CONVERGENCE OF AN ENERGY-PRESERVING SCHEME FOR THE ZAKHAROV EQUATIONS IN ONE SPACE DIMENSION

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ABSTRACT. An energy-preserving, linearly implicit finite difference scheme is presented for approximating solutions to the periodic Cauchy problem for the one-dimensional Zakharov system of two nonlinear partial differential equations. First-order convergence estimates are obtained in a standard "energy" norm in terms of the initial errors and the usual discretization errors.

1. INTRODUCTION

In [11] Zakharov introduced a system of equations to model the propagation of Langmuir waves in a plasma. If we denote by $N(x, t)$ ($x \in \mathbb{R}, t > 0$) the deviation of the ion density from its equilibrium value, and by $E(x, t)$ the envelope of the high-frequency electric field, then the one-dimensional system takes the form

$$(ZS.E) \quad iE_t + E_{xx} = NE,$$

$$(ZS.N) \quad N_{tt} - N_{xx} = \frac{\partial^2}{\partial x^2}(|E|^2).$$

We solve on $\{x \in \mathbb{R}, t > 0\}$ and supplement (ZS) by prescribing initial values for E , N , and N_t :

$$(1) \quad E(x, 0) = E^0(x), \quad N(x, 0) = N^0(x), \quad N_t(x, 0) = N^1(x).$$

Most of the interest to date in (ZS) stems from two particular features. Firstly, (ZS) admits solitary wave solutions [3]. Secondly, in three space dimensions, (ZS) was derived to model the collapse of caverns (cf. [11]). An intriguing and still unresolved question remains in three dimensions as to whether smooth data can generate a solution which becomes singular in *finite* time.

As is well known, (ZS) possesses the two formal invariants

$$(2) \quad \int_{-\infty}^{\infty} |E(x, t)|^2 dx = \int_{-\infty}^{\infty} |E(x, 0)|^2 dx,$$

$$(3) \quad \int_{-\infty}^{\infty} \left(|E_x|^2 + \frac{1}{2}(|v|^2 + N^2) + N|E|^2 \right) dx = \text{const},$$

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where v is given by

$$(4) \quad v = -u_x, \quad u_{xx} = N_t.$$

We know that these are sufficient for global weak existence (cf. [9]). Also from [9] the same conclusion holds in three dimensions under an additional “smallness” condition. Moreover, higher-order estimates from [9] guarantee the existence of a *smooth solution* in one dimension provided smooth data are prescribed.

It is such a smooth solution of (ZS) with periodic boundary conditions which we approximate numerically in this paper. A spectral method is used in [5]; while practical results seem very good, the convergence issue is not rigorously addressed. Our algorithm uses an approximation of “Crank-Nicolson” type on the linear parts of (ZS). We approximate the solution over a fixed but arbitrary time interval $0 \leq t \leq T$.

The nonlinear terms in (ZS) are then approximated in such a way that:

- (i) the discrete L^2 -norm (over a period) of the approximation to E is conserved; and
- (ii) a discrete analogue of the total energy is conserved.

This discrete energy will be shown to be bounded below by a positive definite form. The scheme is linearly implicit and involves only two periodic tridiagonal solvers to advance one step in time. We obtain first-order convergence estimates in the natural “energy norm” in terms of initial errors and standard discretization errors.

In the references we list several papers where conservative schemes have been employed [2, 4, 6, 8]. Related results are to be found in [1, 10].

The standard summation by parts formula is

$$\begin{aligned} \sum_{j=1}^J v_j(u_{j+1} - 2u_j + u_{j-1}) &= v_{J+1}(u_{J+1} - u_J) - v_1(u_1 - u_0) \\ &\quad - \sum_{j=1}^J (v_{j+1} - v_j)(u_{j+1} - u_j). \end{aligned}$$

The “summed” terms cancel whenever $\{u_k\}, \{v_k\}$ are J -periodic mesh functions.

Although [9] treats the Cauchy problem on all of space, the methods given there (i.e., Galerkin) could be extended to deal with the periodic case studied here. Constants depending on T and the Cauchy data are written c_T , while constants depending only on the data are generically written as c . These will change from line to line without explicit mention.

This scheme has been implemented; details will appear elsewhere.

2. THE FINITE DIFFERENCE SCHEME

Let $T > 0$ be arbitrary; we will approximate the solution to the periodic Cauchy problem for (ZS) over the time interval $0 \leq t \leq T$. We first state hypotheses on the Cauchy data and the solution:

(H0) The Cauchy data

$$E(x, 0) = E^0(x), \quad N(x, 0) = N^0(x), \quad N_t(x, 0) = N^1(x)$$

are C^∞ and L -periodic. Moreover,

$$\int_0^L N^1(x) dx = 0,$$

$$\sum_{j=1}^J N^1(jh) = 0 \quad \text{for any } h > 0 \text{ with } Jh = L.$$

(HE) The periodic Cauchy problem possesses a unique smooth global solution.

In order to write the scheme, we define

$$(5') \quad \delta u_k \equiv \Delta x^{-1}(u_{k+1} - u_k),$$

$$(5'') \quad \delta^2 u_k \equiv \Delta x^{-2}(u_{k+1} - 2u_k + u_{k-1}),$$

$$(6) \quad \lambda = \frac{\Delta t}{\Delta x}, \quad \beta = \frac{\Delta t}{\Delta x^2}$$

with $\Delta t, \Delta x > 0$. Now for J a positive integer we choose $\Delta x = \frac{L}{J}$, $\Delta t > 0$ such that

$$(7) \quad n\Delta t \leq T$$

and define $t^l = l\Delta t$, $x_j = j\Delta x$ ($l = 0, \dots, n$; $j = 0, \dots, J$).

Our scheme is

$$(8.E) \quad i \frac{E_k^{n+1} - E_k^n}{\Delta t} + \frac{1}{2} \delta^2 E_k^n + \frac{1}{2} \delta^2 E_k^{n+1} = \frac{1}{4} (N_k^n + N_k^{n+1})(E_k^n + E_k^{n+1}),$$

$$(8.N) \quad \frac{N_k^{n+1} - 2N_k^n + N_k^{n-1}}{\Delta t^2} - \frac{1}{2} \delta^2 N_k^{n+1} - \frac{1}{2} \delta^2 N_k^{n-1} = \delta^2 (|E_k^n|^2).$$

In both relations $k = 1, \dots, J$, $n \geq 0$ in the first and $n \geq 1$ in the second. Here we take E_k^n, N_k^n to be J -periodic mesh functions, i.e.,

$$E_k^n = E_j^n, \quad N_k^n = N_j^n \quad \text{if } k \equiv j \pmod{J}.$$

The scheme is supplemented with the initial values

$$(9) \quad E_k^0 = E^0(x_k),$$

$$(10) \quad N_k^0 = N^0(x_k), \quad N_k^1 = N_k^0 + \Delta t N^1(x_k).$$

We claim that the scheme is uniquely solvable: multiplying (8.N) by Δt^2 , we see that the coefficient matrix for the unknown $\{N_k^{n+1}\}_{k=1}^J$, of order $J \times J$, is

$$(11) \quad A_N = \begin{bmatrix} 1 + \lambda^2 & -\frac{\lambda^2}{2} & 0 & \cdots & -\frac{\lambda^2}{2} \\ -\frac{\lambda^2}{2} & 1 + \lambda^2 & -\frac{\lambda^2}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\lambda^2}{2} & 0 & \cdots & -\frac{\lambda^2}{2} & 1 + \lambda^2 \end{bmatrix},$$

which is invertible by Gerschgorin for any $\lambda > 0$. The coefficient matrix for the unknown $\{E_k^{n+1}\}_{k=1}^J$ has the form

$$(12) \quad iI - A_E,$$

where both matrices are square and of order $J \times J$.

