MATHEMATICS OF COMPUTATION
VOLUME 63, NUMBER 207
JULY 1994, PAGES 307–327

A GENERALIZED SAMPLING THEOREM FOR LOCALLY COMPACT ABELIAN GROUPS

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ABSTRACT. We present a sampling theorem for locally compact abelian groups. The sampling sets are finite unions of cosets of a closed subgroup. This generalizes the well-known case of nonequidistant but periodic sampling on the real line. For nonbandlimited functions an L_1-type estimate for the aliasing error is given. We discuss the application of the theorem to a class of sampling sets in $\mathbf{R}^s$, give a general algorithm for a computer implementation, present a detailed description of the implementation for the s-dimensional torus group, and point out connections to lattice rules for numerical integration.

1. INTRODUCTION

The classical sampling theorem permits reconstruction of a bandlimited function from its values on a set of equidistant points on the real line $\mathbf{R}$. It has been extended in many directions; see the reviews [4, 14, 15] as well as the volumes [24, 25]. An important generalization results from replacing $\mathbf{R}$ by an arbitrary locally compact abelian group G [18], cf. [14, Story 4]. The sampling set is then a coset of a closed subgroup of G. The purpose of this paper is to extend this result to sampling sets which are unions of finitely many cosets of a closed subgroup H, to provide an error analysis for the aliasing error caused by not strictly bandlimited functions, and to discuss the practical application of the results.

Since our sampling sets are invariant under translations by elements of the subgroup H, we will call them periodic sampling sets. The first examples occurred in studies of nonequidistant but periodic sampling on $\mathbf{R}$ [16, 19, 32, 39]. Subsequent generalizations include extensions to $\mathbf{R}^s$ [6, 11], and results for $\mathbf{R}^s \times [0, 2\pi)^r$ [8] with applications to computed tomography [7, 8]. Cheung and Marks [5, 6, 24] have constructed multidimensional periodic sampling sets which permit sampling below the Nyquist density obtained from the classical sampling theorem.

The approach presented here generalizes a method which the author learned from H. J. Landau [20, 21], who attributed it to unpublished work of A. Beurling. A similar method has been used in Kohlenberg's early paper [19].

Received by the editor November 24, 1992 and, in revised form, August 25, 1993.

1991 *Mathematics Subject Classification*. Primary 65T99, 65D30, 94A05, 43A25; Secondary 65D05, 65D32, 65T20, 42B99.

Key words and phrases. Sampling theorem, nonuniform sampling, multidimensional sampling, periodic sampling, bunched sampling, lattice rules, locally compact abelian groups.

A different approach was developed by Butzer and Hinsen [3]. They use the results for nonequidistant but periodic sampling as well as other forms of nonuniform sampling on $\mathbf{R}$ to construct two-dimensional sampling sets. Their results are not encompassed by ours, or vice versa, but the applications described in [3, pp. 77–82] involve periodic sampling sets. We note that the theory to be presented here is not adequate to treat the general case of nonperiodic irregular sampling, where the sampling set only needs to meet some density requirements; see, e.g., [1, 9, 10, 12, 26] and the references given there.

In §2 we introduce the necessary facts about Fourier analysis on locally compact abelian groups (LCA groups). The reason for choosing this general setting is a practical one. On the one hand we obtain results which cover a large class of applications. On the other we are able to use powerful theorems like Pontryagin's duality theorem and its consequences, which lead to simpler and more transparent proofs. We will only need the basic concepts of the theory.

The main results are proved in §3. There, we consider the following problem: Let f be defined on an LCA group G. Compute the Fourier transform $\hat{f}(\xi)$ for ξ in a certain compact set K' from knowledge of f on a finite union of shifted copies of a subgroup $H \subset G$. The sampling theorem then follows by taking an inverse Fourier transform.

Application of the theorem in practice requires the computation of certain auxiliary quantities, which is not always a simple matter. We therefore consider some special cases where these computations simplify considerably. We derive a class of such sampling sets for $G = \mathbf{R}^s$, under the condition that $\hat{f}$ is concentrated in a rectangular set. This class contains the two-dimensional sampling sets discussed in [3, pp. 77–82]. Similar examples are considered in [6].

