FINDING FINITE B_2 -SEQUENCES WITH LARGER $m - a_m^{1/2}$

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ABSTRACT. A sequence of positive integers $a_1 < a_2 < \cdots < a_m$ is called a (finite) B_2 -sequence, or a (finite) Sidon sequence, if the pairwise differences are all distinct. Let

$$K(m) = \max(m - a_m^{1/2}),$$

where the maximum is taken over all m -element B_2 -sequences. Erdős and Turán ask if $K(m) = O(1)$. In this paper we give an algorithm, based on the Bose-Chowla theorem on finite fields, for finding a lower bound of $K(p)$ and a p -element B_2 -sequence with $p - a_p^{1/2}$ equal to this bound, taking $O(p^3 \log^2 p K(p))$ bit operations and requiring $O(p \log p)$ storage, where p is a prime. A search for lower bounds of $K(p)$ for $p \leq p_{145}$ is given, especially $K(p_{145}) > 10.279$, where p_i is the i th prime.

1. INTRODUCTION

A sequence of positive integers $a_1 < a_2 < \cdots < a_m$ is called a (finite) B_2 -sequence, or a (finite) Sidon sequence, if the pairwise differences are all distinct, or, in other words, if all the sums $a_i + a_j$ ($i = j$ is permitted) are different. Let m be the maximum number such that $a_m \leq n$. It is known that

$$n^{1/2}(1 - \varepsilon) < m \leq n^{1/2} + n^{1/4} + 1.$$

The upper bound is due to Lindstrom [6], improving a result of Erdős and Turán [2]. The lower bound is due to Singer [9]. Let

$$(1.1) \quad K(m) = \max(m - a_m^{1/2}),$$

where the maximum is taken over all m -element B_2 -sequences. Erdős and Turán ask whether or not

$$(1.2) \quad K(m) = O(1).$$

Erdős offers \$500 for settling this equation [3, pp. 65–66].

In this paper we do not answer this question, but instead will give an algorithm for finding a lower bound of $K(p)$ and a p -element B_2 -sequence with $p - a_p^{1/2}$ equal to this bound, taking $O(p^3 \log^2 p K(p))$ bit operations and requiring $O(p \log p)$ storage, where p is a prime. A search for lower bounds of

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$K(p)$ for $p \leq p_{145}$ is given, especially $K(p_{145}) > 10.279$, where p_i is the i th prime.

Our algorithm is based on the Bose-Chowla theorem for finite fields $GF(p^2)$. A direct search on a computer shows that probably for any $k > 0$, there would exist a p -element B_2 -sequence with $p - a_p^{1/2} > k$.

2. NOTATIONS AND MAIN RESULTS

We denote by p a prime, and by p_i the i th prime. It is well known that [10, §37] corresponding to each prime p and natural number r there is a unique (up to isomorphism) finite field (Galois field) of p^r elements. We denote this field by $GF(p^r)$. The multiplicative group of the nonzero elements in the Galois field $GF(p^r)$, denoted by $GF^*(p^r)$, is cyclic with $p^r - 1$ elements. If d is a divisor of r , then $GF(p^d)$ is a subfield of $GF(p^r)$ and $GF^*(p^d)$ is a subgroup of $GF^*(p^r)$.

In this paper we need only the case when $r = 2$. In this case, $GF(p)$ is a subfield of $GF(p^2)$ and $GF^*(p)$ is a subgroup of $GF^*(p^2)$.

Let θ be a generator of $GF^*(p^2)$ (denoted by $GF^*(p^2) = \langle \theta \rangle$),

$$(2.1) \quad A(p, \theta) = \{a: 1 \leq a < p^2, \theta^a - \theta \in GF(p)\}$$

and

$$(2.2) \quad \overline{A}(p, \theta) = A(p, \theta) \cup \{p^2\}.$$

Then $\overline{A}(p, \theta)$ has $p + 1$ elements, denoted by $1 = a_1 < a_2 < \cdots < a_p < a_{p+1} = p^2$. Let

$$D(p, \theta) = \{a_{i+1} - a_i: 1 \leq i \leq p\},$$

$$d(p, \theta) = \max\{d: d \in D(p, \theta)\}, \quad d(p) = \max d(p, \theta),$$

where the second maximum is taken over all generators of $GF^*(p^2)$, and define

$$k(p) = p - \sqrt{p^2 - d(p)}.$$

With the above notations, and $K(m)$ defined by (1.1), we state our main results as the following two theorems.

