathcal{A}(\sigma)$ is a determining set of $\overline{\mathcal{D}}(\sigma)$; thus, $c_I = 0 \quad \forall I \in \overline{\mathcal{D}}(\sigma)$, hence

$$(46) \quad c_I = 0 \quad \forall I \in \bigcup_{\sigma \in \mathcal{S} \cup T} \overline{\mathcal{D}}(\sigma),$$

and by Lemma 7 we get that

$$(47) \quad c_I = 0 \quad \forall I \in \mathbf{I}_d.$$

To show that $\mathcal{A}$ is indeed minimal, we give arbitrary Bézier ordinates c_I $\forall I \in \mathcal{A}$, and we construct the rest of the Bézier ordinates in such a way that the piecewise polynomial they determine is a superspline.

So, we assume $c_I \quad \forall I \in \mathcal{A}$ are given, and let $\mathbf{J} \in \mathbf{I}_d$. Then by Lemma 7 there is a unique $\sigma \in \mathcal{S} \cup T$ such that $\mathbf{J} \in \mathcal{D}(\sigma)$.

We define c_J inductively over $\dim \sigma$.

(1) If σ is a vertex, then there exists $K \in \mathcal{T}$ so that $\sigma \prec K$ and $\mathcal{A}(\sigma) \subset K$.

Let σ_J be the subsimplex of K associated with $\mathbf{J}$, $\sigma_J \subset \mathcal{A}(\sigma)$. Then the polynomial p_J is defined; so, for $\mathbf{J} \in \mathcal{D}(\sigma) \cap K'$ with $K' \in \mathcal{T}$, $K' \neq K$, let τ_J be the subsimplex of K' associated with $\mathbf{J}$. Then there is a unique way to give Bézier ordinates to the domain points in τ_J in such a way that they define a polynomial q_J which equals p_J . Furthermore, in this manner, c_J can be defined for all $\mathbf{J} \in \mathcal{D}(\sigma) \setminus \mathcal{A}(\sigma)$.

(2) Suppose c_J has been defined for all $\mathbf{J} \in \bigcup_{\dim \sigma \leq n} \mathcal{D}(\sigma)$.

(3) Let $\mathbf{J} \in \mathcal{D}(\sigma)$ with σ an n -simplex and $K \in \mathcal{T}$ such that $\mathcal{A}(\sigma) \subset K$. And let again σ_J be the subsimplex of K associated with $\mathbf{J}$. Note that

$$(48) \quad \sigma_J \subset \overline{\mathcal{D}}(\sigma) \cap K \subset \mathcal{A}(\sigma) \cup \left(\bigcup_{\tau \prec \sigma} \mathcal{D}(\tau) \cap K \right).$$

Then by the induction hypothesis and by the fact that the Bézier ordinates have been defined on $\mathcal{A}$, we have that all of the Bézier ordinates on σ_J are defined; therefore p_J is defined. So as before, for $\mathbf{J} \in \mathcal{D}(\sigma) \cap K'$, $K' \neq K$, we can define q_J in the same way we did when σ was a vertex. Thus, c_I has been defined for all $\mathbf{I} \in \mathbf{I}_d$.

Next, we need to show that the piecewise polynomial function defined by the c_I 's is well defined.

Suppose $p_L = q_L$ for $\mathbf{L} \in K$ and $\sum_{v \in \xi} l_v = d - \rho$.