The application of our sampling theorem in the general case requires a computer implementation. We therefore devote §4 to this topic. After describing a general algorithm, we discuss in detail all the necessary steps for the case of H being a finite subgroup of the s-dimensional torus group $T^s = \mathbf{R}^s/\mathbf{Z}^s$. This case is of interest for two reasons: First, as we will see below, there exists a relatively simple and fast implementation. Second, even when sampling of nonperiodic functions defined on $\mathbf{R}^s$ is required, only finitely many samples can be processed in practice. This means that one effectively approximates the true function by a function with compact support. Functions with compact support, however, can be viewed as being defined on T^s by means of a simple change of variables. The case of sampling on T^s is therefore both easy to implement and of practical importance.

For $K' = \{0\}$ the problem considered in §3 reduces to numerical integration. It is therefore not surprising that our approach is related to, and in some aspects a generalization of, the class of methods in numerical integration known as lattice rules [37]. We encounter another connection to lattice rules in §4, since the finite subgroups of T^s are precisely the abscissa sets of lattice rules. For an introduction and a review of recent developments in this area see [29, Chapter 5], and also [22, 36].

2. Standard definitions and facts

Let $\mathbf{Z}, \mathbf{R}, \mathbf{C}$ denote the integers, reals, and complex numbers, respectively. For $x, y \in \mathbf{C}^s$ we denote the scalar product by $(x, y) = \sum_{i=1}^{s} x_i \overline{y}_i$. For

$a, b \in \mathbf{R}$, $\operatorname{mod}(a, b)$ is the real number satisfying $0 \le \operatorname{mod}(a, b) < |b|$ and $a - \operatorname{mod}(a, b) \in b\mathbf{Z}$. The fractional part of the real number a is given by $[a] = \operatorname{mod}(a, 1)$. If $x \in \mathbf{R}^s$, then $[x]$ is defined as $([x_1], \dots, [x_s])^T$, x^T denoting the transpose of x. For $X \subset \mathbf{R}^s$, $[X]$ denotes the set $\{[x], x \in X\}$. We will usually write the elements of $\mathbf{R}^s$ as column vectors.

Let G denote a locally compact abelian group written additively. For V a subset of G, let $\overline{V}$ and $|V|$ denote the closure of V and the number of elements in V, respectively. The indicator function χ_V is given by $\chi_V(x) = 1$ for $x \in V$ and $\chi_V(x) = 0$ otherwise. The character group $\widehat{G}$ consists of the continuous homomorphisms of G into the circle group $T = \mathbf{R}/\mathbf{Z}$. The value of the character $\xi \in \widehat{G}$ at the point $x \in G$ is written $\langle x, \xi \rangle$. $\widehat{G}$ has a natural addition and a natural topology relative to which it is also an LCA group. If G is compact, $\widehat{G}$ is discrete. If G is a finite group, then $|G| = |\widehat{G}|$. The Pontryagin duality theorem states that

$$\hat{\hat{G}} = G.$$

Standard examples are: (a) $G = \mathbf{R}^s$, $\widehat{G} = \mathbf{R}^s$; (b) $G = T^s = \mathbf{R}^s/\mathbf{Z}^s$, $\widehat{G} = \mathbf{Z}^s$; (c) $G = \mathbf{Z}^s$, $\widehat{G} = T^s$, where (c) is a consequence of (b) and the duality theorem. In all three cases we have $\langle x, \xi \rangle = [\sum_{i=1}^s x_i \xi_i] = [(x, \xi)]$.

On every LCA group there exists a nonnegative regular measure m_G, the so-called Haar measure of G, which is not identically zero and translation invariant. The Haar measure is uniquely determined up to multiplication by a constant. $m_G(G)$ is finite if and only if G is compact. In this case we normalize m_G so that $m_G(G) = 1$. If G is discrete, m_G will be a multiple of the counting measure. If G is discrete but not compact, we normalize m_G so that it equals the counting measure, i.e., any set consisting of a single point has measure 1. The following useful orthogonality relation is a direct consequence of the translation invariance of the Haar integral. For a proof, see, e.g., [13, §23.19] or [35, p. 10].

