THE RELATIVE CLASS NUMBERS OF IMAGINARY CYCLIC FIELDS OF DEGREES 4, 6, 8, AND 10

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ABSTRACT. We express the relative class number of an imaginary abelian number field K of prime power conductor as a sort of Maillet determinant. Thereby we obtain explicit relative class number formulas for fields K of conductor p , $p \geq 3$ prime, and degree $2d = [K: \mathbb{Q}] \leq 10$, in terms of sums of $2d$ -power residues. In particular, tables are given for $p \leq 10000$.

INTRODUCTION

Let p, m be in $\mathbb{N}$, p prime. In a number of papers the relative class number of the p^m th cyclotomic field has been expressed as a rational determinant (Maillet's determinant; cf. [1, 8, 10, 11], see also [12, 3]). Moreover, an explicit relative class number formula in terms of quartic power residues modulo p has been given for imaginary cyclic quartic fields of conductor p [9, 7]. The aim of the present article is to study a generalization of Maillet's determinant that yields relative class number formulas for *arbitrary* imaginary abelian fields K of conductor p^m (Theorem 1). By specializing these formulas to fields K of degree $[K: \mathbb{Q}] = 2d$ and conductor p , we obtain *explicit* relative class number formulas in the cases $d = 1, 2, 3, 4, 5$ (the formula for $d = 1$ is well known, of course). Our formulas are used to compute relative class number tables for $d = 3, 4, 5$ and $p \leq 10000$ (Tables 1–3 in the Supplement section; the respective table for $d = 2$ can be found in [6]).

1. GENERALIZED MAILLET DETERMINANTS

Let K be an imaginary abelian number field of conductor n . In particular, K is contained in the n th cyclotomic field $\mathbb{Q}(\zeta_n)$, $\zeta_n = e^{2\pi i/n}$. By G_n we denote the Galois group

$$G_n = \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}).$$

Let $(\mathbb{Z}/n\mathbb{Z})^\times$ be the prime residue group mod n . There is a canonical isomorphism

$$(\mathbb{Z}/n\mathbb{Z})^\times \rightarrow G_n$$

which maps the residue class $\bar{k}$, $k \in \mathbb{Z}$, onto σ_k , σ_k being defined by $\sigma_k(\zeta_n) = \zeta_n^k$. For this reason we shall frequently identify $\bar{k}$ with σ_k , and thus the group $(\mathbb{Z}/n\mathbb{Z})^\times$ with G_n .

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Let $H \subseteq G_n$ be the Galois group

$$H = \text{Gal}(\mathbb{Q}(\zeta_n)/K) = \{\sigma \in G_n; \sigma|_K = \text{id}\}.$$

Since K is imaginary, H does not contain complex conjugation $\sigma_{-1} = \overline{}$. Therefore, $K^+ = K \cap \mathbb{R}$ is a proper subfield of K , and $H^+ = \langle \{\overline{-1}\} \cup H \rangle$ is the Galois group $H^+ = \text{Gal}(\mathbb{Q}(\zeta_n)/K^+)$. The group index $[H^+ : H]$ equals 2. We write $d = [K^+ : \mathbb{Q}]$, which means $[K : \mathbb{Q}] = 2d$. Now let X_n be the character group of G_n , $X \subseteq X_n$ the character group of K (i.e., the character group of G_n/H), X^+ the character group of K^+ , and $X^- = X \setminus X^+$. We fix an arbitrary character ψ in X^- . This is the same as saying $\psi(\bar{k}) = 1$ for each $\bar{k} \in H$, and $\psi(\overline{-1}) = -1$.

For a given number $k \in \mathbb{Z}$, let $[k] = [\bar{k}]$ be defined by

$$k \equiv [k] \pmod{n} \quad \text{and} \quad [k] \in \{0, 1, \dots, n-1\}.$$

If $(k, n) = 1$, we put

$$E_k = \psi(\bar{k}) \sum_{\substack{j=1 \\ \bar{j} \in H}}^n (2[kj] - n).$$

Proposition 1. *With the above notations,*

$$E_k = \sum_{\substack{j=1 \\ \bar{j} \in \bar{k}H^+}}^n \psi(\bar{j})[j].$$

In particular, E_k depends on the residue class of $\bar{k}$ modulo H^+ only.