Let $\mathbf{J} \in \mathcal{D}(\sigma) \cap K'$ and suppose that $\mathbf{J} \in \xi_L$ with $\mathbf{J} \neq \mathbf{L}$, where

$$(49) \quad \xi_L = \{\mathbf{I} \in K' : i_v \geq l_v \quad v \in \xi, \quad \xi \in \mathcal{S} \cup T, \text{ and } \sigma \prec \xi\}$$

and

$$(50) \quad \tau_J = \{\mathbf{I} \in K' : i_v \geq j_v \quad v \in \sigma\}.$$

Then $\mathbf{J} \in \xi_{\mathbf{L}}$ implies $j_v \geq l_v$, which in turn implies $\tau_{\mathbf{J}} \subset \xi_{\mathbf{L}}$. Therefore, $q_{\mathbf{J}}$ is a subpolynomial of $q_{\mathbf{L}}$. Similarly, $p_{\mathbf{J}}$ is a subpolynomial of $p_{\mathbf{L}}$. Hence, $p_{\mathbf{J}} = q_{\mathbf{J}}$ since $q_{\mathbf{L}} \cup p_{\mathbf{L}} \in C^\infty(\xi)$. Therefore, the Bézier ordinates are well defined and by construction, the piecewise polynomial defined is a superspline. $\square$

Corollary 10. We have $\dim \mathbb{S}_d^r(\mathcal{T}) = \sum_{\sigma \in \mathcal{S} \cup \mathcal{T}} |\mathcal{A}(\sigma)|$.

In view of the above corollary, to compute $\dim \mathbb{S}_d^r(\mathcal{T})$ we need only to know the cardinality of $\mathcal{A}(\sigma)$ for every $\sigma \in \mathcal{S}$. Clearly, $|\mathcal{A}(\sigma)|$ depends only on $m = \dim \sigma$. The following theorem is proved in Alfeld and Sirvent [5].

Theorem 11. We have $\dim \mathbb{S}_d^r(\mathcal{T}) = \sum_{m=0}^k \phi(m) f_m$, where f_m is the number of m -simplices of $\mathcal{S} \cup \mathcal{T}$, and for $m = 0, \dots, k$, $\phi(m) = \phi_m^k(0)$, where, with $\rho_m = r2^{k-m-1}$, the quantities $\phi_q^m(p)$ are defined recursively by

$$(51) \quad \phi_q^m(p) = \sum_{j=0}^{\rho_q - p} \binom{j+m-q-1}{j} \left[\binom{d-p-j+q}{q} - \sum_{i=0}^{q-1} \binom{q+1}{i+1} \phi_i^q(p+j) \right]$$

if $0 < q < m$, and

$$(52) \quad \phi_0^m(p) = \sum_{j=0}^{\rho_0 - p} \binom{j+m-1}{j} = \binom{\rho_0 - p + m}{m},$$

$$(53) \quad \phi_k^k(0) = \binom{d+k}{k} - \sum_{m=0}^{k-1} \binom{k+1}{m+1} \phi_m^k(0).$$

8. MINIMALLY SUPPORTED BASES

Definition 12. The star of a simplex $\sigma \in \mathcal{S} \cup \mathcal{T}$, denoted $\text{star}(\sigma)$, is the set of all k -simplices $K \in \mathcal{T}$ such that σ is a face of K .

Definition 13. A basis $\{l_\mu : \mu = 1, 2, \dots, \dim \mathbb{S}_d^r(\mathcal{T})\}$ is said to be minimally supported if for each basis function l_μ there exists a simplex $\sigma \in \mathcal{S} \cup \mathcal{T}$ such that the support of l_μ is contained in $\text{star}(\sigma)$.

The basis functions constructed in the proof of Theorem 9 are minimally supported: using the same construction, we can define cardinal supersplines

$$(54) \quad l_{\mathbf{I}} \in \mathbb{S}_d^r(\mathcal{T}) : \quad l_{\mathbf{I}}(\mathbf{J}) = \delta_{\mathbf{IJ}} \quad \forall \mathbf{I}, \mathbf{J} \in \mathcal{A}$$

so that, if $\mathbf{I} \in \mathcal{A}(\sigma)$, then $c_{\mathbf{J}} = 0 \quad \forall \mathbf{J} \in \mathbf{I}_d \setminus \text{star}(\sigma)$ and $c_{\mathbf{I}} = 1$.

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