A_E is symmetric and has the form

$$(13) \quad A_E = \begin{pmatrix} (A_E)_{11} & -\frac{\beta}{2} & 0 & \dots & -\frac{\beta}{2} \\ -\frac{\beta}{2} & (A_E)_{22} & -\frac{\beta}{2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{\beta}{2} & 0 & \dots & -\frac{\beta}{2} & (A_E)_{JJ} \end{pmatrix},$$

where

$$(14) \quad (A_E)_{kk} = \beta + \frac{\Delta t}{4}(N_k^n + N_k^{n+1}).$$

Since A_E has only real eigenvalues, $iI - A_E$ is invertible. Thus the scheme is uniquely solvable at each time step. Indeed, putting $n = 0$ in (8.E), we can solve for $\{E_k^1\}$, since N_k^0, N_k^1, E_k^0 are known from the data. Putting $n = 1$ in (8.N), we can then solve for $\{N_k^2\}$ and, using $\{N_k^2\}$, we can put $n = 1$ in (8.E) and solve for $\{E_k^2\}$, etc.

We summarize with

Lemma 1. *Assume the data satisfy (H0). Then the scheme (8.E), (8.N) is uniquely solvable at each time step.*

Lemma 2. *Let the data satisfy (H0). Define $\{u_k^n\}$ by*

$$\frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{\Delta x^2} = \frac{N_k^{n+1} - N_k^n}{\Delta t}, \quad k = 1, \dots, J-1,$$

$$u_0 = u_J = 0.$$

Extend $\{u_k^n\}$ by defining

$$u_k^n = u_j^n \quad \text{if} \quad k \equiv j \pmod{J}.$$

Then

$$u_k^n = -\Delta x \sum_{j=1}^{J-1} G(x_k, x_j) \frac{N_j^{n+1} - N_j^n}{\Delta t},$$

where

$$G(x, y) = \begin{cases} x(1 - \frac{y}{L}), & 0 \leq x \leq y \leq L, \\ y(1 - \frac{x}{L}), & 0 \leq y \leq x \leq L. \end{cases}$$

Proof. The proof that the given representation is indeed a solution is a straightforward computation and is omitted. The only issue is one of compatibility. Summing the definition of u_k^n , we see that it is required that

$$\sum_{k=1}^J (N_k^{n+1} - N_k^n) = 0.$$

When $n = 0$, this is true by hypotheses (H0) and (10). Using (8.N), we can write

$$N_k^{n+1} - N_k^n = N_k^n - N_k^{n-1} + \frac{\Delta t^2}{2} \delta^2 (N_k^{n+1} + N_k^{n-1} + 2|E_k^n|^2).$$

Using induction, we sum both sides over k . The sum of the first two terms on the right vanishes by the induction hypothesis; the sum of the remaining terms vanishes by periodicity. $\square$

Theorem 1. *Let the data satisfy (H0). Then the scheme (8) possesses the following two invariants:*

(a)

$$\sum_k |E_k^n|^2 \Delta x = \text{const} \quad (n\Delta t \leq T).$$

(b) Define u_k^n as in Lemma 2, so that $\delta^2 u_k^n = (N_k^{n+1} - N_k^n)/\Delta t$. Then

$$\begin{aligned} \mathcal{E}_d^{n+1} \equiv \Delta x \sum_k \left[|\delta E_k^{n+1}|^2 + \frac{1}{2}(\delta u_k^n)^2 + \frac{1}{4}\{(N_k^n)^2 + (N_k^{n+1})^2\} \right. \\ \left. + \frac{1}{2}(N_k^n + N_k^{n+1})|E_k^{n+1}|^2 \right] = \text{const} \end{aligned}$$

for $n\Delta t \leq T$. The sums run over $1 \leq k \leq J$.

Thus the discrete L^2 -norm of E^n over a period is conserved, and the form of $\mathcal{E}_d^n$ is similar to that for the exact solution in (2), (3).

We show that $\mathcal{E}_d^n$ is bounded below by a positive definite form. For this purpose, we put

$$(15) \quad \|E^n\|_2^2 \equiv \sum_k |E_k^n|^2 \Delta x,$$

$$(16) \quad \|\delta E^n\|_2^2 \equiv \sum_k |\delta E_k^n|^2 \Delta x,$$

with similar quantities for N^n . We make note of the discrete Sobolev inequality

$$(17) \quad \sup_k |u_k| \leq c \|u\|_2^{1/2} \|\delta u\|_2^{1/2}$$

valid for periodic mesh functions $\{u_k\}$. Indeed, denoting the Fourier coefficients of the mesh function u by $\{c_m\}$, we write

$$\begin{aligned} |u_k| &\leq c \left(\sum_{|m| \leq M} + \sum_{|m| > M} \right) |c_m| \\ &\leq c M^{1/2} \left(\sum_m |c_m|^2 \right)^{1/2} + c M^{-(1/2)} \left(\sum_m |m|^2 |c_m|^2 \right)^{1/2} \end{aligned}$$

and optimize on M .

The last term $\mathcal{L}$ in $\mathcal{E}_d^n$ is estimable by

$$\begin{aligned} |\mathcal{L}| &\leq \frac{1}{2} \sum_k |N_k^n| |E_k^{n+1}|^2 \Delta x + \frac{1}{2} \sum_k |N_k^{n+1}| |E_k^{n+1}|^2 \Delta x \\ &\leq \frac{\varepsilon}{4} \sum_k ((N_k^n)^2 + (N_k^{n+1})^2) \Delta x + \frac{1}{2\varepsilon} \sum_k |E_k^{n+1}|^4 \Delta x \end{aligned}$$

for any $\varepsilon > 0$. Choosing $\varepsilon = \frac{1}{2}$, we get the bound

$$|\mathcal{L}| \leq \frac{1}{8} \sum_k \Delta x ((N_k^n)^2 + (N_k^{n+1})^2) + \|E^{n+1}\|_4^4.$$

By the Sobolev inequality (17) and part (a) of the theorem,

$$\begin{aligned} \|E^{n+1}\|_4^4 &\leq c\|E^{n+1}\|_2^2\|E^{n+1}\|_\infty^2 \leq c\|E^{n+1}\|_\infty^2 \leq c\|\delta E^{n+1}\|_2^2 \\ &\leq \frac{1}{4}\|\delta E^{n+1}\|_2^2 + c. \end{aligned}$$

This gives us

Lemma 3. *There is a constant c , depending only on the data, such that the solution of the discrete scheme (8.E), (8.N) satisfies*

$$\sum_k \Delta x [|E_k^{n+1}|^2 + |\delta E_k^{n+1}|^2 + (\delta u_k^n)^2 + (N_k^n)^2 + (N_k^{n+1})^2] \leq c,$$

and hence $\sup_k |E_k^n| \leq c$.

Proof of Theorem 1. As is well known, part (a) is obtained by multiplying (8.E) by $\overline{E}_k^{n+1} + \overline{E}_k^n$, summing over k , $k = 1, \dots, J$, and taking the imaginary part.