Proof. Since $\psi(\bar{j}) = 1$ for all elements $\bar{j} \in H$, one obtains

$$E_k = \sum_{\bar{j} \in H} \psi(\bar{k}\bar{j})(2[kj] - n).$$

Now E_k can be rewritten as

$$\begin{aligned} E_k &= \frac{1}{2} \sum_{\bar{j} \in H} (\psi(\bar{k}\bar{j})(2[kj] - n) + \psi(\overline{-k}\bar{j})(2[-kj] - n)) \\ &= \frac{1}{2} \sum_{\bar{j} \in \bar{k}H^+} \psi(\bar{j})(2[j] - n) \\ &= \sum_{\bar{j} \in \bar{k}H^+} \psi(\bar{j})[j] - \frac{n}{2} \psi(\bar{k}) \sum_{\bar{j} \in H^+} \psi(\bar{j}). \end{aligned}$$

However, the last sum is 0, since $\psi(\bar{j}) = 1$ for $\bar{j} \in H$, $\psi(\bar{j}) = -1$ for $\bar{j} \in H^+ \setminus H$, and $|H| = |H^+ \setminus H| = d$. $\square$

In view of Proposition 1 we may write $E_k = E_r$, where r is the residue class of $\bar{k}$ modulo H^+ .

Let $\mathcal{R} \subseteq \mathbb{Z}$ be a system of representatives of G_n/H^+ . In particular, $|\mathcal{R}| = d$. Suppose, moreover, that $\mathcal{R}$ is ordered in some way. We put

$$D = \det(E_{kj})_{k, j \in \mathcal{R}} = \det(E_{rs})_{r, s \in G_n/H^+},$$

and

$$D^* = \det(E_{rs-1})_{r,s \in G_n/H^+}.$$

Finally, let $\delta = |\{k \in \mathcal{R}; \bar{k}^2 \in H^+\}|$. We get

Proposition 2. *In the above situation,*

$$D = (-1)^{(d-\delta)/2} \cdot D^*.$$

Proof. Consider the permutation

$$\rho: G_n/H^+ \rightarrow G_n/H^+$$

of G_n/H^+ , defined by $\rho(r) = r^{-1}$. Clearly, $D = \text{sign}(\rho) \cdot D^*$. But $\text{sign}(\rho) = (-1)^\varepsilon$, with $\varepsilon = |\{r \in G_n/H^+; r \neq r^{-1}\}|/2 = (d - |\{r \in G_n/H^+; r^2 = 1\}|)/2 = (d - \delta)/2$. $\square$

For a prime divisor p of n , let e_p (resp. e_p^+) be the ramification index of p in K (resp. K^+). Similarly, g_p (resp. g_p^+) denotes the number of prime divisors of p in K (resp. K^+).

Theorem 1. *Let the above notations hold. Then $D^* = 0$ if there is a p , $p|n$, with $g_p = 2g_p^+$. Otherwise,*

$$D^* = (-2n)^d 2^\kappa h^- / (Q \cdot w),$$

with

$$\begin{aligned} \kappa &= \sum \{g_p^+; p|n, g_p = g_p^+, e_p = e_p^+\}, \\ h^- &= \text{relative class number of } K, \\ Q &= \text{unit index of } K, \\ w &= \text{number of roots of unity in } K \text{ (for notation, cf. [5])}. \end{aligned}$$

Proof. By its definition, D^* is a group determinant belonging to the abelian group G_n/H^+ , which means (cf. [5, p. 23])

$$D^* = \prod_{\chi \in X^+} \left(\sum_{r \in G_n/H^+} \chi(r) \cdot E_r \right).$$