Theorem 1. *Given $k > 0$, if there exists a prime p with $k(p) > k$, or, in other words, with $d(p) > 2kp - k^2$, then there exists a p -element B_2 -sequence $1 = b_1 < b_2 < \cdots < b_p = p^2 - d(p)$ with*

$$K(p) \geq k(p) = p - b_p^{1/2} > k.$$

Theorem 2. *There exists an algorithm for finding $k(p)$ (or $d(p)$) and a p -element B_2 -sequence $\{b_i\}$ with $p - b_p^{1/2} = k(p)$, taking $O(p^3 \log^2 p K(p))$ bit operations and requiring $O(p \log p)$ storage.*

3. PROOF OF THEOREM 1

To prove Theorem 1, we need four lemmas. The first lemma is just a special case of the Bose-Chowla theorem [1] obtained in 1962. Although the proof can be found in either [1] or [4, Chapter 2], we rewrite it here, since the idea in the proof will be used in the proofs of our theorems.

Lemma 3.1 (Bose-Chowla). *Both $A(p, \theta)$ and $\bar{A}(p, \theta)$ are B_2 -sequences.*

Proof. Let $A(p, \theta) = \{a_i : 1 \leq i \leq p\}$ and $c(a) = \theta^a - \theta \in GF(p)$ for $a \in A(p, \theta)$. If $\{i, j\} \neq \{i', j'\}$, $1 \leq i \leq j \leq p$, $1 \leq i' \leq j' \leq p$, then

$$(3.1) \quad (\theta + c(a_i))(\theta + c(a_j)) - (\theta + c(a_{i'}))(\theta + c(a_{j'})) \neq 0.$$

For the left-hand side of (3.1), considered as a polynomial in θ with coefficients in $GF(p)$, is of degree at most one in θ and does not vanish identically, since there is at most one factorization of a monic polynomial into monic linear factors; whilst θ is of degree 2 over $GF(P)$. Thus, $\theta^{a_i+a_j} \neq \theta^{a_{i'}+a_{j'}}$, and then

$$(3.2) \quad a_i + a_j \not\equiv a_{i'} + a_{j'} \pmod{p^2 - 1}.$$

Therefore $a_i + a_j \neq a_{i'} + a_{j'}$, i.e., $A(p, \theta)$ is a B_2 -sequence.

Now let $a_i, a_j, a_k \in A(p, \theta)$, $1 \leq i, j, k \leq p$. If $a_i + a_j = a_k + p^2$, then $\{i, j\} \neq \{1, k\}$ and

$$a_i + a_j \equiv a_k + 1 = a_k + a_1 \pmod{p^2 - 1},$$

which contradicts (3.2). Thus, $\bar{A}(p, \theta)$ is also a B_2 -sequence. $\square$

Lemma 3.2. *Let $\bar{A}(p, \theta) = \{1 = a_1 < a_2 < \dots < a_p < a_{p+1} = p^2\}$. Then*

$$\{a_{i+1}, a_{i+2}, \dots, a_p, a_{p+1} = a_1 + p^2 - 1, a_2 + p^2 - 1, \dots, a_i + p^2 - 1\}$$

is also a B_2 -sequence for any i with $1 \leq i \leq p$.

Proof. This follows easily by (3.2). $\square$

Lemma 3.3. *If $\{a_i\}$ is a B_2 -sequence and $h < a_1$, then so is $\{a_i - h\}$.*

Proof. Obvious. $\square$

Lemma 3.4. *Let $\bar{A}(p, \theta) = \{1 = a_1 < a_2 < \dots < a_p < a_{p+1} = p^2\}$. Given t with $1 \leq t \leq p$, let $h = a_{t+1} - 1$. Then*

$$\{1 = b_1 < b_2 < \dots < b_p = p^2 - (a_{t+1} - a_t)\}$$

is a B_2 -sequence, where

$$b_i = \begin{cases} a_{t+i} - h, & 1 \leq i \leq p - t, \\ a_{t+i-p} + p^2 - 1 - h, & p - t < i \leq p. \end{cases}$$

Proof. This follows by Lemmas 3.2 and 3.3. $\square$

Remark. In the above lemma, if we choose t such that $d(p, \theta) = a_{t+1} - a_t$, then the sequence $\{b_i : 1 \leq i \leq p\}$ associated with this t has larger $p - b_p^{1/2}$.

Example 3.1. Let $p = 7$ and $\theta^2 = \theta - 3$; then $GF^*(p^2) = (\theta)$. We have $\bar{A}(p, \theta) = \{a_i\} = \{1, 2, 5, 11, 31, 36, 38, 49\}$ and $d(p, \theta) = a_5 - a_4$. Let

$$b_i = \begin{cases} a_{4+i} - 30, & 1 \leq i \leq 3, \\ a_{i-3} + 18, & 4 \leq i \leq 7. \end{cases}$$

Then $\{b_i\} = \{1, 6, 8, 19, 20, 23, 29\}$ is a B_2 -sequence with $p - b_p^{1/2} = 1.614\dots$

Example 3.2. Let $p = 11$ and $\theta^2 = 9\theta - 6$; then $GF^*(p^2) = (\theta)$. We have $\bar{A}(p, \theta) = \{a_i\} = \{1, 7, 17, 32, 34, 45, 52, 66, 71, 74, 75, 121\}$ and

$d(p, \theta) = a_{12} - a_{11}$. Then $\{b_i\} = A(p, \theta) = \{1, 7, 17, 32, 34, 45, 52, 66, 71, 74, 75\}$ is a B_2 -sequence with $p - b_p^{1/2} = 2.339 \dots$.