In order to verify (b), we multiply (8.E) by $\overline{E}_k^{n+1} - \overline{E}_k^n$ and sum on k . Adding this to its conjugate, we obtain

$$(18) \quad I_n + I_{n+1} = \frac{1}{4} \sum_k (N_k^{n+1} + N_k^n) \cdot 2 \operatorname{Re}(E_k^{n+1} + E_k^n)(\overline{E}_k^{n+1} - \overline{E}_k^n),$$

where

$$I_m = \frac{1}{\Delta x^2} \operatorname{Re} \sum_k (\overline{E}_k^{n+1} - \overline{E}_k^n)(E_{k+1}^m - 2E_k^m + E_{k-1}^m) \quad (m = n, n+1).$$

The right side of (18) equals

$$(19) \quad \frac{1}{2} \sum_k (|E_k^{n+1}|^2 - |E_k^n|^2)(N_k^{n+1} + N_k^n).$$

Summing by parts, we get for the left side of (18)

$$(20) \quad I_n + I_{n+1} = -\frac{1}{\Delta x^2} \sum_k |E_{k+1}^{n+1} - E_k^{n+1}|^2 + \frac{1}{\Delta x^2} \sum_k |E_{k+1}^n - E_k^n|^2.$$

Thus (19), (20) yield the identity

$$(21) \quad -\sum_k |\delta E_k^{n+1}|^2 + \sum_k |\delta E_k^n|^2 = \frac{1}{2} \sum_k (|E_k^{n+1}|^2 - |E_k^n|^2)(N_k^{n+1} + N_k^n).$$

We obtain the contribution from $\{N_k^n\}$ by recalling from Lemma 2 that

$$(22) \quad \delta^2 u_k^n \equiv \frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{\Delta x^2} = \frac{N_k^{n+1} - N_k^n}{\Delta t}$$

and by multiplying (8.N) by $\frac{1}{2}(u_k^n + u_k^{n-1})$ and then summing on k . There results

$$(23) \quad \text{I} - \text{II} = \text{III},$$

where

$$\begin{aligned} \text{I} &= \frac{1}{2} \sum_k \frac{(N_k^{n+1} - 2N_k^n + N_k^{n-1})}{\Delta t^2} (u_k^n + u_k^{n-1}), \\ \text{II} &= \frac{1}{4} \sum_k \frac{(u_k^n + u_k^{n-1})}{\Delta x^2} [N_{k+1}^{n+1} - 2N_k^{n+1} + N_{k-1}^{n+1} + N_{k+1}^{n-1} - 2N_k^{n-1} + N_{k-1}^{n-1}], \\ \text{III} &= \frac{1}{2} \sum_k \frac{(u_k^n + u_k^{n-1})}{\Delta x^2} [|E_{k+1}^n|^2 - 2|E_k^n|^2 + |E_{k-1}^n|^2]. \end{aligned}$$

Term III is summed by parts:

$$\begin{aligned} \text{III} &= -\frac{1}{2\Delta x^2} \sum_k [(u_{k+1}^n + u_{k+1}^{n-1}) - (u_k^n + u_k^{n-1})] [|E_{k+1}^n|^2 - |E_k^n|^2] \\ (24) \quad &= -\frac{1}{2\Delta x^2} \sum_k [u_k^n + u_k^{n-1} - u_{k-1}^n - u_{k-1}^{n-1}] |E_k^n|^2 \\ &\quad + \frac{1}{2\Delta x^2} \sum_k [u_{k+1}^n + u_{k+1}^{n-1} - u_k^n - u_k^{n-1}] |E_k^n|^2, \end{aligned}$$

where we have shifted $k \rightarrow k-1$ to obtain the first sum. Thus, by (22),

$$\begin{aligned} \text{III} &= \frac{1}{2\Delta x^2} \sum_k |E_k^n|^2 [(u_{k+1}^n - 2u_k^n + u_{k-1}^n) + (u_{k+1}^{n-1} - 2u_k^{n-1} + u_{k-1}^{n-1})] \\ (25) \quad &= \frac{1}{2} \sum_k |E_k^n|^2 \left[\frac{N_k^{n+1} - N_k^n}{\Delta t} + \frac{N_k^n - N_k^{n-1}}{\Delta t} \right] \\ &= \frac{1}{2\Delta t} \sum_k |E_k^n|^2 (N_k^{n+1} - N_k^{n-1}). \end{aligned}$$

To evaluate I, we note that by (22)

$$\delta^2 u_k^n - \delta^2 u_k^{n-1} = \frac{N_k^{n+1} - N_k^n}{\Delta t} - \left(\frac{N_k^n - N_k^{n-1}}{\Delta t} \right) = \frac{N_k^{n+1} - 2N_k^n + N_k^{n-1}}{\Delta t}.$$

Thus,

$$\text{I} = \frac{1}{2\Delta t} \sum_k (u_k^n + u_k^{n-1}) [\delta^2 u_k^n - \delta^2 u_k^{n-1}]$$

and, summing this by parts, we get

$$(26) \quad \text{I} = -\frac{1}{2\Delta t} \sum_k (\delta u_k^n)^2 + \frac{1}{2\Delta t} \sum_k (\delta u_k^{n-1})^2.$$

Summing II now by parts, we find

$$\begin{aligned} \text{II} &= -\frac{1}{4\Delta x^2} \sum_k [(u_{k+1}^n + u_{k+1}^{n-1}) - (u_k^n + u_k^{n-1})] \\ &\quad \cdot [(N_{k+1}^{n+1} - N_k^{n+1}) + (N_{k+1}^{n-1} - N_k^{n-1})] \\ &= -\frac{1}{4\Delta x^2} \sum_k [u_k^n + u_k^{n-1} - u_{k-1}^n - u_{k-1}^{n-1}] [N_k^{n+1} + N_k^{n-1}] \\ &\quad + \frac{1}{4\Delta x^2} \sum_k [u_{k+1}^n + u_{k+1}^{n-1} - u_k^n - u_k^{n-1}] [N_k^{n+1} + N_k^{n-1}], \end{aligned}$$

where we have again shifted $k \rightarrow k-1$ to get the first sum. Thus, by (22),

$$\begin{aligned} \text{II} &= \frac{1}{4\Delta x^2} \sum_k (N_k^{n+1} + N_k^{n-1}) [(u_{k+1}^n - 2u_k^n + u_{k-1}^n) + (u_{k+1}^{n-1} - 2u_k^{n-1} + u_{k-1}^{n-1})] \\ &= \frac{1}{4} \sum_k (N_k^{n+1} + N_k^{n-1}) \left[\frac{N_k^{n+1} - N_k^n}{\Delta t} + \frac{N_k^n - N_k^{n-1}}{\Delta t} \right] \\ &= \frac{1}{4\Delta t} \sum_k [(N_k^{n+1})^2 - (N_k^{n-1})^2]. \end{aligned}$$

Therefore, equation (23) yields

$$\begin{aligned} & -\frac{1}{2\Delta t} \sum_k (\delta u_k^n)^2 - \frac{1}{4\Delta t} \sum_k (N_k^{n+1})^2 \\ (27) \quad & = -\frac{1}{2\Delta t} \sum_k (\delta u_k^{n-1})^2 - \frac{1}{4\Delta t} \sum_k (N_k^{n-1})^2 \\ & \quad + \frac{1}{2\Delta t} \sum_k |E_k^n|^2 (N_k^{n+1} - N_k^{n-1}). \end{aligned}$$

Now multiply this by Δt and add the result to (21) to get

$$\begin{aligned} & -\frac{1}{2} \sum_k (\delta u_k^n)^2 - \frac{1}{4} \sum_k (N_k^{n+1})^2 - \sum_k |\delta E_k^{n+1}|^2 \\ (28) \quad & = -\frac{1}{2} \sum_k (\delta u_k^{n-1})^2 - \frac{1}{4} \sum_k (N_k^{n-1})^2 - \sum_k |\delta E_k^n|^2 \\ & \quad + \frac{1}{2} \sum_k [|E_k^n|^2 (N_k^{n+1} - N_k^{n-1}) + (|E_k^{n+1}|^2 - |E_k^n|^2) (N_k^{n+1} + N_k^n)]. \end{aligned}$$

The last term here equals

$$\frac{1}{2} \sum_k |E_k^{n+1}|^2 (N_k^{n+1} + N_k^n) - \frac{1}{2} \sum_k |E_k^n|^2 (N_k^n + N_k^{n-1}).$$

Therefore, when we define $\mathcal{E}_d^{n+1}$ as in part (b) of Theorem 1, (28) implies $\mathcal{E}_d^{n+1} = \mathcal{E}_d^n$ and hence $\mathcal{E}_d^n = \mathcal{E}_d^0$ and energy is conserved. $\square$

In order to state the main theorem, we define the *errors* by

$$(29) \quad e_k^n = E(x_k, t^n) - E_k^n,$$

$$(30) \quad \eta_k^n = N(x_k, t^n) - N_k^n.$$

Here, E_k^n , N_k^n are computed from the scheme (8.E), (8.N) for $n\Delta t \leq T$, $1 \leq k \leq J$.

