IMPROVED LOWER BOUNDS FOR THE DISCREPANCY OF INVERSIVE CONGRUENTIAL PSEUDORANDOM NUMBERS

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ABSTRACT. The inversive congruential method with prime modulus for generating uniform pseudorandom numbers is studied. Lower bounds for the discrepancy of k -tuples of successive pseudorandom numbers are established, which improve earlier results of Niederreiter. Moreover, the present proof is substantially simpler than the earlier one.

1. INTRODUCTION AND MAIN RESULTS

A particularly promising approach of generating uniform pseudorandom numbers in the interval $[0, 1)$ is the inversive congruential method with prime modulus. A review of several nonlinear congruential methods is given in the survey articles [1, 5, 6] and in H. Niederreiter's excellent monograph [7].

Let $p \geq 5$ be a prime, and identify $\mathbf{Z}_p = \{0, 1, \dots, p-1\}$ with the finite field of order p . For $z \in \mathbf{Z}_p^* := \mathbf{Z}_p \setminus \{0\}$ let $\bar{z}$ denote the multiplicative inverse of z modulo p , and put $\bar{0} := 0$. For integers $a, c \in \mathbf{Z}_p^*$ an *inversive congruential sequence* $(y_n)_{n \geq 0}$ of elements of $\mathbf{Z}_p$ is defined by

$$y_{n+1} \equiv ac^2\bar{y}_n + c \pmod{p}, \quad n \geq 0.$$

A sequence $(x_n)_{n \geq 0}$ of *inversive congruential pseudorandom numbers* in the interval $[0, 1)$ is obtained by $x_n = y_n/p$ for $n \geq 0$. Observe that these sequences are always purely periodic. In [2], sequences having maximal period length p are characterized. In particular, it follows from [2, Theorem 2] that this property depends only on $a \in \mathbf{Z}_p^*$, but not on the specific value of $c \in \mathbf{Z}_p^*$. Let $\mathbf{M}_p^*$ be the set of all $a \in \mathbf{Z}_p^*$ which belong to inversive congruential sequences with maximal period length p .

For assessing statistical independence properties the *discrepancy* of the k -tuples

$$\mathbf{x}_n = (x_n, x_{n+1}, \dots, x_{n+k-1}) \in [0, 1)^k, \quad 0 \leq n < p,$$

of successive inversive congruential pseudorandom numbers can be used, which is defined by

$$D_p^{(k)} = \sup_J |F_p(J) - V(J)|,$$

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where the supremum is extended over all subintervals J of $[0, 1]^k$, $F_p(J)$ is p^{-1} times the number of points among $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{p-1}$ falling into J , and $V(J)$ denotes the k -dimensional volume of J . The following two theorems from [4] provide lower bounds for $D_p^{(k)}$. Let φ be Euler's totient function and $\omega(m)$ be the number of different prime factors of a positive integer m . Let

$$t(p) = \left(1 - \frac{1}{p}(p^{1/2} + 2)2^{\omega(p-1)}\right)^{1/2}$$

and

$$A_p(t) = \frac{(1 - t^2)p - (p^{1/2} + 2)2^{\omega(p-1)}}{(4 - t^2)p + 4p^{1/2} + 1}$$

for $0 < t \leq t(p)$. Note that [2, Corollary 1] implies that an inversive congruential sequence has maximal period length p if $z^2 - cz - ac^2$ is a primitive polynomial over $\mathbf{Z}_p$.

Theorem 1. *There are at least $\varphi(p+1)$ primitive polynomials $z^2 - cz - ac^2$ over $\mathbf{Z}_p$ such that the discrepancy $D_p^{(k)}$ for the corresponding inversive congruential generator satisfies*

$$D_p^{(k)} > \frac{1}{2(\pi + 2)}(p^{-1/2} - 2p^{-3/5})$$

for all dimensions $k \geq 2$.

Theorem 2. *Let $0 < t \leq t(p)$. Then there are more than $A_p(t)\varphi(p^2 - 1)/2$ primitive polynomials $z^2 - cz - ac^2$ over $\mathbf{Z}_p$ such that the discrepancy $D_p^{(k)}$ for the corresponding inversive congruential generator satisfies*

$$D_p^{(k)} > \frac{t}{2(\pi + 2)}p^{-1/2}$$

for all dimensions $k \geq 2$.

In the present paper the following improved lower bounds for $D_p^{(k)}$ are established. These results have two main advantages. They apply to all inversive congruential sequences with maximal period length p and not only to those belonging to a primitive polynomial, and they provide information on the subclasses of inversive congruential generators which correspond to the different values of $a \in \mathbf{M}_p^*$. Moreover, the proof of these results, which is given in the third section, is much simpler than the one of Theorems 1 and 2 in [4]. Let

$$\tilde{t}(p) = \left(\frac{p-3}{p-1}\right)^{1/2}$$

and

$$\tilde{A}_p(t) = \frac{(1 - t^2)p - 2p(p-1)^{-1}}{(4 - t^2)p + 4p^{1/2} + 1}$$

for $0 < t \leq \tilde{t}(p)$.

Result 1. *Let $a \in \mathbf{M}_p^*$. Then there exists a $c \in \mathbf{Z}_p^*$ such that the discrepancy $D_p^{(k)}$ for the corresponding inversive congruential generator satisfies*

$$D_p^{(k)} \geq \frac{\tilde{t}(p)}{2(\pi + 2)}p^{-1/2}$$

for all dimensions $k \geq 2$.