We are now ready to prove Theorem 1.

Proof of Theorem 1. Given $k > 0$, suppose there exists a prime p with $k(p) > k$. Let $GF^*(p^2) = (\theta)$ such that $d(p) = d(p, \theta)$. Let $\bar{A}(p, \theta) = \{1 = a_1 < a_2 < \dots < a_p < a_{p+1} = p^2\}$ and $d(p, \theta) = a_{t+1} - a_t$ for some t with $1 \leq t \leq p$. Then the sequence $\{b_i\}$ in Lemma 3.4 is just what we want. $\square$

4. PROOF OF THEOREM 2

In this section, for a given prime p , we denote by $\text{order}(a)$ the order of a in $GF^*(p)$ or in $GF^*(p^2)$. To prove Theorem 2, we need seven lemmas.

Lemma 4.1. Let $GF^*(p^2) = (\theta)$ and $\theta^2 = u\theta - v$. Then we have

- (I) $\theta^p + \theta = u$, $\theta^{p+1} = v$;
- (II) $\theta^i \notin GF(p)$ for $1 \leq i \leq p$;
- (III) $\text{order}(v) = p - 1$.

Proof. (I) This follows from the fact that θ and θ^p are two roots of $x^2 = ux - v$.

(II) This follows from the facts that $\text{order}(\theta) = p^2 - 1$ and the order of an element of $GF^*(p)$ is at most $p - 1$.

(III) This follows by (I) and the fact that $\text{order}(\theta) = p^2 - 1$. $\square$

Lemma 4.2. Let $\theta \in GF(p^2)$, $\theta^2 = u\theta - v$ with $u, v \in GF(p)$ and $\text{order}(v) = p - 1$, and $\theta^i \notin GF(p)$ for $1 \leq i \leq p$. Then we have $GF^*(p^2) = (\theta)$.

Proof. Since $\theta^i \notin GF(p)$ for $1 \leq i \leq p$, we have that θ and θ^p are two different roots of $x^2 = ux - v$. Thus, $\theta^{p+1} = v$. Suppose $\text{order}(\theta) = m < p^2 - 1$. Let $m = (p+1)q + r$ with $0 \leq r \leq p$. If $r = 0$, then $\text{order}(\theta^{p+1}) \leq q < p - 1$, which contradicts the condition that $\text{order}(v) = p - 1$. If $1 \leq r \leq p$, then $\theta^r = \theta^{m-(p+1)q} = (\theta^{p+1})^{-q} \in GF(p)$, which contradicts the condition that $\theta^i \notin GF(p)$ for $1 \leq i \leq p$.

Thus, we have $\text{order}(\theta) = p^2 - 1$, i.e., $GF^*(p^2) = (\theta)$. $\square$

Lemma 4.3. Let $\theta \in GF(p^2)$ and $\theta^2 = u\theta - v$ with $u, v \in GF(p)$. Then a necessary and sufficient condition for $GF^*(p^2) = (\theta)$ is that

$$\text{order}(v) = p - 1 \quad \text{and} \quad \theta^i \notin GF(p) \quad \text{for } 1 \leq i \leq p.$$

Proof. This follows by Lemmas 4.1 and 4.2. $\square$

Lemma 4.4. We have $(p^2 - 1)/2 + p \in A(p, \theta)$.

Proof. By Lemma 4.1 we have $\theta^{(p^2-1)/2+p} - \theta = -\theta^p - \theta \in GF(p)$. $\square$

Lemma 4.5. Let $\bar{A}(p, \theta) = \{1 = a_1 < a_2 < \dots < a_p < a_{p+1} = p^2\}$. Then for any a_i with $p+1 < a_i < p^2$, there exists some r_i with $1 \leq r_i \leq p$ such that $\theta^{r_i+a_{i+1}-a_i} - \theta^{r_i} \in GF(p)$. Moreover, we have $\theta^b - \theta^{r_i} \notin GF(p)$ for any b with $r_i < b < r_i + a_{i+1} - a_i$.

Proof. Let $a_i = (p+1)t_i + r_i$ with $0 \leq r_i \leq p$. Since $\theta^{p+1}, \theta^{a_i} - \theta \in GF(p)$, we have $r_i \neq 0$, i.e., $1 \leq r_i \leq p$. Then $\theta^{r_i+a_{i+1}-a_i} - \theta^{r_i} = \theta^{a_{i+1}-(p+1)t_i} - \theta^{a_i-(p+1)t_i} = (\theta^{a_{i+1}} - \theta^{a_i})\theta^{(p+1)(-t_i)} \in GF(p)$.