It is easy to see that

$$\sum_{r \in G_n/H^+} \chi(r) \cdot E_r = \sum_{\substack{k=1 \\ (k,n)=1}}^n \chi \psi(\bar{k}) \cdot k,$$

whence

$$D^* = \prod_{\chi \in X^+} \left(\sum_{\substack{k=1 \\ (k,n)=1}}^n \chi(\bar{k}) \cdot k \right).$$

Now let f_χ denote the conductor of χ , and χ_f the primitive character of $(\mathbb{Z}/f_\chi\mathbb{Z})^\times$ that belongs to χ . A reduction formula of Hasse (cf. [5, p. 18]) says

$$\sum_{\substack{k=1 \\ (k,n)=1}}^n \chi(\bar{k}) \cdot k = n \cdot \prod_{p|n} (1 - \chi_f(p)) \cdot B_\chi,$$

where B_χ is the generalized Bernoulli number

$$B_\chi = \sum_{k=1}^{f_\chi} \chi_f(\bar{k}) \cdot k / f_\chi.$$

The product $\prod_{\chi \in X^-} (1 - \chi_f(p))$ can be evaluated in a well-known way (cf., e.g., [2]). We obtain

$$\prod_{\chi \in X^-} (1 - \chi_f(p)) = \begin{cases} 0 & \text{if } g_p = 2g_p^+, \\ 1 & \text{if } e_p = 2e_p^+, \\ 2 & \text{if } g_p = g_p^+, e_p = e_p^+. \end{cases}$$

Finally,

$$\prod_{\chi \in X^-} B_\chi = (-2)^d \cdot h^- / (Q \cdot w)$$

(cf. [5, p. 12]). On putting these results together, one gets the theorem. $\square$

Let us now consider the *special case* $n = p^m$, p an odd prime. Then K is a cyclic extension of $\mathbb{Q}$ with $[K : \mathbb{Q}] = 2d$. Since $e_p = 2e_p^+$, the number κ is 0. According to [5, p. 68], the unit index Q equals 1. The number w is given by the following

Proposition 3. *With the above conventions,*

$$w = \begin{cases} 2 \cdot p^m & \text{if } K = \mathbb{Q}(\zeta_n), \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Assume that K contains a root of unity different from ± 1 . Then K contains a root of unity of p -power order. Therefore, $\zeta_p \in K$. But then $[\mathbb{Q}(\zeta_n) : K] = p^k$ for some k , $0 \leq k \leq m-1$. Since $\mathbb{Q}(\zeta_n)$ is cyclic over $\mathbb{Q}$, there is only one subfield K of $\mathbb{Q}(\zeta_n)$ with this property, viz., $K = \mathbb{Q}(\zeta_{p^{m-k}})$. However, $n = p^m$ is the conductor of K ; hence $k = 0$. $\square$

On collecting the above observations, we obtain the

Corollary to Theorem 1. *Let $p \geq 3$ be prime, $n = p^m$, $m \geq 1$, and K be an imaginary abelian field of conductor n and degree $2d = [K : \mathbb{Q}]$. Then*

$$D^* = \begin{cases} (-1)^d (2n)^{d-1} \cdot h^- & \text{if } K = \mathbb{Q}(\zeta_n), \\ (-n)^d \cdot 2^{d-1} \cdot h^- & \text{otherwise.} \end{cases}$$

2. THE CASE OF A PRIME CONDUCTOR

Let, in particular, $n = p \geq 3$ be prime, which implies that $2d | (p-1)$. Then

$$H = G_p^{2d} = \{\bar{k}^{2d}; \bar{k} \in G_p\}.$$

Since $\overline{-1}$ is not in H , we get $(-1)^{(p-1)/(2d)} \equiv -1 \pmod{p}$, and $p \equiv 1 + 2d \pmod{4d}$. If, conversely, $p \equiv 1 + 2d \pmod{4d}$, there is a uniquely determined subfield K of $\mathbb{Q}(\zeta_p)$ with $[K : \mathbb{Q}] = 2d$, and K is imaginary. We now choose a number $g \in \mathbb{Z} \setminus p\mathbb{Z}$ such that $\mathcal{R} = \{1, g, g^2, \dots, g^{d-1}\}$ is a system of representatives for G_p/H^+ . This is the same as saying that

$$(*) \quad g^{k(p-1)/d} \not\equiv 1 \pmod{p}$$

for each $k \in \{1, \dots, d-1\}$. We define

$$(**) \quad F_k = \sum_{j=1}^{p-1} (2[g^k j^{2d}] - p), \quad k \in \mathbb{Z}.$$