Lemma 2.1. *If G is compact and its Haar measure is normalized so that $m_G(G) = 1$, then*

$$\int_G e^{2\pi i \langle x, \xi \rangle}\, dm_G(x) = \begin{cases} 1, & \xi = 0, \\ 0, & \xi \neq 0. \end{cases}$$

$L_p(G)$ denotes the space of all Borel functions on G such that $\|f\|_p = (\int_G |f(x)|^p\, dm_G(x))^{1/p}$ is finite. $C_c(G)$ is the space of all continuous functions on G with compact support. The Fourier transform of a function $f \in L_1(G)$ is the continuous function $\hat{f}$ on $\widehat{G}$ defined by

$$\hat{f}(\xi) = \int_G f(x) e^{-2\pi i \langle x, \xi \rangle}\, dm_G(x).$$

We will always normalize the Haar measure $m_{\widehat{G}}$ such that the following holds.

Theorem 2.2 (Fourier inversion formula). *If $f \in L_1(G)$ is continuous and $\hat{f} \in L_1(\widehat{G})$, then*

$$f(x) = \int_{\widehat{G}} \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle}\, dm_{\widehat{G}}(\xi) = \hat{\hat{f}}(-x). \tag{1}$$

The Fourier transform can be extended to a linear isomorphism of $L_2(G)$ onto $L_2(\widehat{G})$ by means of the Plancherel theorem (cf. [13, §31.18]). The convolution $f * g$ of two functions $f, g \in L_2(G)$ is given by $(f * g)(x) = \int_G f(y)g(x-y)\,dm_G(y)$. If $f * g \in L_2(G)$, we have $(f * g)^\frown(\xi) = \hat{f}(\xi)\hat{g}(\xi)$. The inverse formula

$$(f * g)(x) = \int_{\widehat{G}} \hat{f}(\xi)\hat{g}(\xi)e^{2\pi i\langle x, \xi\rangle}\,dm_{\widehat{G}}(\xi) \tag{2}$$

holds for all $f, g \in L_2(G)$ and is equivalent to the Parseval Identity.

Let H be a closed subgroup of an LCA group G. The annihilator of H is the set $H^\perp \subset \widehat{G}$ given by

$$H^\perp = \{\eta \in \widehat{G} : \langle y, \eta\rangle = 0 \text{ for all } y \in H\}.$$

$H^\perp$ is a closed subgroup of $\widehat{G}$ and is isomorphically homeomorphic to the character group of G/H, i.e.,

$$H^\perp = (G/H)^\frown.$$

Furthermore, we have that

$$(H^\perp)^\perp = H \quad \text{and} \quad \widehat{H} = \widehat{G}/H^\perp.$$

The following technical lemma, which we will need later on, is a consequence of the identification of $H^\perp$ with $(G/H)^\frown$.

Lemma 2.3. *Let G be an LCA group, H a closed subgroup of G such that G/H is compact, and K a compact subset of $\widehat{G}$. Then $|H^\perp \cap K|$ is finite.*

Proof. Since G/H is compact, the topology of $(G/H)^\frown = H^\perp$ is discrete; i.e., all sets $\{\eta\}$ whose only element is the point $\eta \in H^\perp$ are open. Since the topology on $H^\perp$ is the relative topology induced in $H^\perp$ by $\widehat{G}$, it follows that for each $\eta \in H^\perp$ there is an open set $U_\eta \subset \widehat{G}$ such that $H^\perp \cap U_\eta = \{\eta\}$. Therefore, the open set $\widehat{G}\backslash H^\perp$ and the sets U_η for $\eta \in H^\perp \cap K$ provide an open covering of K. Since K is compact, there is a finite subcovering C. We obtain

$$|H^\perp \cap K| = \left|\bigcup_{U_\eta \in C} H^\perp \cap U_\eta\right| = \left|\bigcup_{U_\eta \in C} \{\eta\}\right| < \infty. \qquad \square$$

For $f \in L_1(G)$ define the function $R_H f : G/H \to \mathbf{C}$ by

$$R_H f(x + H) = \int_H f(x + y)\,dm_H(y).$$

Note that the integral on the right-hand side does not change when x is replaced by $x+h$ with $h \in H$, so $R_H f$ is indeed a function of the coset $x+H$. According to Theorem 28.54 in [13], $R_H f$ belongs to $L_1(G/H)$. If $f \in C_c(G)$, then $R_H f \in C_c(G/H)$; see [13, Theorem 15.21]. We normalize the Haar measures on H, G/H, and $H^\perp$ so that

$$\begin{aligned}&\int_{G/H} R_H f(x + H)\,dm_{G/H}(x + H)\\ &\quad = \int_{G/H}\int_H f(x + y)\,dm_H(y)\,dm_{G/H}(x + H) = \int_G f\,dm_G,\end{aligned} \tag{3}$$