Now suppose $\theta^b - \theta^{r_i} \in GF(p)$ for some b with $r_i < b < r_i + a_{i+1} - a_i$. Let $b' = b + (p+1)t_i$. Then we have $a_i < b' < a_{i+1}$ and

$$\theta^{b'} - \theta^{a_i} = \theta^{(p+1)t_i}(\theta^b - \theta^{r_i}) \in GF(p),$$

which contradicts (2.1) or (2.2). $\square$

Lemma 4.6. *For a pair of $u, v \in GF(p)$ with $\text{order}(v) = p-1$, let $\theta^2 = u\theta - v$. Then it takes $O(p \log^2 p)$ bit operations to check if $GF^*(p^2) = (\theta)$. Moreover, if $GF^*(p^2) = (\theta)$ has been checked, then it takes $O(p \log^2 pk(p))$ bit operations to get $d(p, \theta)$.*

Proof. Let $\theta^i = u_i\theta - v_i$ with $u_i, v_i \in GF(p)$. Then $u_1 = 1$, $u_2 = u$, and

$$(4.1) \quad u_{i+1} = u_i u - u_{i-1} v \pmod{p} \quad \text{for } i \geq 2.$$

By a conventional algorithm [5, Chapter 4], [8, pp. 33–44], it takes $O(\log^2 p)$ bit operations for computing each u_i . By Lemma 4.3, to check if $GF^*(p^2) = (\theta)$, we need (only) to compute u_i for $1 \leq i \leq p$ and to check that none of them is zero. This can be done in $O(p \log^2 p)$ bit operations.

Now suppose $GF^*(p^2) = (\theta)$ has been checked. We use $O(p \log p)$ storage to save all u_i for $1 \leq i \leq p$.

For $i > p$, let $i = (p+1)t + r$, where $0 \leq r \leq p$; then

$$(4.2) \quad u_i = \begin{cases} 0, & r = 0, \\ v^t u_r \pmod{p}, & 1 \leq r \leq p, \end{cases}$$

by Lemma 4.1. Since u_r ($1 \leq r \leq p$) are stored, they need not be recomputed. The quantity v^t can be computed by recurrence: $v^t = v^{t-1}v \pmod{p}$.

By Lemma 4.5 and the above descriptions of the computation of u_i , we have

$$(4.3) \quad d(p, \theta) = \max_{1 \leq i \leq p} \{s - i : s \text{ is the least integer such that } s > i \text{ and } u_s = u_i\}.$$

We use the following procedure in pseudocode to get $d(p, \theta)$:

BEGIN

$j \leftarrow 0; s \leftarrow 1; \alpha(1) \leftarrow 1; d(p, \theta) \leftarrow 0;$

For $i := 2$ To $p-1$ Do $\alpha(i) \leftarrow 0;$

While $j < p$ Do

 Begin

$s \leftarrow s + 1; \text{ If } (p+1)|s \text{ Then } s \leftarrow s + 1;$

 If $s > p$ Then calculate u_s by (4.2);

 If $\alpha(u_s) > 0$ Then

 begin

$j \leftarrow j + 1; b \leftarrow s - \alpha(u_s);$

 If $b > d(p, \theta)$ Then $d(p, \theta) \leftarrow b;$

 If $s > p$ Then $\alpha(u_s) \leftarrow 0$

 end;

 If $s \leq p$ Then $\alpha(u_s) \leftarrow s$

 End

END;

In the procedure the values of $\alpha(i)$ for $1 \leq i \leq p$ are stored and changed from time to time. It requires $O(p \log p)$ storage, since $0 \leq \alpha(i) \leq p$. By (4.3), to get $d(p, \theta)$, the u_s are calculated for s at most equal to $s = p + d(p, \theta)$. Thus, the procedure will be terminated in $O(d(p, \theta) \log^2 p)$ or $O(p \log^2 p k(p))$ bit operations. $\square$

Remark 4.1. In the procedure, as s increases, there may be several values of s ($\leq p$) for which the u_s have a same value, say, w . Assign $\alpha(w)$ to be the latest value of s with $u_s = w$. When we find a larger s with $u_s = w$, we get a new member of the set of (4.3): $b = s - \alpha(w)$. The variable j is the number of integers in the set of (4.3) which have been compared for taking the maximum for $d(p, \theta)$.