Then $\psi(\bar{g}^k) \cdot F_k = 2d \cdot E_{g^k}$. By Proposition 2 and the corollary of Theorem 1,

$$(***) \quad \det(\psi(\bar{g}^{j+k}) F_{j+k})_{j,k=0,\dots,d-1} = (2d)^d \cdot D = c \cdot h^-,$$

with

$$c = \begin{cases} (-1)^{(3d-1)/2} \cdot 2^{2d-1} \cdot p^{d-1} \cdot d^d & \text{if } d = (p-1)/2, \\ & p \equiv 3 \pmod{4}, \\ (-1)^{(3d-2)/2} \cdot 2^{2d-1} \cdot p^{d-1} \cdot d^d & \text{if } d = (p-1)/2, \\ & p \equiv 1 \pmod{4}, \\ (-1)^{(3d-1)/2} \cdot 2^{2d-1} \cdot (pd)^d & \text{if } d < (p-1)/2, \\ & d \text{ odd}, \\ (-1)^{(3d-2)/2} \cdot 2^{2d-1} \cdot (pd)^d & \text{if } d < (p-1)/2, \\ & d \text{ even}. \end{cases}$$

Examples. For small numbers d it is easy to write down the determinant in (***) term by term. We do so for $d = 1, \dots, 5$.

1. Let $d = 1$, i.e., $p \equiv 3 \pmod{4}$, and $p > 3$. In this case (***) means

$$F_0 = \sum_{j=1}^{p-1} (2[j^2] - p) = -2p \cdot h^-.$$

This is well known (cf. [4, p. 387]).

2. Let $d = 2$, i.e., $p \equiv 5 \pmod{8}$, and $p > 5$. Because of $(\frac{2}{p}) \equiv 2^{(p-1)/2} \equiv -1 \pmod{p}$, the number $g = 2$ has property (*), and the character ψ can be defined by

$$\psi(\bar{2}) = i, \quad \psi(\bar{k}) = 1,$$

for all $k \in H = G_p^4$. With F_k defined as in (**), formula (***) reads as

$$\det \begin{pmatrix} F_0 & iF_1 \\ iF_1 & F_0 \end{pmatrix} = F_0^2 + F_1^2 = 32 \cdot p^2 \cdot h^-.$$

3. Let $d = 3$, i.e., $p \equiv 7 \pmod{12}$, and $p > 7$. Choose a number $g' \in \mathbb{Z} \setminus p\mathbb{Z}$ such that $g'^{(p-1)/3} \not\equiv 1 \pmod{p}$. Then put $g = g'^2$. Moreover, since $p \equiv 3 \pmod{4}$, the Legendre symbol $\psi = (\frac{\cdot}{p})$ is an odd character with $H = G_p^6$ contained in $\text{Ker } \psi$. However, $\psi(\bar{g}^k) = 1$, $k = 0, 1, 2$, and, with F_k as in (**), formula (***) takes the form

$$3 \cdot F_0 \cdot F_1 \cdot F_2 - (F_0^3 + F_1^3 + F_2^3) = 864 \cdot p^3 \cdot h^-.$$