and the Fourier inversion formula holds for the Fourier transform on G/H. Multiplying $f(x)$ in (3) by $e^{-2\pi i\langle x,\eta\rangle}$ with $\eta \in H^\perp$ gives the following relation between the Fourier transforms of f and $R_H f$:

$$(R_H f)\hat{}(\eta) = \hat{f}(\eta), \qquad \eta \in H^\perp. \tag{4}$$

Note that on the left-hand side the Fourier transform is taken with respect to G/H, while it is taken with respect to G on the right-hand side.

The Fourier inversion formula for $R_H f$ gives rise to the Poisson summation formula (cf. [13, §31.46(e)]):

Theorem 2.4. *Suppose that $f \in L_1(G)$, that every function $y \to f(x+y)$ belongs to $L_1(H)$, that $R_H f$ is a continuous function on G/H, and that $(R_H f)\hat{} \in L_1(H^\perp)$. Then*

$$\int_H f(x+y)\, dm_H(y) = \int_{H^\perp} \hat{f}(\eta) e^{2\pi i\langle x,\eta\rangle}\, dm_{H^\perp}(\eta). \tag{5}$$

Proof. Using (4), we see that (5) can be written as

$$R_H f(x+H) = \int_{H^\perp} (R_H f)\hat{}(\eta) e^{2\pi i\langle x,\eta\rangle}\, dm_{H^\perp}(\eta).$$

Hence, (5) is established by applying Theorem 2.2 with f, G, and $\widehat{G}$ replaced by $R_H f$, G/H, and $H^\perp$, respectively. □

If Theorem 2.4 can be applied to the function $x \to f(x)e^{-2\pi i\langle x,\xi\rangle}$, $\xi \in \widehat{G}$, the following version of the Poisson summation formula results:

$$\int_H f(x+y) e^{-2\pi i\langle x+y,\xi\rangle}\, dm_H(y) = \int_{H^\perp} \hat{f}(\xi+\eta) e^{2\pi i\langle x,\eta\rangle}\, dm_{H^\perp}(\eta). \tag{6}$$

Let us consider some examples. (i) The Haar measure on $G = \mathbf{R}^s$ as well as on $\widehat{G} = \mathbf{R}^s$ is the Lebesgue measure, and the Fourier transform is given by

$$\hat{f}(\xi) = \int_{\mathbf{R}^s} f(x) e^{-2\pi i(x,\xi)}\, dx.$$

Let $\theta \in \mathbf{R}^s$ with $(\theta, \theta) = 1$. The hyperplane $H = \{x \in \mathbf{R}^s,\ (x, \theta) = 0\}$ is a closed subgroup of $\mathbf{R}^s$. We see that $H^\perp = \{\tau\theta,\ \tau \in \mathbf{R}\}$, and because of $x+H = (x, \theta)\theta + H$ we have $\mathbf{R}^s/H = \{t\theta + H,\ t \in \mathbf{R}\}$. $R_H f(t\theta + H) = \int_{(x,\theta)=t} f(x)\, dx$ is called the Radon transform of f. Writing $R_\theta f(t)$ for $R_H f(t\theta + H)$, we see that (4) reads

$$\begin{aligned}\hat{f}(\tau\theta) = (R_H f)\hat{}(\tau\theta) &= \int_{\mathbf{R}} R_H f(t\theta + H) e^{-2\pi i(t\theta,\tau\theta)}\, dt \\ &= \int_{\mathbf{R}} R_\theta f(t) e^{-2\pi i\tau t}\, dt = (R_\theta f)\hat{}(\tau).\end{aligned}$$

This is the well-known "projection-slice" theorem. In computerized tomography it is the basis of some standard algorithms for reconstructing f from measurements of $R_\theta f$ [27].