Example 4.1. Let $p = 7$ and $\theta^2 = \theta - 3$; then $GF^*(p^2) = (\theta)$ as in Example 3.1. Let $\alpha(1) = 1$ and $\alpha(i) = 0$ for $2 \leq i \leq 7$. Then the variables in the procedure will be evaluated as follows:

s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
u_s	1	1	5	2	1	2	6	0	3	3	1	6	3	6	4	0	2	2	3	4	2	4	5
$\alpha(u_s)$	1	2	3	4	5	6	7				0	0					0						0
b		1			3	2					6	5					11						20
j	0	1			2	3					4	5					6						7
$d(p, \theta)$	0	1			3						6						11						20

We obtain, without calculating a_i , $d(7, \theta) = 20$, as in Example 3.1.

Lemma 4.7. Given $u, v \in GF(p)$ such that $GF^*(p^2) = (\theta)$ and $d(p, \theta) = d(p)$ with $\theta^2 = u\theta - v$, it takes $O(p^2 \log^2 p)$ bit operations to get a p -element B_2 -sequence $\{b_i\}$ with $b_p = p^2 - d(p)$.

Proof. Let u_i be defined as in the proof of Lemma 4.6. Then by (2.1) and (2.2) we have

$$\overline{A}(p, \theta) = \{i: 1 \leq i \leq p^2, u_i = 1\}.$$

By (4.1) and (4.2), it takes $O(p^2 \log^2 p)$ bit operations to calculate u_i and to check if $u_i = 1$ for $1 \leq i \leq p^2$. After $\overline{A}(p, \theta)$ is known, the time for getting a B_2 -sequence $\{b_i\}$ with $b_p = p^2 - d(p)$ is negligible ($O(p \log p)$ bit operations) by Lemma 3.4. $\square$

Now we are ready to prove Theorem 2.

Proof of Theorem 2. Given a prime p , let $g = g(p)$ be the least primitive root of p . Let $v = g^s \pmod{p}$ for odd s with $1 \leq s < p - 1$ and $(s, p - 1) = 1$; then $\text{order}(v) = p - 1$. The number of pairs of u, v with $\text{order}(v) = p - 1$ is $(p - 1)\varphi(p - 1) < p^2$, where $\varphi(\cdot)$ is the Euler φ -function. Thus, by Lemma 4.6 it takes $O(p^3 \log^2 p)$ bit operations to find all pairs of u, v such that $GF^*(p^2) = (\theta)$ with $\theta^2 = u\theta - v$. It is clear that the time for calculating $g^s \pmod{p}$ by recurrence and for checking if $(s, p - 1) = 1$ by the Euclidean algorithm [8, pp. 58–68] is negligible.

Since there are in total $\varphi(p^2 - 1) < p^2$ generators in $GF^*(p^2)$, it takes another $O(p^3 \log^2 p k(p))$ bit operations to find $d(p)$ or $k(p)$ by Lemma 4.6.

Since $k(p) \leq K(p)$, the total time for finding $k(p)$ or $d(p)$ is $O(p^3 \log^2 p K(p))$ bit operations, while the time for finding a B_2 -sequence $\{b_i\}$ with $p - b_p^{1/2} = k(p)$ is negligible by Lemma 4.7.

By the proofs of Lemmas 4.6 and 4.7, we see that the space required is $O(p \log p)$ bytes. $\square$

Remark 4.2. The least primitive root $g = g(p)$ of p is usually found quite fast in the manner described in [7, pp. 105–106].

5. DESCRIPTION OF THE ALGORITHM

In this section we will implement the algorithm for finding $k(p)$. To speed things up, we need some more lemmas.

Lemma 5.1. *We have $D(p, \theta) = D(p, \theta^{-p})$.*

Proof. Let $h = (p^2 - 1)/2 + p$; then $h \in A(p, \theta) \cap A(p, \theta^{-p})$ by Lemma 4.4. Let

$$\overline{A}(p, \theta) = \{1 = a_1 < a_2 < \cdots < a_p < a_{p+1} = p^2\}$$

and $h = a_t$ for some t with $1 < t \leq p$. Then

$$\begin{aligned} (\theta^{-p})^{h-(a_i-a_1)} - (\theta^{-p})^h &= (\theta^{-h+a_i-a_1} - \theta^{-h})^p \\ &= (-\theta^{-p+a_i-a_1} + \theta^{-p})^p = \left(\frac{\theta - \theta^{a_i}}{\theta^{p+1}} \right)^p \in GF(p) \end{aligned}$$

for $1 \leq i \leq t$ by Lemma 4.1. Similarly,

$$(\theta^{-p})^{h+p^2-1-(a_i-a_1)} - (\theta^{-p})^h \in GF(p)$$

for $t \leq i \leq p$. Thus,

$$\overline{A}(p, \theta^{-p}) = \{h - (a_i - a_1) : 1 \leq i \leq t\} \cup \{h + p^2 - 1 - (a_i - a_1) : t \leq i \leq p\};$$

therefore, $D(p, \theta) = D(p, \theta^{-p})$. $\square$

Example 5.1. Let $p = 7$ and $\theta^2 = \theta - 3$ as in Example 3.1. Then $h = 31$, $t = 5$, $(\theta^{-p})^2 = 5\theta^{-p} - 5$, $\overline{A}(p, \theta^{-p}) = \{1, 21, 27, 30, 31, 42, 44, 49\}$, $D(p, \theta) = \{1, 3, 6, 20, 5, 2, 11\} = D(p, \theta^{-p})$.