4. Let $d = 4$, i.e., $p \equiv 9 \pmod{16}$, and $p > 9$. Choose $g \in \mathbb{Z} \setminus p\mathbb{Z}$ such that $(\frac{g}{p}) \equiv g^{(p-1)/2} \equiv -1 \pmod{p}$. Then g has property (*), and ψ can be defined by $\psi(\bar{g}) = \zeta_8 = e^{\pi i/4}$, $\psi(\bar{k}) = 1$ for all $\bar{k} \in H = G_p^8$. With F_k as in (**), our

formula (***) reads as

$$-(F_0^4 + F_1^4 + F_2^4 + F_3^4) - 2(F_0^2 F_2^2 + F_1^2 F_3^2) \\ - 4(F_0^2 F_1 F_3 - F_1^2 F_0 F_2 - F_2^2 F_1 F_3 + F_3^2 F_0 F_2) = -32768 \cdot p^4 \cdot h^-.$$

5. Finally, let $d = 5$, i.e., $p \equiv 11 \pmod{20}$, and put $p > 11$. Choose a number $g' \in \mathbb{Z}/p\mathbb{Z}$ such that $g'^{(p-1)/5} \not\equiv 1 \pmod{p}$. Then $g = g'^2$ has property (*). Again, the Legendre symbol $\psi = (\frac{\cdot}{p})$ is an odd character of G_p , with $H \subseteq \text{Ker } \psi$ and $\psi(\bar{g}) = 1$. We obtain

$$F_0^5 + F_1^5 + F_2^5 + F_3^5 + F_4^5 \\ - 5\{F_0^3(F_1 F_4 + F_2 F_3) + F_1^3(F_0 F_2 + F_3 F_4) + F_2^3(F_0 F_4 + F_1 F_3) \\ + F_3^3(F_0 F_1 + F_2 F_4) + F_4^3(F_0 F_3 + F_1 F_2)\} \\ + 5\{F_0(F_1^2 F_4^2 + F_2^2 F_3^2) + F_1(F_0^2 F_2^2 + F_3^2 F_4^2) \\ + F_2(F_0^2 F_4^2 + F_1^2 F_3^2) + F_3(F_0^2 F_1^2 + F_2^2 F_4^2) + F_4(F_0^2 F_3^2 + F_1^2 F_2^2)\} \\ - 5 \cdot F_0 F_1 F_2 F_3 F_4 = -1600000 \cdot p^5 \cdot h^-.$$

Remarks. 1. Of course it is possible to give analogous relative class number formulas for $d \geq 6$, too. In the case $d = 6$, however, the determinant $\det(\psi(\bar{g})^{j+k} F_{j+k})$ consists of 68 monomials in $F_0, \dots, F_5$. Therefore, the formula is too complicated to be written in full.

2. Let $d = 2^q \cdot d'$, d' odd. Then X contains a character ψ of order 2^{q+1} . Obviously, $\psi(-1) = -1$ and $\psi(\bar{k}) = 1$ for all $\bar{k} \in H = G_p^{2d}$. Thus, ψ has the required properties, and we may say that there is always an appropriate character ψ of G_p of 2-power order.

3. TABLES

We have used the formulas of Examples 1, ..., 5 to compute the relative class numbers h_{2d}^- of imaginary subfields $K \subseteq \mathbb{Q}(\zeta_p)$ with $[K: \mathbb{Q}] = 2d$, $d \in \{2, 3, 4, 5\}$ and $2d + 1 < p < 500000$. In the Supplement we display the result for $d = 3, 4, 5$ and $p < 10000$ (Tables 1, 2, 3). The respective table for $[K: \mathbb{Q}] = 4$ can be found in [6].