Lemma 4. *Let the data satisfy (H0). Define $\{U_k^n\}$ by*

$$(31) \quad \frac{U_{k+1}^n - 2U_k^n + U_{k-1}^n}{\Delta x^2} = \frac{\eta_k^{n+1} - \eta_k^n}{\Delta t}, \quad k = 1, \dots, J-1, \\ U_0 = U_J = 0.$$

Extend $\{U_k^n\}$ by defining

$$U_k^n = U_j^n \quad \text{if } k \equiv j \pmod{J}.$$

Then

$$U_k^n = -\Delta x \sum_{j=1}^{J-1} G(x_k, x_j) \frac{\eta_j^{n+1} - \eta_j^n}{\Delta t},$$

where

$$(32) \quad G(x, y) = \begin{cases} x(1 - \frac{y}{L}), & 0 \leq x \leq y \leq L, \\ y(1 - \frac{x}{L}), & 0 \leq y \leq x \leq L. \end{cases}$$

Proof. The actual computation showing that the given representation is a solution is easy and is omitted. As in Lemma 2, there remains the compatibility question. Using the definition (30) of η_k^n , we have

$$\delta^2 U_k^n = \Delta t^{-1} [N(x_k, t^{n+1}) - N_k^{n+1} - N(x_k, t^n) + N_k^n] \\ = -\delta^2 u_k^n + \Delta t^{-1} [N(x_k, t^{n+1}) - N(x_k, t^n)].$$

Therefore, as in Lemma 2, we require that

$$S \equiv \sum_{k=1}^J [N(x_k, t^{n+1}) - N(x_k, t^n)] = 0.$$

We expand $N(x, t)$ in a Fourier series with Fourier coefficients $\{c_m\}$:

$$N(x, t) = \sum_m c_m(t) \exp\left(\frac{2im\pi x}{L}\right).$$

Thus, $c_0(t)$ is proportional to $\int_0^L N(x, t) dx$. Integrating (ZS.N) over a period, we see that this integral is a linear function of t . In fact, $c_0(t)$ is constant in time in view of (H0). Now we write

$$\sum_{k=1}^J N(x_k, t) = \sum_m c_m(t) \sum_{k=1}^J \exp\left(\frac{2im\pi x_k}{L}\right)$$

and evaluate the inner sum explicitly. Using $x_k = k\Delta x = kL/J$, we see that this sum over k vanishes unless $m = 0$, in which case

$$\sum_{k=1}^J N(x_k, t) = Jc_0(t).$$

Hence $S = 0$ as desired. $\square$

The norms are defined, e.g., as $\|e^n\|_2^2 = \sum_{k=1}^J |e_k^n|^2 \Delta x$, etc.

Theorem 2. Let $T > 0$; assume (HE) and that the data satisfy (H0). Given any positive integer J , let $J\Delta x = L$ and choose $\Delta t = \Delta x$. Let E_k^n, N_k^n be computed from the scheme (8.E), (8.N), (9), (10) for $n\Delta t \leq T$. Define

$$(33) \quad \mathcal{E}^n = \frac{1}{2} [\|e^{n+1}\|_2^2 + \|\delta e^{n+1}\|_2^2 + \|\delta U^n\|_2^2 + \frac{1}{2}(\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2)].$$

(Thus, $\mathcal{E}^n$ is the (square of the) “energy norm” of the errors.)

Then there exists a constant c_T depending only on the data and T , with the property that for Δx sufficiently small, we have

$$\mathcal{E}^n \leq c_T [\mathcal{E}^0 + \Delta x^2].$$

Moreover, $\mathcal{E}^0 = O(\Delta x^2)$, and hence

$$\mathcal{E}^n \leq c_T \Delta x^2 \quad \text{as } \Delta x \rightarrow 0.$$

The proof of Theorem 2 will be given in the next section.

Remark. The choice $\Delta t = \Delta x$ allows us to easily combine several estimates. It is seen from the proof that the same estimates can be obtained provided Δt is bounded both above and below by a constant times Δx .

3. CONVERGENCE ESTIMATES, PROOF OF THE MAIN THEOREM

We begin by defining the standard discretization errors

$$(34) \quad \begin{aligned} \tau_k^n &= \frac{i}{\Delta t} (E(x_k, t^{n+1}) - E(x_k, t^n)) \\ &\quad + \frac{1}{2\Delta x^2} (E(x_{k+1}, t^n) - 2E(x_k, t^n) + E(x_{k-1}, t^n)) \\ &\quad + \frac{1}{2\Delta x^2} (E(x_{k+1}, t^{n+1}) - 2E(x_k, t^{n+1}) + E(x_{k-1}, t^{n+1})) \\ &\quad - \frac{1}{4} (N(x_k, t^n) + N(x_k, t^{n+1})) (E(x_k, t^n) + E(x_k, t^{n+1})) \end{aligned}$$

and

$$(35) \quad \begin{aligned} \sigma_k^n &= \frac{1}{\Delta t^2} (N(x_k, t^{n+1}) - 2N(x_k, t^n) + N(x_k, t^{n-1})) \\ &\quad - \frac{1}{2\Delta x^2} (N(x_{k+1}, t^{n+1}) - 2N(x_k, t^{n+1}) + N(x_{k-1}, t^{n+1})) \\ &\quad - \frac{1}{2\Delta x^2} (N(x_{k+1}, t^{n-1}) - 2N(x_k, t^{n-1}) + N(x_{k-1}, t^{n-1})) \\ &\quad - \frac{1}{\Delta x^2} (|E(x_{k+1}, t^n)|^2 - 2|E(x_k, t^n)|^2 + |E(x_{k-1}, t^n)|^2). \end{aligned}$$

As usual, these measure the amount by which the exact solutions fail to satisfy the approximate equations.

Recall that E, N are smooth solutions.

Lemma 5. We have $|\tau_k^n| + |\sigma_k^n| = O(\Delta t^2 + \Delta x^2)$ as $\Delta x, \Delta t \rightarrow 0$.

Proof. By Taylor’s theorem and (ZS.E) we can write the first three terms τ_3

in τ_k^n as

$$\begin{aligned}
\tau_3 &= i \left(E_t(x_k, t^n) + \frac{1}{2} \Delta t E_{tt}(x_k, \beta_k^n) \right) + \frac{1}{2} (E_{xx}(x_k, t^n) + O(\Delta x^2)) \\
&\quad + \frac{1}{2} (E_{xx}(x_k, t^{n+1}) + O(\Delta x^2)) \quad (t^n < \beta_k^n < t^{n+1}) \\
&= i E_t(x_k, t^n) + \frac{i \Delta t}{2} E_{tt}(x_k, \beta_k^n) + O(\Delta x^2) \\
&\quad + \frac{1}{2} [N(x_k, t^n) E(x_k, t^n) - i E_t(x_k, t^n)] \\
&\quad + \frac{1}{2} [N(x_k, t^{n+1}) E(x_k, t^{n+1}) - i E_t(x_k, t^{n+1})] \\
&= \frac{N(x_k, t^n) E(x_k, t^n) + N(x_k, t^{n+1}) E(x_k, t^{n+1})}{2} + O(\Delta x^2) \\
&\quad + \frac{i \Delta t}{2} E_{tt}(x_k, \beta_k^n) + \frac{i}{2} [E_t(x_k, t^n) - E_t(x_k, t^{n+1})] \\
&= \frac{N(x_k, t^n) E(x_k, t^n) + N(x_k, t^{n+1}) E(x_k, t^{n+1})}{2} + O(\Delta t^2 + \Delta x^2).
\end{aligned}$$

Now the result for τ_k^n will follow if

$$\begin{aligned}
&\frac{1}{2} (N(x_k, t^n) E(x_k, t^n) + N(x_k, t^{n+1}) E(x_k, t^{n+1})) \\
&\quad - \frac{1}{4} (N(x_k, t^n) + N(x_k, t^{n+1})) (E(x_k, t^n) + E(x_k, t^{n+1})) \\
&= O(\Delta t^2 + \Delta x^2).
\end{aligned}$$

Simple algebra shows that this expression equals

$$\frac{1}{4} (E(x_k, t^{n+1}) - E(x_k, t^n)) (N(x_k, t^{n+1}) - N(x_k, t^n)),$$

and hence is $O(\Delta t^2)$.

