AN EXTENSION OF THE THEOREM ON PRIMITIVE DIVISORS IN ALGEBRAIC NUMBER FIELDS

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ABSTRACT. The theorem about primitive divisors in algebraic number fields is generalized in the following manner. Let A, B be algebraic integers, $(A, B) = 1$, $AB \neq 0$, A/B not a root of unity, and ζ_k a primitive root of unity of order k . For all sufficiently large n , the number $A^n - \zeta_k B^n$ has a prime ideal factor that does not divide $A^m - \zeta_k^j B^m$ for arbitrary $m < n$ and $j < k$.

The analogue of Zsigmondy's theorem in algebraic number fields [3] asserts the following.

If A, B are algebraic integers, $(A, B) = 1$, $AB \neq 0$, and A/B of degree d is not a root of unity, there exists a constant $n_0(d)$ such that for $n > n_0(d)$, $A^n - B^n$ has a prime ideal factor that does not divide $A^m - B^m$ for $m < n$.

This theorem will be extended as follows:

Theorem. Let K be an algebraic number field, A, B integers of K , $(A, B) = 1$, $AB \neq 0$, A/B of degree d not a root of unity, and ζ_k a primitive k th root of unity in K . For every $\varepsilon > 0$ there exists a constant $c(d, \varepsilon)$ such that if $n > c(d, \varepsilon)(1 + \log k)^{1+\varepsilon}$, there exists a prime ideal of K that divides $A^n - \zeta_k B^n$, but does not divide $A^m - \zeta_k^j B^m$ for $m < n$ and arbitrary j .

The above theorem implies the finiteness of the number of solutions of generalized cyclotomic equations considered by Browkin [1, p. 236].

The proof will follow closely the proof given in [3]. Let $\mathbb{Q}(A/B) = K_0$, $\frac{A}{B} = \frac{\alpha}{\beta}$, where $\alpha, \beta \in K_0$, α, β are integers, and $(\alpha, \beta) = \mathfrak{d}$. Let S and S_0 be the set of all isomorphic injections of $K_0(\zeta_k)$ and K_0 , respectively, in the complex field, and set

$$w(\alpha/\beta) = \log \prod_{\sigma \in S_0} \max\{|\alpha^\sigma|, |\beta^\sigma|\} - \log N\mathfrak{d},$$

where N denotes the absolute norm in K_0 . Here, $w(\alpha/\beta)$ is the logarithm of the Mahler measure of α/β and so it is independent of the choice of α, β in K_0 .

Lemma 1. If $|\alpha| = |\beta|$, but α/β is not a root of unity, then

$$\log |\alpha^n - \zeta_k \beta^n| = n \log |\beta| + O(d + w(\alpha/\beta)) \log kn,$$

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where the constant in the O -symbol depends only on d and is effectively computable.

Lemma 2. If $|\alpha| \neq |\beta|$, then

$$\log |\alpha^n - \zeta_k \beta^n| = n \log \max\{|\alpha|, |\beta|\} + O(d^2 + dw(\alpha/\beta)),$$

where the constant in the O -symbol is absolute and effectively computable.

The next lemma is just quoted from [3], where it occurs as Lemma 4.

Lemma 3. Let $\phi_n(x, y)$ be the n th cyclotomic polynomial in homogeneous form. If $\mathfrak{P}$ is a prime ideal of K , $n > 2(2^d - 1)$, $\mathfrak{P} \nmid \phi_n(A, B)$, and $\mathfrak{P}$ is not a primitive divisor of $A^n - B^n$, then

$$\text{ord}_{\mathfrak{P}} \phi_n(A, B) \leq \text{ord}_{\mathfrak{P}} n.$$

Finally, we prove

Lemma 4. Let

$$\psi_n(x, y; \zeta_k) = \prod_{\substack{(j, n)=1 \\ j \equiv 1 \pmod k}} (x - \zeta_{kn}^j y).$$

We have

$$(1) \quad \psi_n(x, y; \zeta_k) = \prod_{\substack{m|n \\ (m, k)=1}} (x^{n/m} - \zeta_k^{\overline{m}} y^{n/m})^{\mu(m)},$$

where $m\overline{m} \equiv 1 \pmod k$ and

$$\deg \psi_n = \varphi(n) \frac{(k, n)}{\varphi((k, n))}.$$

Proof. The right-hand side of (1) can be written as

$$\prod_{\substack{m|n \\ (m, k)=1}} \prod_{i=0}^{n/m-1} (x - \zeta_{n/m}^i \zeta_{kn/m}^{\overline{m}} y)^{\mu(m)}.$$

A factor $x - \zeta_{kn}^j y$ occurs in this product with the exponent

$$E = \sum_{\substack{m|n \\ (m, k)=1}} \mu(m) \sum_{\substack{i=0 \\ m(ki+\overline{m}) \equiv j \pmod{kn}}}^{n/m-1} 1.$$

Now,

$$\sum_{\substack{i=0 \\ m(ki+\overline{m}) \equiv j \pmod{kn}}}^{n/m-1} 1 = \begin{cases} \sum_{\substack{i=0 \\ ki+\overline{m} \equiv j/m \pmod{kn/m}}}^{n/m-1} 1 & \text{if } m|j, \\ 0 & \text{otherwise,} \end{cases}$$

and if $m|j$,

$$\sum_{\substack{i=0 \\ ki+\overline{m} \equiv j/m \pmod{kn/m}}}^{n/m-1} 1 = \begin{cases} 1 & \text{if } j \equiv 1 \pmod k, \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$E = \begin{cases} \sum_{m|n, m|j} \mu(m) & \text{if } j \equiv 1 \pmod{k}, \\ 0 & \text{otherwise,} \end{cases}$$

and finally

$$E = \begin{cases} 1 & \text{if } (n, j) = 1, \quad j \equiv 1 \pmod{k}, \\ 0 & \text{otherwise,} \end{cases}$$

which proves the first part of the lemma.