(ii) Let G be the s-dimensional torus group $T^s = (\mathbf{R}/\mathbf{Z})^s$. We choose the s-dimensional hypercube $[0, 1)^s$ with addition modulo 1 in each component as a model for T^s. The Haar measure m_G is the restriction of the Lebesgue measure on $\mathbf{R}^s$ to $[0, 1)^s$, and $m_{\widehat{G}}$ is the counting measure. For $f \in L_1(G)$ we have

$$\hat{f}(\xi) = \int_{[0,1)^s} f(x) e^{-2\pi i \langle x, \xi \rangle} \, dx .$$

Since $\xi \in \mathbf{Z}^s$, and because of

$$e^{-2\pi i \langle x, \xi \rangle} = e^{-2\pi i \sum_{k=1}^s x_k \xi_k},$$

$\hat{f}(\xi)$ is just the usual Fourier coefficient of f. Hence, the condition $\hat{f} \in L_1(\widehat{G})$ means that f has an absolutely convergent Fourier series.

Let H be a finite subgroup of T^s. It can be shown (see Proposition 4.2 below) that there exists an $s \times s$ matrix W such that W^{-1} is an integer matrix and, as a set, $H = W\mathbf{Z}^s \cap [0, 1)^s = [W\mathbf{Z}^s]$. Hence, H represents an integration lattice in the sense of Sloan and Kachoyan [37]. The annihilator $H^\perp$ is given by the "reciprocal lattice" $W^{-T}\mathbf{Z}^s$, where W^{-T} denotes the transpose of W^{-1}. The Haar measure on H is $1/|H|$ times the counting measure. $m_{H^\perp}$ is equal to the counting measure. The integral over H,

$$\int_H f(y) \, dm_H(y) = \frac{1}{|H|} \sum_{y \in H} f(y),$$

is now a so-called lattice rule for numerical integration, which gives an approximation for $\int_G f(x) \, dm_G(x) = \int_{[0,1)^s} f(x) \, dx$. The error analysis is furnished by the Poisson summation formula (5), which reads as follows: Let f be continuous and have an absolutely convergent Fourier series. Then

$$\frac{1}{|H|} \sum_{y \in H} f(x + y) = \sum_{\eta \in H^\perp} \hat{f}(\eta) e^{2\pi i \langle x, \eta \rangle}.$$

Letting $x = 0$ and remembering that $\hat{f}(0) = \int_{[0,1)^s} f(x) \, dx$, we obtain the formula

$$\frac{1}{|H|} \sum_{y \in H} f(y) - \int_{[0,1)^s} f(x) \, dx = \sum_{\eta \in H^\perp, \ \eta \neq 0} \hat{f}(\eta), \tag{7}$$

which is the basis of analyzing the integration error; see [37]. For example, assume that $\hat{f}(\xi) = 0$ outside a compact set $K \subset \mathbf{Z}^s$. If the lattice H is chosen such that

$$H^\perp \cap K \subseteq \{0\}, \tag{8}$$

the right-hand side of (7) will vanish and the numerical integration is exact. Special choices of K can serve as criteria for constructing lattice rules. For details, see, e.g., [22] and the references given there.

3. Sampling theorems

We will assume throughout this section that H is a closed subgroup of an LCA group G, and that G/H is compact. This implies that $H^\perp = (G/H)^\wedge$ is discrete. We assume m_G to be given and normalize the Haar measures on H and G/H so that $m_{G/H}(G/H) = 1$ and (3) holds. The Fourier inversion formula then requires $m_{H^\perp}$ to be equal to the counting measure on $H^\perp$. Furthermore, we assume that $f \in L_1(G)$ is continuous, every function $y \to f(x+y)$

belongs to $L_1(H)$, and $\hat{f}(\xi) = 0$ outside a compact set $K \subset \widehat{G}$. It follows that the Poisson summation formula (6) holds for all $\xi \in \widehat{G}$.

In the following we study the problem of computing $\hat{f}(\xi)$ for ξ in a compact set $K' \subseteq K$ from the values of f on finitely many cosets $x_n + H$. The sampling theorem then follows from an inverse Fourier transform; see Theorem 3.5 below. We first derive a result for the case $K' = \{0\}$, i.e., for approximating $\int_G f\, dm_G$ from finitely many values of $R_H f(x+H)$. Equation (3) suggests the following approach:

$$\int_G f(x)\, dm_G(x) = \int_{G/H} R_H f(x+H)\, dm_{G/H}(x+H)$$
$$\simeq \sum_{n=0}^{N-1} \beta_n R_H f(x_n + H),$$

with coefficients $\beta_n \in \mathbf{C}$. This amounts to performing a numerical integration of $R_H f$ over G/H. We will use our a priori information about $\hat{f}$ to develop criteria on how to choose the β_n and x_n. The simplest case is to use $R_H(0 + H) = \int_H f(y)\, dm_H(y)$ as our approximation. The example at the end of the previous section suggests calling this a generalized lattice rule. (A different generalization for compact but not necessarily abelian groups has been given by Niederreiter [30].) The Poisson summation formula yields the general form of equation (7):

$$\int_H f(y)\, dm_H(y) - \int_G f(x)\, dm_G(x) = \sum_{\eta \in H^\perp,\ \eta \neq 0} \hat{f}(\eta).$$