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Supplement to
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Table 1
The Relative Class Number h_6^- of Sextic Fields
With Prime Conductor p , $7 < p < 10000$

p	h_6^-	p	h_6^-	p	h_6^-
19	1	1279	299	2731	407
31	9	1291	675	2767	11907
43	1	1303	77	2791	20475
67	1	1327	1755	2803	324
79	5	1399	1296	2851	1067
103	5	1423	1323	2887	2275
127	65	1447	713	2971	209
139	9	1459	341	3019	343
151	7	1471	3197	3067	343
163	4	1483	427	3079	1763
199	27	1531	407	3163	567
211	63	1543	475	3187	427
223	301	1567	405	3259	189
271	11	1579	189	3271	8748
283	9	1627	91	3307	837
307	9	1663	323	3319	1025
331	27	1699	308	3331	405
367	27	1723	35	3343	3667
379	117	1747	545	3391	12025
439	405	1759	567	3463	10393
463	49	1783	323	3499	1001
487	49	1831	6631	3511	2993
499	9	1867	95	3547	1809
523	65	1879	324	3559	1755
547	36	1951	1188	3571	1665
571	95	1987	553	3583	2813
607	208	1999	4617	3607	3781
619	35	2011	931	3631	7009
631	325	2083	847	3643	1053
643	9	2131	676	3691	1807
691	65	2143	6097	3727	868
727	481	2179	49	3739	671
739	65	2203	65	3823	9077
751	135	2239	3185	3847	851
787	65	2251	133	3907	469
811	133	2287	1943	3919	2223
823	81	2311	812	3931	2387
859	217	2347	155	3943	11907
883	117	2371	1027	3967	693
907	189	2383	8033	4003	3679
919	1273	2467	679	4027	567
967	143	2503	1323	4051	539
991	119	2539	671	4099	540
1039	437	2551	2009	4111	2925
1051	95	2647	855	4159	2077
1063	247	2659	1339	4219	4860
1087	81	2671	2231	4231	9639
1123	185	2683	335	4243	675
1171	343	2707	511	4327	931
1231	567	2719	3731	4339	323

Table2
The Relative Class Number h_g^* of Octic Fields
With Prime Conductor p , $9 < p < 10000$

p	h_g^*	p	h_g^*	p	h_g^*	p	h_g^*	p	h_g^*
4363	2781	6151	63071	8311	15067	41	1	2969	284489
4423	13167	6199	704	8419	3097	73	89	3001	5521
4447	1343	6211	7371	8431	11741	89	113	3049	1921
4483	1161	6247	4095	8443	4829	137	17	3209	3961
4507	5317	6271	2752	8467	7695	281	1433	3257	302609
4519	34307	6287	44523	8527	2107	233	233	3433	14089
4567	17028	6343	2475	8539	4607	313	233	3449	549225
4591	9751	6367	3589	8563	2457	409	425	3529	1225
4603	1729	6379	4811	8599	8127	457	41	3593	6497
4639	7956	6427	729	8623	14841	521	97	3673	26937
4651	527	6451	2363	8647	11284	569	521	3769	7489
4663	13167	6547	3773	8707	585	601	34153	3833	3825
4723	837	6571	8505	8731	29309	617	73	3881	20737
4759	54835	6607	8235	8731	6188	761	257	3929	5553
4783	575	6619	2821	8779	2205	809	641	4057	105993
4831	2673	6679	47245	8803	1701	857	401	4073	29889
4903	3159	6691	6489	8839	7007	937	4001	4153	16393
4951	1147	6703	1817	8863	35177	953	361	4201	647593
4987	2025	6763	3429	8887	3268	1033	289	4217	4129
4999	6237	6823	2673	8923	10507	1049	193	4297	1747657
5011	1323	6871	8235	8971	7201	1097	233	4409	959769
5023	1975	6883	729	9007	6125	1129	369	4441	1889
5059	2983	6907	4913	9043	1665	1193	193	4457	5193
5107	2401	6967	16731	9067	1053	1289	26401	4649	22257
5119	3627	6991	4757	9091	8001	1433	30937	4729	12897
5167	16137	7027	2948	9103	2223	1481	1153	4889	7481
5179	1331	7039	4687	9127	11457	1609	189473	4937	11689
5227	1395	7159	5915	9151	16549	1657	12809	4969	76289
5323	585	7207	5249	9187	2331	1721	25297	5081	197969
5347	2197	7219	3555	9199	6579	1801	10001	5113	27801
5407	4171	7243	1963	9283	6116	1913	112361	5209	249841
5419	637	7351	2079	9319	6068	1993	1009	5273	30449
5431	5301	7411	6475	9343	7497	2137	977	5417	12841
5443	2187	7459	4095	9391	10780	2089	69649	5449	32617
5479	8428	7507	869	9403	1397	2153	7913	5641	23137
5503	31675	7591	3185	9439	40725	2281	673	5657	26489
5527	703	7603	2981	9463	64915	2297	5617	5737	396009
5563	8145	7639	12400	9511	130617	2377	11657	5801	29449
5623	2079	7687	3248	9547	6331	2393	1377	5849	27425
5659	1444	7699	2997	9619	1957	2441	387433	5881	4793
5683	2717	7759	513	9631	74921	2473	401	5897	32977
5719	5668	7879	8428	9679	9843	2521	1241	59921	59921
5791	1881	7927	64881	9787	1133	2613	8281	6089	2251433
5827	2205	7951	2907	9811	1575	2853	6473	6121	4953
5839	1591	7963	1183	9859	3087	2879	8281	6217	53017
5851	1575	8011	1300	9883	7259	2929	5417	6329	152881
5923	3073	8059	1701	9907	8145	2777	7225	6361	456073
6043	41067	8167	27027	9931	8541	3857	1577	6473	70193
6067	1125	8191	32560	9967					
6079	25308	8263	5977						
6091	1215	8287	33345						