As for σ_k^n , we use Taylor's theorem again to write

$$\begin{aligned}
\sigma_k^n &= (N_{tt}(x_k, t^n) + O(\Delta t^2)) - \frac{1}{2} (N_{xx}(x_k, t^{n+1}) + O(\Delta x^2)) \\
&\quad - \frac{1}{2} (N_{xx}(x_k, t^{n-1}) + O(\Delta x^2)) - \left(\frac{\partial^2}{\partial x^2} |E(x_k, t^n)|^2 + O(\Delta x^2) \right).
\end{aligned}$$

The result follows from (ZS.N), since

$$N_{xx}(x_k, t^n) - \frac{1}{2} (N_{xx}(x_k, t^{n+1}) + N_{xx}(x_k, t^{n-1})) = O(\Delta t^2). \quad \square$$

Recall that the errors are defined by (29), (30). In order to obtain the error equations we subtract (8.E) from the definition (34) of τ_k^n to get

$$\begin{aligned}
&i \left(\frac{e_k^{n+1} - e_k^n}{\Delta t} \right) + \frac{1}{2} \delta^2 e_k^n + \frac{1}{2} \delta^2 e_k^{n+1} \\
&= \tau_k^n + \frac{1}{4} [N(x_k, t^n) + N(x_k, t^{n+1})] [E(x_k, t^n) + E(x_k, t^{n+1})] \\
&\quad - \frac{1}{4} [N_k^n + N_k^{n+1}] [E_k^n + E_k^{n+1}] \\
&= \tau_k^n + \frac{1}{4} [(\eta_k^n + \eta_k^{n+1}) (E(x_k, t^n) + E(x_k, t^{n+1})) \\
&\quad + (N_k^n + N_k^{n+1}) (e_k^n + e_k^{n+1})].
\end{aligned} \tag{36}$$

Subtracting (8.N) from (35), the definition of σ_k^n , we get similarly

$$(37) \quad \frac{\eta_k^{n+1} - 2\eta_k^n + \eta_k^{n-1}}{\Delta t^2} - \frac{1}{2}\delta^2\eta_k^{n+1} - \frac{1}{2}\delta^2\eta_k^{n-1} \\ = \sigma_k^n + \delta^2(|E(x_k, t^n)|^2 - |E_k^n|^2).$$

In a sequence of lemmas we will derive energy estimates on e and η .

Lemma 6 (L^2 -estimate of e). *There are constants c, c_T such that for $\Delta x, \Delta t$ sufficiently small,*

$$\|e^{n+1}\|_2^2 \leq (1 + c\Delta t)\|e^n\|_2^2 + c_T(\Delta t^2 + \Delta x^2)^2\Delta t \\ + c\Delta t(\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2).$$

Proof. As in Theorem 1(a), we multiply (36) by $\bar{e}_k^{n+1} + \bar{e}_k^n$, sum on k , and take the imaginary part to get

$$(38) \quad \text{I} + \text{II} = \text{III} + \text{IV},$$

where

$$\text{I} = \frac{1}{\Delta t} \operatorname{Re} \sum_k (e_k^{n+1} - e_k^n)(\bar{e}_k^{n+1} + \bar{e}_k^n) = \frac{1}{\Delta t} \sum_k (|e_k^{n+1}|^2 - |e_k^n|^2), \\ \text{II} = \frac{1}{2} \operatorname{Im} \sum_k (\bar{e}_k^{n+1} + \bar{e}_k^n)(\delta^2 e_k^{n+1} + \delta^2 e_k^n), \\ \text{III} = \operatorname{Im} \sum_k (\bar{e}_k^{n+1} + \bar{e}_k^n)\tau_k^n, \\ \text{IV} = \frac{1}{4} \operatorname{Im} \sum_k (\bar{e}_k^{n+1} + \bar{e}_k^n)[(\eta_k^n + \eta_k^{n+1})(E(x_k, t^n) + E(x_k, t^{n+1}))],$$

the last simplifying since N is real. All sums are taken over indices k with $1 \leq k \leq J$.

Term I is as desired. For III, we have from Lemma 5

$$|\text{III}| \leq c \sum_k (|e_k^{n+1}|^2 + |e_k^n|^2) + c \sum_k |\tau_k^n|^2 \\ \leq c\Delta x^{-1}(\|e^{n+1}\|_2^2 + \|e^n\|_2^2) + c_T(\Delta t^2 + \Delta x^2)^2 \cdot J,$$

and IV is easily estimable by

$$|\text{IV}| \leq c \sup_{x, t \leq T} |E(x, t)| \cdot \sum_k \frac{(|e_k^{n+1}| + |e_k^n|)\Delta x^{1/2} \cdot (|\eta_k^{n+1}| + |\eta_k^n|)\Delta x^{1/2}}{\Delta x} \\ \leq c\Delta x^{-1}[\|e^{n+1}\|_2^2 + \|e^n\|_2^2 + \|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2].$$

As before, term II vanishes upon summation by parts. Now we multiply (38) by $\Delta t \Delta x$ and use the bounds derived above to get

$$(39) \quad \|e^{n+1}\|_2^2 \leq \|e^n\|_2^2 + c\Delta t(\|e^{n+1}\|_2^2 + \|e^n\|_2^2) + c_T(\Delta t^2 + \Delta x^2)^2 \cdot J\Delta t\Delta x \\ + c\Delta t(\|\eta^{n+1}\|_2^2 + \|e^n\|_2^2 + \|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2).$$

Thus, we have

$$(40) \quad (1 - c\Delta t)\|e^{n+1}\|_2^2 \leq (1 + c\Delta t)\|e^n\|_2^2 + c_T(\Delta t^2 + \Delta x^2)^2\Delta t \\ + c\Delta t(\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2),$$

and the result follows. $\square$

When estimating the energy, we will need bounds on the discrete potentials u_k^n from Lemma 2 and U_k^n from Lemma 4.

Lemma 7. *There is a constant c depending only on the data such that*

$$\sup_k |u_k^n| \leq c.$$

Proof. We write, using the boundary condition $u_0^n = 0$,

$$|u_k^n| = \left| \sum_{j=1}^k (u_j^n - u_{j-1}^n) \right| = \left| \Delta x \sum_{j=1}^k \delta u_{j-1}^n \right| \leq \|\delta u^n\|_2 (J \Delta x)^{1/2},$$

and this is bounded by Lemma 3 and the definition of J . $\square$

Lemma 8. *Let U_k^n be defined as in Lemma 4. There is a constant c such that*

$$\sup_k |U_k^n| \leq c(\mathcal{E}^n)^{1/2}.$$

Proof. The proof is the same as that of Lemma 7, but in the last step we use the definition of $\mathcal{E}^n$ from Theorem 2. $\square$

Lemma 9 (Energy of e). *Let $h = \Delta t = \Delta x$, and define*

$$\begin{aligned} \text{II}^n &= \frac{1}{2} \text{Re} \sum_k (E(x_k, t^n) + E(x_k, t^{n+1})) (\eta_k^{n+1} + \eta_k^n) \bar{e}_k^{n+1}, \\ \text{III}^n &= \frac{1}{2} \sum_k (N_k^{n+1} + N_k^n) |e_k^{n+1}|^2. \end{aligned}$$