Note that the Haar integral over $H^\perp$ is a series since $H^\perp$ is discrete. Again the lattice rule will be exact if the condition (8), i.e., $H^\perp \cap K \subseteq \{0\}$, is satisfied.

Now assume that (8) does not hold. Since $|H^\perp \cap K|$ is finite because of Lemma 2.3, there is an integer m such that

$$(H^\perp \cap K) \backslash \{0\} = \{\eta_1, \ldots, \eta_{m-1}\}.$$

Using (5) with $x = x_n$ yields

$$\begin{aligned} \sum_{n=0}^{N-1} \beta_n R_H f(x_n + H) - \sum_{n=0}^{N-1} \beta_n \hat{f}(0) &= \sum_{\eta \in H^\perp,\ \eta \neq 0} \hat{f}(\eta) \sum_{n=0}^{N-1} \beta_n e^{2\pi i \langle x_n, \eta \rangle} \\ &= \sum_{j=1}^{m-1} \hat{f}(\eta_j) \sum_{n=0}^{N-1} \beta_n e^{2\pi i \langle x_n, \eta_j \rangle}. \end{aligned} \tag{9}$$

Now we try to choose points x_n and weights β_n such that the right-hand side vanishes and $\sum_{n=0}^{N-1} \beta_n = 1$. We obtain

Theorem 3.1. *Let* $(H^\perp \backslash \{0\}) \cap K = \{\eta_1, \ldots, \eta_{m-1}\}$. *If*

$$\begin{aligned} &\sum_{n=0}^{N-1} \beta_n = 1, \\ &\sum_{n=0}^{N-1} \beta_n e^{2\pi i \langle x_n, \eta_j \rangle} = 0, \qquad j = 1, \ldots, m-1, \end{aligned} \tag{10}$$

then

$$\sum_{n=0}^{N-1} \beta_n R_H f(x_n + H) = \int_G f(x)\, dm_G(x).$$

Proof. Insert (10) into (9). □

The question arises under what conditions the system of equations (10) admits a solution. An obvious necessary and sufficient condition is that the first row of the system matrix must not be an element of the subspace of $\mathbf{C}^N$ spanned by the other rows. The next two propositions give a necessary condition and a sufficient condition, respectively, in terms of the x_n and η_j. Let U denote the smallest subgroup of G/H containing $\{x_0 + H, \dots, x_{N-1} + H\}$ and let us write M_0 for $\{\eta_1, \dots, \eta_{m-1}\}$.

Proposition 3.2. *If* (10) *admits a solution, then* $M_0 \cap U^\perp = \varnothing$.

Proof. Assume there is $1 \le k \le m-1$ with $\eta_k \in M_0 \cap U^\perp$. Then $\langle x_n, \eta_k \rangle = 0$ for $n = 0, \dots, N-1$, and the equation for $j = k$ reads

$$\sum_{n=0}^{N-1} \beta_n e^{2\pi i \langle x_n, \eta_k \rangle} = \sum_{n=0}^{N-1} \beta_n = 0,$$

which contradicts the first equation of (10). □

If the $(x_n + H)$ form a subgroup of G/H, the necessary condition of the previous proposition is also sufficient:

Proposition 3.3. *If* $U = \{x_0 + H, \dots, x_{N-1} + H\}$ *and* $M_0 \cap U^\perp = \varnothing$, *then the choice* $\beta_n = 1/N$, $n = 0, \dots, N-1$, *gives a solution of* (10).