Table 3
The Relative Class Number h_{10}^- of Decimic Fields
With Prime Conductor p , $11 < p < 10000$

p	h_{10}^-	p	h_{10}^-	p	h_{10}^-
31	3	2791	17589	6151	1381249
71	7	2851	67771	6211	1642575
131	25	2971	102311	6271	1885521
151	1967	3011	35301	6311	10429999
191	143	3191	482379	6451	468367
211	123	3251	14291	6491	521761
251	77	3271	126387	6551	1429857
271	341	3331	14325	6571	124575
311	779	3371	117411	6591	1210461
331	243	3391	4607647	6791	104462611
431	161931	3491	109043	6871	16842375
491	11979	3511	151741	6911	984057
571	275	3571	194325	6971	2536875
631	2483	3631	1814213	6991	516241
691	1025	3671	190431	7151	3918925
751	2325	3691	17173	7211	1873375
811	4417	3851	1275125	7331	7711143
911	61721	3911	196793	7351	277233
971	2325	3931	169136	7411	2585125
991	7667	4051	26531	7451	17395175
1031	1925	4091	285483	7591	253825
1051	1375	4111	43329	7691	2346553
1091	262667	4211	24173	7951	1597371
1151	13981	4231	253011	8011	1186375
1171	9177	4331	539825	8111	1246421
1231	12177	4371	391919	8171	513051
1291	7929	4391	178669	8191	29343875
1451	17693	4451	2390269	8231	1583707
1471	3473	4651	343247	8291	1472087
1511	135289	4691	453621	8311	2376001
1531	23111	4751	6460181	8431	3235907
1571	11407	4831	207273	8731	4161107
1811	5543	4871	14635621	8831	593380304
1831	30419	4931	59675	8851	5879925
1871	74475	4951	51491	8971	3280369
1931	52731	5011	669081	9011	51679023
1951	42603	5051	485489	9091	422751
2011	3787	5091	406175	9151	1590647
2111	411649	5171	426875	9311	5518427
2251	8983	5231	2268423	9371	2005129
2311	21917	5351	658977	9391	2552275
2331	37439	5431	17877871	9431	48072661
2351	2024883	5531	344333	9491	16328475
2371	133393	5591	29355579	9511	63802299
2411	7613	5651	792391	9551	2848449
2531	62407	5711	38986139	9631	4752517
2551	59081	5791	420783	9791	1528079
2591	641877	5851	590331	9811	4511451
2671	129053	6011	3139587	9851	21762225
2711	76903	6091	108075	9871	1843919
2731	13541	6131	468131	9931	1622903