Then

$$\begin{aligned} & \frac{1}{2} \|\delta e^n\|_2^2 + h(\text{II}^{n-1} + \text{III}^{n-1}) - (\frac{1}{2} \|\delta e^{n+1}\|_2^2 + h(\text{II}^n + \text{III}^n)) \\ &= O[h(\mathcal{E}^n + \mathcal{E}^{n-1}) + h^3]. \end{aligned}$$

Proof. As in Theorem 1(b), we multiply (36) by $(\bar{e}_k^{n+1} - \bar{e}_k^n)$, sum over k , $k = 1, \dots, J$, add the result to its conjugate, and take the real part. There results the identity

$$\text{I}_0 = \text{I} + \text{II} + \text{III},$$

where

$$\begin{aligned} \text{I}_0 &= \text{Re} \sum_k (\bar{e}_k^{n+1} - \bar{e}_k^n) (\delta^2 e_k^n + \delta^2 e_k^{n+1}), \\ |\text{I}| &= \left| 2 \text{Re} \sum_k \tau_k^n (\bar{e}_k^{n+1} - \bar{e}_k^n) \right| \\ &\leq c_T h^2 J^{1/2} h^{-1/2} (\|e^{n+1}\|_2 + \|e^n\|_2) \leq c_T h (\mathcal{E}^n + \mathcal{E}^{n-1})^{1/2}, \\ \text{II} &= \frac{1}{2} \text{Re} \sum_k (\eta_k^{n+1} + \eta_k^n) (E(x_k, t^n) + E(x_k, t^{n+1})) (\bar{e}_k^{n+1} - \bar{e}_k^n), \\ \text{III} &= \frac{1}{2} \sum_k (N_k^n + N_k^{n+1}) (|e_k^{n+1}|^2 - |e_k^n|^2). \end{aligned}$$

We sum I_0 by parts to get

$$(41) \quad I_0 = \frac{1}{2} \sum_k |\delta e_k^n|^2 - \frac{1}{2} \sum_k |\delta e_k^{n+1}|^2.$$

Next, we rewrite term III as

$$\begin{aligned} \text{III} &= \frac{1}{2} \sum_k [(N_k^{n+1} + N_k^n) |e_k^{n+1}|^2 - (N_k^n + N_k^{n-1}) |e_k^n|^2 + (N_k^{n-1} - N_k^{n+1}) |e_k^n|^2] \\ &\equiv \text{III}^n - \text{III}^{n-1} + \frac{1}{2} \sum_k (N_k^{n-1} - N_k^{n+1}) |e_k^n|^2, \end{aligned}$$

where

$$(42) \quad \text{III}^n = \frac{1}{2} \sum_k (N_k^{n+1} + N_k^n) |e_k^{n+1}|^2.$$

Recall from the definition (Lemma 2) of u_k^n that

$$\delta^2 u_k^n = \frac{N_k^{n+1} - N_k^n}{h}.$$

Thus,

$$\delta^2 (u_k^n + u_k^{n-1}) = \frac{N_k^{n+1} - N_k^{n-1}}{h},$$

and therefore

$$\text{III} = \text{III}^n - \text{III}^{n-1} - \frac{1}{2} h \sum_k |e_k^n|^2 \delta^2 (u_k^n + u_k^{n-1}).$$

We sum by parts to get for the last term the bound

$$\begin{aligned} O \left(h \sum_k |e_k^n| |\delta e_k^n| (|\delta u_k^n| + |\delta u_k^{n-1}|) \right) &= O(\|e^n\|_\infty \|\delta e^n\|_2 (\|\delta u^n\|_2 + \|\delta u^{n-1}\|_2)) \\ &= O(\|e^n\|_2^{1/2} \|\delta e^n\|_2^{3/2}), \end{aligned}$$

where we have used Lemma 3. Hence,

$$(43) \quad \text{III} = \text{III}^n - \text{III}^{n-1} + O(\mathcal{E}^{n-1}).$$

Consider now term II. For brevity we set

$$(44) \quad w_k^n = E(x_k, t^n) + E(x_k, t^{n+1}),$$

so that

$$w_k^n - w_k^{n-1} = E(x_k, t^{n+1}) - E(x_k, t^{n-1}) = O(h).$$

We write term II as

$$\begin{aligned} \text{II} &= \frac{1}{2} \text{Re} \sum_k (\eta_k^{n+1} + \eta_k^n) w_k^n (\bar{e}_k^{n+1} - \bar{e}_k^n) \\ &= \frac{1}{2} \text{Re} \sum_k w_k^n \eta_k^{n+1} \bar{e}_k^{n+1} - \frac{1}{2} \text{Re} \sum_k w_k^{n-1} \eta_k^n \bar{e}_k^n - \frac{1}{2} \text{Re} \sum_k (w_k^n - w_k^{n-1}) \eta_k^n \bar{e}_k^n \\ &\quad + \frac{1}{2} \text{Re} \sum_k w_k^n \eta_k^n \bar{e}_k^{n+1} - \frac{1}{2} \text{Re} \sum_k w_k^n \eta_k^{n+1} \bar{e}_k^n. \end{aligned}$$

Now we add and subtract the expression

$$\frac{1}{2} \operatorname{Re} \sum_k w_k^{n-1} \eta_k^{n-1} \bar{e}_k^n$$

and define

$$(45) \quad \Pi^n = \frac{1}{2} \operatorname{Re} \sum_k w_k^n \eta_k^{n+1} \bar{e}_k^{n+1} + \frac{1}{2} \operatorname{Re} \sum_k w_k^n \eta_k^n \bar{e}_k^{n+1}.$$

Then, using Lemma 4, we can write Π as

$$(46) \quad \begin{aligned} \Pi &= \Pi^n - \Pi^{n-1} + O(\mathcal{E}^{n-1}) \\ &\quad - \frac{1}{2} \operatorname{Re} \sum_k \bar{e}_k^n [(w_k^n - w_k^{n-1}) \eta_k^{n+1} + w_k^{n-1} (\eta_k^{n+1} - \eta_k^{n-1})] \\ &= \Pi^n - \Pi^{n-1} + O(\mathcal{E}^{n-1}) + O((\mathcal{E}^{n-1})^{1/2} (\mathcal{E}^n)^{1/2}) \\ &\quad - \frac{1}{2} \operatorname{Re} \sum_k h \bar{e}_k^n w_k^{n-1} \delta^2 (U_k^n + U_k^{n-1}). \end{aligned}$$

We sum the last term here once by parts; it equals

$$\begin{aligned} &\frac{1}{2} \operatorname{Re} \sum_k h \delta (U_k^n + U_k^{n-1}) (w_k^{n-1} \delta \bar{e}_k^n + \bar{e}_{k+1}^n \delta w_k^{n-1}) \\ &= O[(\|\delta U^n\|_2 + \|\delta U^{n-1}\|_2) (\|E(t^{n-1})\|_\infty \|\delta e^n\|_2 + \|E_x(t^{n-1})\|_\infty \|e^n\|_2)] \\ &= O((\mathcal{E}^n + \mathcal{E}^{n-1})). \end{aligned}$$