Result 2. Let $0 < t \leq \tilde{i}(p)$ and $a \in \mathbf{M}_p^*$. Then there are more than $\tilde{A}_p(t)(p-1)$ values of $c \in \mathbf{Z}_p^*$ such that the discrepancy $D_p^{(k)}$ for the corresponding inversive congruential generator satisfies

$$D_p^{(k)} \geq \frac{t}{2(\pi+2)} p^{-1/2}$$

for all dimensions $k \geq 2$.

2. AUXILIARY RESULTS

First, some further notation is necessary. Let $e(t) = e^{2\pi it}$ for $t \in \mathbb{R}$ and $\chi(z) = e(z/p)$ for $z \in \mathbf{Z}$. For fixed $a \in \mathbf{Z}_p^*$ and $c \in \mathbf{Z}_p$, an exponential sum is defined by

$$S(c) = \sum_{y \in \mathbf{Z}_p} \chi(c(y + a\bar{y})).$$

Lemma 1. Let $a \in \mathbf{Z}_p^*$. Then

$$\sum_{c \in \mathbf{Z}_p^*} |S(c)|^2 \geq p(p-3).$$

Proof. Easy calculations show that

$$\begin{aligned} \sum_{c \in \mathbf{Z}_p} |S(c)|^2 &= \sum_{c \in \mathbf{Z}_p} \sum_{y, z \in \mathbf{Z}_p} \chi(c(y - z + a(\bar{y} - \bar{z}))) \\ &= \sum_{y, z \in \mathbf{Z}_p} \sum_{c \in \mathbf{Z}_p} \chi(c(y - z + a(\bar{y} - \bar{z}))) \\ &= p \cdot \#\{(y, z) \in \mathbf{Z}_p \times \mathbf{Z}_p \mid y - z + a(\bar{y} - \bar{z}) \equiv 0 \pmod{p}\} \\ &\geq p(\#\{(y, z) \in \mathbf{Z}_p^* \times \mathbf{Z}_p^* \mid (y - z)(1 - a\bar{y}\bar{z}) \equiv 0 \pmod{p}\} + 1) \\ &= p(\#\{(y, z) \in \mathbf{Z}_p^* \times \mathbf{Z}_p^* \mid y = z \text{ or } y \equiv a\bar{z} \pmod{p}\} + 1) \\ &\geq p(2p - 3), \end{aligned}$$

where the last inequality follows from the fact that there are at most two values of $z \in \mathbf{Z}_p^*$ with $z \equiv a\bar{z} \pmod{p}$. Since $S(0) = p$, one obtains at once

$$\sum_{c \in \mathbf{Z}_p^*} |S(c)|^2 \geq p(2p - 3) - p^2 = p(p - 3). \quad \square$$

Lemma 2. Let $0 < t \leq \tilde{i}(p)$ and $a \in \mathbf{Z}_p^*$. Then there are more than $\tilde{A}_p(t)(p-1)$ values of $c \in \mathbf{Z}_p^*$ such that

$$|S(c)| \geq tp^{1/2}.$$

Proof. The lemma is proved by contradiction. Suppose that $|S(c)| \geq tp^{1/2}$ for at most $\tilde{A}_p(t)(p-1)$ values of $c \in \mathbf{Z}_p^*$. Then $|S(c)| < tp^{1/2}$ for at least $(1 - \tilde{A}_p(t))(p-1)$ values of $c \in \mathbf{Z}_p^*$. Now, observe that $S(c) = K(\chi; c, ac) + 1$, where $K(\chi; \cdot, \cdot)$ denotes the Kloosterman sum defined in [3, Definition 5.42]. Hence, it follows from the classical bound for Kloosterman sums (cf. [3,

Theorem 5.45]) that $|S(c)| \leq 2p^{1/2} + 1$ for all $c \in \mathbf{Z}_p^*$. Therefore, one obtains

$$\begin{aligned} \sum_{c \in \mathbf{Z}_p^*} |S(c)|^2 &< (1 - \tilde{A}_p(t))(p-1)t^2p + \tilde{A}_p(t)(p-1)(2p^{1/2} + 1)^2 \\ &= p(p-3), \end{aligned}$$

which is a contradiction to Lemma 1. $\square$

3. PROOF OF THE RESULTS

First, Lemma 1 in [4] is applied with $N = p$, $\mathbf{t}_n = \mathbf{x}_n$ for $0 \leq n < p$, $\mathbf{h} = (1, 1, 0, \dots, 0) \in \mathbf{Z}^k$, and hence $m = 2$. This yields

$$\begin{aligned} D_p^{(k)} &\geq \frac{1}{2(\pi+2)p} \left| \sum_{n=0}^{p-1} e(x_n + x_{n+1}) \right| \\ &= \frac{1}{2(\pi+2)p} \left| \sum_{n=0}^{p-1} \chi(y_n + ac^2 \bar{y}_n) \right|. \end{aligned}$$

Since $(y_n)_{n \geq 0}$ has maximal period length p , i.e., $\{y_0, y_1, \dots, y_{p-1}\} = \mathbf{Z}_p$, one obtains

$$D_p^{(k)} \geq \frac{1}{2(\pi+2)p} \left| \sum_{z \in \mathbf{Z}_p} \chi(z + ac^2 \bar{z}) \right|.$$

Now, the transformation $z \equiv cy \pmod{p}$ yields

$$D_p^{(k)} \geq \frac{1}{2(\pi+2)p} \left| \sum_{y \in \mathbf{Z}_p} \chi(c(y + a\bar{y})) \right| = \frac{1}{2(\pi+2)p} |S(c)|.$$

Therefore, Result 2 follows at once from Lemma 2. Finally, Result 1 is obtained from Result 2 with $t = \tilde{t}(p)$.

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