In order to prove the second part, we notice that there are exactly $\varphi(n) \frac{(k, n)}{\varphi((k, n))}$ positive integers $j \leq kn$ such that $(n, j) = 1$, $j \equiv 1 \pmod{k}$. $\square$

Lemma 5. For every $\varepsilon > 0$ there exists $c(d, \varepsilon)$ such that, if

$$n > c(d, \varepsilon)(1 + \log k)^{1+\varepsilon},$$

then we have

$$(2) \quad |N_{K/\mathbb{Q}} \psi_n(A, B; \zeta_k)| > (nk)^{[K:\mathbb{Q}]}. \quad \text{Proof. By Lemma 4,}$$

$$\psi_n(A, B; \zeta_k) = \left(\frac{B}{\beta}\right)^{\varphi(n)(k, n)/\varphi((k, n))} \psi_n(\alpha, \beta; \zeta_k),$$

and since $\left(\frac{B}{\beta}\right) = \mathfrak{d}^{-1}$, we have

$$(\psi_n(A, B; \zeta_k)) = \mathfrak{d}^{-\varphi(n)(k, n)/\varphi((k, n))} \psi_n(\alpha, \beta; \zeta_k),$$

$$\begin{aligned} & \frac{1}{[K:K_0(\zeta_k)]} \log |N_{K/\mathbb{Q}} \psi_n(A, B; \zeta_k)| \\ &= \log |N_{K_0(\zeta_k)/\mathbb{Q}} \psi_n(\alpha, \beta; \zeta_k)| - [K_0(\zeta_k):K_0] \varphi(n) \frac{(k, n)}{\varphi((k, n))} \log N\mathfrak{d} \\ &= \sum_{\sigma \in S} \sum_{\substack{m|n \\ (m, k)=1}} \mu(m) \log |(\alpha^\sigma)^{n/m} - \zeta_k^{\overline{m}} (\beta^\sigma)^{n/m}| \\ &\quad - [K_0(\zeta_k):K_0] \varphi(n) \frac{(k, n)}{\varphi((k, n))} \log N\mathfrak{d} \\ &= \sum_{\sigma \in S} \sum_{\substack{m|n \\ (m, k)=1}} \mu(m) \left(\frac{n}{m} \log \max\{|\alpha^\sigma|, |\beta^\sigma|\} + O\left(d + w\left(\frac{\alpha}{\beta}\right)\right) \log kn \right) \\ &\quad - [K_0(\zeta_k):K_0] \varphi(n) \frac{(k, n)}{\varphi((k, n))} \log N\mathfrak{d} \\ &= [K_0(\zeta_k):K_0] \left(\varphi(n) \frac{(k, n)}{\varphi((k, n))} w\left(\frac{\alpha}{\beta}\right) + O\left(d + w\left(\frac{\alpha}{\beta}\right)\right) 2^{\nu(n)} \log kn \right), \end{aligned}$$

where the constant in O depends only on d and is effectively computable. Now, by Dobrowolski's theorem [2], if α/β is an integer, then

$$\begin{aligned} w\left(\frac{\alpha}{\beta}\right) &= \log \prod_{\sigma \in S_0} \max\left\{\left|\frac{\alpha^\sigma}{\beta^\sigma}\right|, 1\right\} \\ &\geq \log \left(1 + c_1 \left(\frac{\log \log ed}{\log d}\right)^3\right) \geq c_2 \left(\frac{\log \log ed}{\log d}\right)^3, \end{aligned}$$

where c_1 and c_2 are absolute constants.

If α/β is not an integer, then $(\beta) \neq \mathfrak{d}$ and

$$w\left(\frac{\alpha}{\beta}\right) \geq \log N\beta - \log N\mathfrak{d} \geq \log 2.$$

Thus, in both cases,

$$w\left(\frac{\alpha}{\beta}\right) \geq c_2 \left(\frac{\log \log ed}{\log d}\right)^3,$$

provided $c_2 \leq \log 2$.

Since for every $\varepsilon > 0$

$$\frac{\varphi(n)}{2^{\nu(n)}} > c_3(\varepsilon)n^{1-\varepsilon},$$

it follows that for $n > c(d, \varepsilon)(1 + \log k)^{1+\varepsilon}$

$$\log |N_{K/\mathbb{Q}}\psi_n(A, B; \zeta_k)| > [K : \mathbb{Q}] \log nk,$$

which proves the lemma. $\square$

Proof of the theorem. By Lemma 5, for $n > c(d, \varepsilon)(\log k)^{1+\varepsilon}$ we have (2), and thus $\psi_n(A, B; \zeta_k)$ has a prime ideal factor $\mathfrak{P}$ in K such that

$$\text{ord}_{\mathfrak{P}} \psi_n(A, B; \zeta_k) > \text{ord}_{\mathfrak{P}} kn.$$

But $\mathfrak{P} | \psi_n(A, B; \zeta_k) | \phi_{kn}(A, B)$, hence by Lemma 3 we have that $\mathfrak{P}$ is a primitive prime divisor of $A^{kn} - B^{kn}$ and thus does not divide $A^m - \zeta_k^j B^m$ for $m < n$ and arbitrary j . On the other hand,

$$\mathfrak{P} | \psi_n(A, B; \zeta_k) | A^n - \zeta_k B^n,$$

thus $\mathfrak{P}$ has the desired property. $\square$

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