Using these estimates in (46), we have

$$(47) \quad \Pi = \Pi^n - \Pi^{n-1} + O(\mathcal{E}^n + \mathcal{E}^{n-1}).$$

Finally, we multiply the relation

$$I_0 = I + \Pi + \text{III}$$

by h and use the estimates for each of these terms derived above to get

$$(48) \quad \begin{aligned} \frac{1}{2} \|\delta e^n\|_2^2 - \frac{1}{2} \|\delta e^{n+1}\|_2^2 &= O(h^3) + O[h(\mathcal{E}^n + \mathcal{E}^{n-1})] \\ &\quad + \Pi^n h + \text{III}^n h - \Pi^{n-1} h - \text{III}^{n-1} h, \end{aligned}$$

or

$$(49) \quad \begin{aligned} \frac{1}{2} \|\delta e^n\|_2^2 + h(\Pi^{n-1} + \text{III}^{n-1}) - (\frac{1}{2} \|\delta e^{n+1}\|_2^2 + h(\Pi^n + \text{III}^n)) \\ = O(h(\mathcal{E}^n + \mathcal{E}^{n-1}) + h^3), \end{aligned}$$

and this is the statement of Lemma 9. $\square$

Lemma 10 (η -energy). *Let $h = \Delta t = \Delta x$. Then*

$$\begin{aligned} &-\frac{1}{2} \|\delta U^n\|_2^2 - \frac{1}{4} (\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2) + \frac{1}{2} \|\delta U^{n-1}\|_2^2 + \frac{1}{4} (\|\eta^n\|_2^2 + \|\eta^{n-1}\|_2^2) \\ &= O(h^5 + h(\mathcal{E}^n + \mathcal{E}^{n-1})). \end{aligned}$$

Proof. Recall from Lemma 4 the relation

$$\delta^2 U_k^n = \frac{U_{k+1}^n - 2U_k^n + U_{k-1}^n}{h^2} = \frac{\eta_k^{n+1} - \eta_k^n}{h}.$$

We multiply the η -equation (37) by $\frac{1}{2}(U_k^n + U_k^{n-1})$ and sum over k to get the identity

$$(50) \quad I_1 - I_2 - I_3 = I_4 + I_5,$$

where

$$\begin{aligned} I_1 &= \frac{1}{2} \sum_k (U_k^n + U_k^{n-1}) \frac{(\eta_k^{n+1} - 2\eta_k^n + \eta_k^{n-1})}{h^2}, \\ I_2 &= \frac{1}{4} \sum_k (U_k^n + U_k^{n-1}) \delta^2 \eta_k^{n-1}, \\ I_3 &= \frac{1}{4} \sum_k (U_k^n + U_k^{n-1}) \delta^2 \eta_k^{n+1}, \\ I_4 &= \frac{1}{2} \sum_k \sigma_k^n (U_k^n + U_k^{n-1}) = O(h(\mathcal{E}^n + \mathcal{E}^{n-1})^{1/2}) \quad (\text{by Lemma 8}), \\ I_5 &= \frac{1}{2} \sum_k (U_k^n + U_k^{n-1}) \delta^2 \{|E(x_k, t^n)|^2 - |E_k^n|^2\}. \end{aligned}$$

We sum $I_2 + I_3$ by parts, with the result

$$(51) \quad I_2 + I_3 = -\frac{1}{4} \sum_k \delta(\eta_k^{n+1} + \eta_k^{n-1}) \delta(U_k^n + U_k^{n-1}).$$

Expansion of this yields

$$\begin{aligned} & -\frac{1}{4h^2} \sum_k (\eta_{k+1}^{n+1} + \eta_{k+1}^{n-1} - \eta_k^{n+1} - \eta_k^{n-1})(U_{k+1}^n + U_{k+1}^{n-1} - U_k^n - U_k^{n-1}) \\ &= -\frac{1}{4h^2} \sum_k (\eta_k^{n+1} + \eta_k^{n-1})(U_k^n + U_k^{n-1} - U_{k-1}^n - U_{k-1}^{n-1}) \\ & \quad + \frac{1}{4h^2} \sum_k (\eta_k^{n+1} + \eta_k^{n-1})(U_{k+1}^n + U_{k+1}^{n-1} - U_k^n - U_k^{n-1}), \end{aligned}$$

where we put $k \rightarrow k-1$ to get the first sum. Thus,

$$\begin{aligned} I_2 + I_3 &= \frac{1}{4h^2} \sum_k (\eta_k^{n+1} + \eta_k^{n-1}) [U_{k+1}^n - 2U_k^n + U_{k-1}^n + U_{k+1}^{n-1} - 2U_k^{n-1} + U_{k-1}^{n-1}] \\ &= \frac{1}{4} \sum_k (\eta_k^{n+1} + \eta_k^{n-1}) [\delta^2 U_k^n + \delta^2 U_k^{n-1}] \\ &= \frac{1}{4h} \sum_k (\eta_k^{n+1} + \eta_k^{n-1}) [(\eta_k^{n+1} - \eta_k^n) + (\eta_k^n - \eta_k^{n-1})] \\ &= \frac{1}{4h} \sum_k ((\eta_k^{n+1})^2 - (\eta_k^{n-1})^2) \\ &= \frac{1}{4h} \sum_k ((\eta_k^{n+1})^2 + (\eta_k^n)^2) - \frac{1}{4h} \sum_k ((\eta_k^n)^2 + (\eta_k^{n-1})^2). \end{aligned}$$

Term I_5 is summed once by parts, with the result

$$(52) \quad I_5 = -\frac{1}{2h^2} \sum_k (U_{k+1}^n + U_{k+1}^{n-1} - U_k^n - U_k^{n-1}) \cdot (|E(x_{k+1}, t^n)|^2 - |E_{k+1}^n|^2 - |E(x_k, t^n)|^2 + |E_k^n|^2),$$

and further expansion yields

$$(53) \quad \begin{aligned} I_5 &= -\frac{1}{2h} \operatorname{Re} \sum_k (\delta U_k^n + \delta U_k^{n-1}) \\ &\quad \cdot [(E(x_{k+1}, t^n) - E_{k+1}^n)(\overline{E}(x_{k+1}, t^n) + \overline{E}_{k+1}^n) \\ &\quad - (E(x_k, t^n) - E_k^n)(\overline{E}(x_k, t^n) + \overline{E}_k^n)] \\ &= -\frac{1}{2h} \operatorname{Re} \sum_k (\delta U_k^n + \delta U_k^{n-1}) \\ &\quad \cdot [e_{k+1}^n(\overline{E}(x_{k+1}, t^n) + \overline{E}_{k+1}^n) - e_k^n(\overline{E}(x_k, t^n) + \overline{E}_k^n)] \\ &= -\frac{1}{2h} \operatorname{Re} \sum_k (\delta U_k^n + \delta U_k^{n-1}) [(e_{k+1}^n - e_k^n)(\overline{E}(x_{k+1}, t^n) + \overline{E}_{k+1}^n) \\ &\quad + e_k^n(\overline{E}(x_{k+1}, t^n) - \overline{E}(x_k, t^n) + \overline{E}_{k+1}^n - \overline{E}_k^n)] \\ &= O\left(\sum_k (|\delta U_k^n| + |\delta U_k^{n-1}|)(|\delta e_k^n| + |e_k^n|(c_T + |\delta E_k^n|))\right) \\ &= O(h^{-1}(\mathcal{E}^n + \mathcal{E}^{n-1}) + h^{-1}\|e^n\|_\infty(\mathcal{E}^n + \mathcal{E}^{n-1})^{1/2}\|\delta E^n\|_2) \\ &= O(h^{-1}(\mathcal{E}^n + \mathcal{E}^{n-1})) \end{aligned}$$

by the Sobolev inequality applied to $\|e^n\|_\infty$.