Example 5.2. Let $p = 11$ and $\theta^2 = 9\theta - 6$ as in Example 3.2. Then $h = 71$, $t = 9$, $(\theta^{-p})^2 = 7\theta^{-p} - 2$, $\overline{A}(p, \theta^{-p}) = \{1, 6, 20, 27, 38, 40, 55, 65, 71, 117, 118, 121\}$, $D(p, \theta) = \{6, 10, 15, 2, 11, 7, 14, 5, 3, 1, 46\} = D(p, \theta^{-p})$.

Lemma 5.2. *If $GF^*(p^2) = (\theta)$, $\theta^2 = u\theta - v$ with $u, v \in GF(p)$, then the minimum polynomial of θ^{-p} over $GF(p)$ is $x^2 = wx - v^{-1}$ for some $w \in GF(p)$.*

Proof. Let $x^2 = wx - t$ be the minimum polynomial of θ^{-p} over $GF(p)$ with $w, t \in GF(p)$. Then by Lemma 4.1 we have

$$t = (\theta^{-p})^{p+1} = (\theta^{p+1})^{-p} = (v^p)^{-1} = v^{-1}. \quad \square$$

Lemma 5.3. *Given a prime p , let $g = g(p)$ be the least primitive root of p . To find $d(p)$, we need only compare those $d(p, \theta)$ with $\theta^2 = u\theta - v$, $u, v \in GF(p)$, and*

$$(5.1) \quad v = g^s \pmod{p} \text{ for some } s \text{ with } (s, p-1) = 1 \text{ and } 1 \leq s < \frac{p-1}{2}.$$

Proof. This follows by Lemmas 4.1, 5.1, and 5.2. $\square$

Lemma 5.4. *We have $A(p, \theta) = A(p, \theta^p)$.*

Proof. This follows from the fact that both θ and θ^p have the same minimum polynomial over $GF(p)$. $\square$

Lemma 5.5. *Given a prime p , let m_1 be the number of distinct sets $D(p, \theta)$ and m_2 the number of pairs of $u, v \in GF(p)$ such that*

$$v \text{ satisfying (5.1), if } \theta^2 = u\theta - v, \text{ then } GF^*(p^2) = (\theta).$$

Then we have $m_1 \leq m_2 = \phi(p^2 - 1)/4$.

Proof. This follows by Lemmas 5.1, 5.2, 5.3, 5.4 and the fact that the group $GF^*(p^2)$ has $\phi(p^2 - 1)$ generators. $\square$

Lemma 5.6. *Given a prime p , if $GF^*(p^2) = (\theta)$, $\theta^2 = u\theta - v$, with $u, v \in GF(p)$, then*

$$(5.2) \quad (u2^{-1})^2 - v \text{ is a quadratic nonresidue of } p.$$

Proof. This follows from the fact that

$$x^2 - ux + v = (x - u2^{-1})^2 - ((u2^{-1})^2 - v)$$

is irreducible over $GF(p)$. $\square$

Remark 5.1. From Lemmas 5.3 and 5.6 we see that, given a prime p , to find $d(p)$, we need only, for those pairs of $u, v \in GF(p)$ satisfying (5.1) and (5.2), check if $GF^*(p^2) = (\theta)$ with $\theta^2 = u\theta - v$, and find $d(p, \theta)$ by Lemma 4.6; then take the maximum of them.

With the above preparation, we describe our algorithm in the following pseudocode:

REPEAT

 read a prime p and its least primitive root $g = g(p)$ from a disk file;

$d(p) \leftarrow 0$; $count \leftarrow 0$;

 For each pair of $u, v \in GF(p)$ satisfying (5.1) and (5.2) do

 BEGIN

 Check if $GF^*(p^2) = (\theta)$ with $\theta^2 = u\theta - v$ by Lemma 4.3;

 (cf. the proof of Lemma 4.6)

 If $GF^*(p^2) = (\theta)$ with $\theta^2 = u\theta - v$ then

 Begin

$count \leftarrow count + 1$; find $d(p, \theta)$ by Lemma 4.6;

 If $d(p, \theta) > d(p)$ then

 begin

$d(p) \leftarrow d(p, \theta)$; save u, v

 end

 End

 END;

$k(p) \leftarrow p - \sqrt{p^2 - d(p)}$;

 output $p, g, count, d(p), k(p), u, v$

UNTIL $p = p_{145}$;

Remark 5.2. If $count \neq \phi(p^2 - 1)/4$ for some p in the output, then there must be some errors in the program by Lemma 5.5.