Lastly, for the term I_1 we note from (31) that

$$\delta^2 U_k^n - \delta^2 U_k^{n-1} = \frac{1}{h} (\eta_k^{n+1} - \eta_k^n - (\eta_k^n - \eta_k^{n-1})) = \frac{\eta_k^{n+1} - 2\eta_k^n + \eta_k^{n-1}}{h},$$

and hence

$$(54) \quad I_1 = \frac{1}{2h} \sum_k (U_k^n + U_k^{n-1}) \delta^2 (U_k^n - U_k^{n-1}).$$

Summing by parts we get

$$(55) \quad \begin{aligned} I_1 &= -\frac{h^{-2}}{2h} \sum_k [U_{k+1}^n + U_{k+1}^{n-1} - U_k^n - U_k^{n-1}] \\ &\quad \cdot [U_{k+1}^n - U_{k+1}^{n-1} - (U_k^n - U_k^{n-1})]. \end{aligned}$$

This can be rewritten as

$$(56) \quad I_1 = -\frac{1}{2h} \sum_k [(\delta U_k^n)^2 - (\delta U_k^{n-1})^2] = -\frac{1}{2h^2} [\|\delta U^n\|_2^2 - \|\delta U^{n-1}\|_2^2].$$

Returning now to (50), we multiply it by h^2 to get

$$(57) \quad \begin{aligned} &-\frac{1}{2}\|\delta U^n\|_2^2 + \frac{1}{2}\|\delta U^{n-1}\|_2^2 \\ &- \frac{1}{4}(\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2) + \frac{1}{4}(\|\eta^n\|_2^2 + \|\eta^{n-1}\|_2^2) \\ &= O(h(\mathcal{E}^n + \mathcal{E}^{n-1}) + h^5). \end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 2. Let us define $h = \Delta t = \Delta x$ and

$$(58) \quad H^{n-1} = \frac{1}{2} \|\delta e^n\|_2^2 + \frac{1}{2} \|\delta U^{n-1}\|_2^2 + \frac{1}{4} (\|\eta^n\|_2^2 + \|\eta^{n-1}\|_2^2).$$

Recall the definitions of the terms II^n , III^n from Lemma 9. Adding the conclusions of Lemmas 9 and 10, we get

$$(59) \quad \begin{aligned} H^n + h(\text{II}^n + \text{III}^n) &= H^{n-1} + h(\text{II}^{n-1} + \text{III}^{n-1}) \\ &\quad + O(h(\mathcal{E}^n + \mathcal{E}^{n-1}) + h^3), \end{aligned}$$

where, from (33),

$$(60) \quad \mathcal{E}^n = \frac{1}{2} \|e^{n+1}\|_2^2 + H^n.$$

Now, for a (large) positive constant γ (to be chosen below) set

$$(61) \quad \widehat{\mathcal{E}}^n \equiv \gamma \|e^{n+1}\|_2^2 + H^n + h(\text{II}^n + \text{III}^n).$$

From (59) and Lemma 6 it follows that

$$(62) \quad \begin{aligned} \widehat{\mathcal{E}}^n &\leq \gamma(1 + ch) \|e^n\|_2^2 + \gamma c_T h^5 + c\gamma h (\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2) \\ &\quad + H^{n-1} + h(\text{II}^{n-1} + \text{III}^{n-1}) + O(h(\mathcal{E}^n + \mathcal{E}^{n-1}) + h^3). \end{aligned}$$

Now we estimate II^n , III^n easily by

$$(63) \quad \begin{aligned} h|\text{II}^n| &= \left| \frac{h}{2} \text{Re} \sum_k (E(x_k, t^n) + E(x_k, t^{n+1})) (\eta_k^{n+1} + \eta_k^n) \bar{e}_k^{n+1} \right| \\ &\leq c(\|E(t^n)\|_\infty + \|E(t^{n+1})\|_\infty) \|\eta^{n+1} + \eta^n\|_2 \|e^{n+1}\|_2 \\ &\leq \frac{1}{16} (\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2) + c \|e^{n+1}\|_2^2 \end{aligned}$$

(with a constant c depending only on the data), and

$$\begin{aligned} h|\text{III}^n| &\leq \left| \frac{h}{2} \sum_k (N_k^{n+1} + N_k^n) |e_k^{n+1}|^2 \right| \\ &\leq c \|e^{n+1}\|_\infty \|N^{n+1} + N^n\|_2 \|e^{n+1}\|_2 \\ &\leq c (\|N^{n+1}\|_2 + \|N^n\|_2) \|e^{n+1}\|_2^{3/2} \|\delta e^{n+1}\|_2^{1/2} \end{aligned}$$

by the Sobolev inequality. Since the first factor is bounded by Lemma 3, we obtain

$$(64) \quad h|\text{III}^n| \leq \frac{1}{8} \|\delta e^{n+1}\|_2^2 + c \|e^{n+1}\|_2^2$$

with c depending only on the data. Adding (63) to (64), we obtain

$$(65) \quad \begin{aligned} h(|\text{II}^n| + |\text{III}^n|) &\leq \frac{1}{8} \|\delta e^{n+1}\|_2^2 + \frac{1}{16} (\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2) + c \|e^{n+1}\|_2^2 \\ &\leq \frac{1}{4} H^n + c \|e^{n+1}\|_2^2 \end{aligned}$$

by the definition (58) of H^n . It follows that $\widehat{\mathcal{E}}^n$ is strictly positive for a sufficiently large choice of γ , depending only on the data.

In fact, we can choose γ large enough so that $\gamma > 1$ and

$$(66) \quad \widehat{\mathcal{E}}^n \geq \frac{c}{2} \|e^{n+1}\|_2^2 + \frac{3}{4} H^n$$

with a constant $c > 0$ depending only on the data and on γ .

Hence, from (62),

$$(67) \quad \widehat{\mathcal{E}}^n \leq \widehat{\mathcal{E}}^{n-1} + c_T \gamma h (\mathcal{E}^n + \mathcal{E}^{n-1}) + c_T \gamma h^3.$$

Now from its definition, we have, since $\gamma > 1$,

$$(68) \quad \begin{aligned} \mathcal{E}^n &= \frac{1}{2} \|e^{n+1}\|_2^2 + H^n < \gamma \|e^{n+1}\|_2^2 + H^n \\ &= \widehat{\mathcal{E}}^n - h(\text{II}^n + \text{III}^n) \leq \widehat{\mathcal{E}}^n + \frac{1}{4} H^n + c \|e^{n+1}\|_2^2, \end{aligned}$$

where we have used (65). Since $H^n \leq \mathcal{E}^n$ by (60), we conclude that

$$(69) \quad \frac{3}{4} \mathcal{E}^n \leq \widehat{\mathcal{E}}^n + c \|e^{n+1}\|_2^2 \leq c_\gamma \widehat{\mathcal{E}}^n$$

in view of (66). For any such (fixed) choice of γ , we obtain from (67)

$$(1 - c_T h) \widehat{\mathcal{E}}^n \leq (1 + c_T h) \widehat{\mathcal{E}}^{n-1} + c_T h^3.$$

It follows that for $h = \Delta t = \Delta x$ sufficiently small, depending only on T and the data, we have

$$\widehat{\mathcal{E}}^n \leq c_T [\widehat{\mathcal{E}}^0 + h^2].$$

Since $(\widehat{\mathcal{E}}^n)^{1/2}$ is equivalent to $(\mathcal{E}^n)^{1/2}$, the first part of the proof is complete.

It remains to estimate $\mathcal{E}^0$. From (29), (30) and (9), (10) we have

$$e_k^0 = 0, \quad \eta_k^0 = 0, \quad \eta_k^1 = O(h^2).$$

Thus, $\|\eta^1\|_2^2 + \|\eta^0\|_2^2 = O(h^4)$. From Lemma 6 with $n = 0$, $\|e^1\|_2^2 = O(h^5)$, and hence

$$\|\delta e^1\|_2^2 = h^{-1} \sum_{k=1}^J |e_{k+1}^1 - e_k^1|^2 \leq 4h^{-1} \sum_{k=1}^J |e_k^1|^2 = O(h^3).$$

Finally, we bound $\|\delta U_k^n\|_2$. We multiply the definition of U_k^n by U_k^n , sum over k , and then sum by parts to get

$$\|\delta U^0\|_2^2 = - \sum_{k=1}^J U_k^0 (\eta_k^1 - \eta_k^0) = \sum_{k=1}^{J-1} \sum_{j=1}^{J-1} G(x_k, x_j) \eta_k^1 \eta_j^1,$$

where we have used Lemma 4 again. Since G is continuous, it follows from general considerations (or from explicit computation, using $\eta_k^1 = O(h^2)$) that the last expression is $O(h^2)$, and this completes the proof. $\square$

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