6. NUMERICAL RESULTS: $k(p)$ FOR $p \leq p_{145}$

On an SCC486 (a compatible IBM PC/AT486), it takes about 120 hours to get $k(p)$ and related values for $p \leq p_{145}$ in Table 1.

TABLE 1

i	p_i	$g(p_i)$	$count$	$d(p_i)$	$k(p_i)$	u	v
3	5	2	2	13	1.535 ...	1	2
4	7	3	4	20	1.614 ...	1	3
5	11	2	8	46	2.339 ...	7	2
6	13	2	12	57	2.416 ...	2	6
7	17	3	24	89	2.857 ...	9	5
8	19	2	24	103	2.937 ...	9	13
9	23	5	40	140	3.276 ...	4	20
10	29	2	48	201	3.701 ...	7	19
11	31	3	64	195	3.323 ...	1	24
12	37	2	108	250	3.548 ...	6	32
13	41	6	96	307	3.932 ...	15	19
14	43	3	120	341	4.167 ...	4	19
15	47	5	176	404	4.514 ...	11	23
16	53	2	216	429	4.214 ...	44	26
17	59	2	224	439	3.845 ...	7	42
18	61	2	240	586	5.008 ...	58	10
19	67	2	320	617	4.774 ...	12	18
20	71	7	288	699	5.106 ...	6	59
21	73	5	432	646	4.567 ...	22	5
22	79	3	384	717	4.676 ...	61	74
23	83	2	480	793	4.923 ...	76	15
24	89	3	480	818	4.720 ...	23	59
25	97	5	672	1000	5.299 ...	63	58
26	101	2	640	1024	5.203 ...	6	53
27	103	5	768	912	4.526 ...	11	86
28	107	2	936	1018	4.867 ...	56	72
29	109	6	720	1128	5.303 ...	3	40
30	113	3	864	1121	5.074 ...	108	43
31	127	3	1152	1364	5.488 ...	23	109
32	131	2	960	1271	4.944 ...	10	119
33	137	3	1408	1417	5.273 ...	92	6
34	139	2	1056	1741	6.410 ...	74	26
35	149	2	1440	1618	5.532 ...	109	51
36	151	6	1440	1874	6.338 ...	135	140
37	157	5	1872	1649	5.342 ...	93	142
38	163	2	2160	1737	5.418 ...	76	122
39	167	5	1968	2279	6.968 ...	21	159
40	173	2	2352	1997	5.871 ...	89	46
41	179	2	2112	2055	5.835 ...	88	165
42	181	2	1728	2151	6.042 ...	126	128
43	191	19	2304	2306	6.135 ...	158	63
44	193	5	3072	2460	6.481 ...	116	153
45	197	2	2520	2283	5.882 ...	68	179
46	199	3	2400	2283	5.821 ...	143	148
47	211	2	2496	2888	6.958 ...	22	174
48	223	3	3456	2930	6.669 ...	200	20
49	227	2	4032	3093	6.918 ...	108	66
50	229	6	3168	3527	7.834 ...	80	194
51	233	3	4032	2899	6.306 ...	98	155
52	239	7	3072	2977	6.311 ...	115	173
53	241	7	3520	3527	7.432 ...	116	68
54	251	6	3600	3059	6.169 ...	208	202
55	257	3	5376	3413	6.728 ...	120	132

TABLE 1 (continued)

i	p_i	$g(p_i)$	$count$	$d(p_i)$	$k(p_i)$	u	v
56	263	5	5200	3467	6.675 ...	113	194
57	269	2	4752	3249	6.108 ...	7	132
58	271	6	4608	3502	6.540 ...	203	210
59	277	5	6072	3567	6.515 ...	17	179
60	281	3	4416	3809	6.861 ...	265	42
61	283	3	6440	3838	6.864 ...	191	226
62	293	2	6048	3891	6.716 ...	168	42
63	307	5	5760	4589	7.567 ...	245	267
64	311	17	5760	4455	7.246 ...	225	103
65	313	10	7488	4208	6.795 ...	178	14
66	317	2	8112	4222	6.730 ...	123	237
67	331	3	6560	4587	7.003 ...	59	90
68	337	10	7488	4661	6.987 ...	116	248
69	347	2	9632	5226	7.613 ...	54	264
70	349	2	6720	4429	6.404 ...	242	166
71	353	3	9280	5229	7.485 ...	6	212
72	359	7	8544	5270	7.416 ...	292	183
73	367	6	10560	5512	7.587 ...	121	341
74	373	2	9600	5408	7.321 ...	64	135
75	379	2	7776	4995	6.648 ...	242	284
76	383	5	12160	5223	6.880 ...	60	140
77	389	2	9216	5712	7.412 ...	219	375
78	397	5	11880	6649	8.464 ...	222	46
79	401	3	10560	5577	7.015 ...	10	19
80	409	21	10240	6290	7.763 ...	90	132
81	419	2	8640	6141	7.393 ...	219	96
82	421	2	10080	6102	7.310 ...	231	39
83	431	7	12096	6582	7.704 ...	57	426
84	433	5	12960	6403	7.457 ...	419	201
85	439	15	11520	6517	7.486 ...	11	74
86	443	2	13824	7167	8.164 ...	82	332
87	449	3	11520	6830	7.671 ...	237	166
88	457	13	16416	6562	7.236 ...	92	328
89	461	2	10560	6647	7.266 ...	198	251
90	463	3	13440	6512	7.086 ...	265	349
91	467	2	16704	6761	7.295 ...	339	295
92	479	13	15232	8175	8.610 ...	176	13
93	487	3	19440	7606	7.872 ...	106	368
94	491	2	13440	7584	7.784 ...	267	447
95	499	7	16400	8624	8.717 ...	208	417
96	503	5	18000	7206	7.214 ...	99	266
97	509	2	16128	7608	7.529 ...	38	440
98	521	3	16128	7779	7.519 ...	282	239
99	523	2	21840	8655	8.340 ...	520	446
100	541	2	19440	8535	7.946 ...	139	2
101	547	2	19584	8626	7.942 ...	510	241
102	557	2	24840	8541	7.720 ...	466	346
103	563	2	25760	9215	8.244 ...	234	388
104	569	3	20160	8655	7.656 ...	200	149
105	571	3	17280	9260	8.166 ...	473	537
106	577	5	26112	8991	7.844 ...	326	137
107	587	2	24528	10535	9.043 ...	188	11
108	593	3	25920	9371	7.954 ...	417	41
109	599	7	21120	9467	7.955 ...	329	62
110	601	7	20160	10156	8.509 ...	72	317
111	607	3	28800	11562	9.599 ...	263	345
112	613	2	29376	9583	7.866 ...	416	362
113	617	3	24480	10881	8.881 ...	542	12
114	619	2	24480	10272	8.353 ...	120	488
115	631	3	22464	9804	7.817 ...	411	270

TABLE 1 (continued)

i	p_i	$g(p_i)$	$count$	$d(p_i)$	$k(p_i)$	u	v
116	641	3	27136	12084	9.496 ...	461	384
117	643	11	27984	11568	9.059 ...	251	126
118	647	5	31104	12778	9.951 ...	488	511
119	653	2	34992	11480	8.850 ...	505	399
120	659	2	22080	10055	7.673 ...	370	594
121	661	2	26400	10021	7.624 ...	643	333
122	673	5	32256	12220	9.140 ...	656	290
123	677	2	34944	11634	8.647 ...	348	515
124	683	5	32400	11277	8.305 ...	286	79
125	691	3	30272	11769	8.569 ...	501	507
126	701	2	25920	12928	9.282 ...	458	523
127	709	2	32480	11043	7.830 ...	661	282
128	719	11	34368	12114	8.474 ...	80	674
129	727	5	31680	12768	8.834 ...	300	475
130	733	6	43920	12512	8.585 ...	187	92
131	739	3	34560	11965	8.140 ...	669	590
132	743	5	37440	13163	8.911 ...	700	467
133	751	3	36800	12105	8.102 ...	500	257
134	757	2	40824	13094	8.698 ...	132	656
135	761	6	36288	14077	9.305 ...	625	198
136	769	11	30720	12146	7.938 ...	462	247
137	773	2	48384	14417	9.382 ...	143	273
138	787	2	50960	13937	8.904 ...	452	588
139	797	2	42768	13333	8.408 ...	480	537
140	809	3	43200	13307	8.266 ...	82	653
141	811	3	36288	14428	8.944 ...	625	346
142	821	2	43520	13836	8.470 ...	186	233
143	823	3	55488	15272	9.331 ...	763	221
144	827	2	45936	13987	8.500 ...	87	708
145	829	2	43296	16938	10.279 ...	825	306

7. SUMMARY

Since $k(p_{145}) = 10.279\dots$ and $p_{145} = 829$, there exists an 829-element B_2 -sequence $\{b_i\}$ with $829 - b_{829}^{1/2} > 10.279$. By the proofs of Theorems 1 and 2, it is easy (actually it takes 4'16'' on an IBM PC/XT) to get all elements of $\{b_i\}$. To save space, we give only the first and the last ten elements as follows:

1 1738 3183 3419 4949 5710 6177 6522 7229 8380

 664432 664834 665138 665902 666010 667081 667206 668286 670235 670303

From Table 1 in §6, it is reasonable to conjecture that

(7.1) given $k > 0$, there exists an integer m such that $K(m) > k$.

Clearly, (7.1) contradicts (1.2). We hope that in a future paper, either (7.1) or (1.2) will be proved (i.e., the other will be disproved).

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