Proof. U is a discrete, compact group; hence the Haar measure m_U equals $1/N$ times the counting measure. We have $\widehat{U} = (G/H)^\wedge/U^\perp = H^\perp/U^\perp$, and the value of the character $\eta + U^\perp \in \widehat{U}$ at the point $x_n + H$ is given by $\langle x_n + H, \eta + U^\perp \rangle = \langle x_n, \eta \rangle$. The assertion now follows from applying Lemma 2.1 with $G = U$ for $\xi = 0$ and $\xi = \eta_j + U^\perp$, $j = 1, \dots, m-1$, respectively. □

We will now derive extensions of Theorem 3.1. If $f(x)$ satisfies our general assumptions, so does $g(x) = f(x)e^{-2\pi i \langle x, \xi \rangle}$. We see that $\hat{g}(\xi') = 0$ for $\xi' \notin ((-\xi) + K)$. Applying Theorem 3.1 to g gives

Proposition 3.4. *For* $\xi \in \widehat{G}$ *let* M_ξ *be the finite set*

$$(H^\perp \backslash \{0\}) \cap (K - \xi) = \{\eta_1, \dots, \eta_{m-1}\}.$$

If $x_n + H \in G/H$, $\beta_n \in \mathbf{C}$ *satisfy* (10), *then*

$$\hat{f}(\xi) = \sum_{n=0}^{N-1} \beta_n \int_H f(x_n + y) e^{-2\pi i \langle x_n + y, \xi \rangle}\, dm_H(y). \tag{11}$$

Our next goal is to use Proposition 3.4 to compute $\hat{f}(\xi)$ for ξ in a compact set $K' \subseteq K$ with fixed points x_n independent of ξ. This is more complicated because the coefficients β_n depend on M_ξ and therefore on ξ. However, for $\xi \in K'$, $M_\xi = (H^\perp \backslash \{0\}) \cap (K - \xi)$ is contained in $H^\perp \cap (K - K')$, which is a finite set since $K - K'$ is compact. Hence, as ξ runs through K', M_ξ will assume only finitely many different values $M_1, \dots, M_L$. The relation $\xi \equiv \xi' \Leftrightarrow M_\xi = M_{\xi'}$ is an equivalence relation on K' induced by the subgroup H. The equivalence classes are $K_l = \{\xi \in K' : M_\xi = M_l\}$, $l = 1, \dots, L$. The sets K_l are mutually disjoint and we have $K' = \bigcup_{l=1}^{L} K_l$. Each K_l consists of the points ξ for which $\xi + \eta \in K$ if $\eta \in M_l \cup \{0\}$, and $\xi + \eta \notin K$ if $\eta \in H^\perp \backslash (M_l \cup \{0\})$.

Theorem 3.5. *Assume that $f \in L_1(G)$ is continuous, every function $y \to f(x+y)$ belongs to $L_1(H)$, and that $\hat{f}$ vanishes outside a compact set $K \subset \widehat{G}$. Let $M_l = \{\eta_1^{(l)}, \dots, \eta_{m_l-1}^{(l)}\}$, $l = 1, \dots, L$, be the values assumed by $(H^\perp \backslash \{0\}) \cap (K - \xi)$ as ξ runs through the compact set $K' \subseteq K$. Let χ_{K_l} be the indicator function of $K_l = \{\xi \in K' : M_\xi = M_l\}$. Assume $x_0 + H, \dots, x_{N-1} + H \in G/H$ are such that for $l = 1, \dots, L$ the systems of equations*

$$\tag{12} \begin{gathered} \sum_{n=0}^{N-1} \beta_n^{(l)} = 1, \\ \sum_{n=0}^{N-1} \beta_n^{(l)} e^{2\pi i \langle x_n, \eta_j^{(l)} \rangle} = 0, \qquad j = 1, \dots, m_l - 1, \end{gathered}$$

admit solutions $\beta_n^{(l)}$, $n = 0, \dots, N-1$. Let $F \in L_2(G)$ with $\operatorname{supp}(\widehat{F}) \subseteq K'$, *and define*

$$\tag{13} (Sf)^\frown(\xi) = \widehat{F}(\xi) \sum_{n=0}^{N-1} \sum_{l=1}^{L} \beta_n^{(l)} \chi_{K_l}(\xi) \int_H f(x_n + y) e^{-2\pi i \langle x_n + y, \xi \rangle} \, dm_H(y)$$

and $Sf(x) = \int_{\widehat{G}} (Sf)^\frown(\xi) e^{2\pi i \langle x, \xi \rangle} \, dm_{\widehat{G}}(\xi)$. Then

$$\tag{14} \widehat{F}(\xi) \hat{f}(\xi) = (Sf)^\frown(\xi) \quad \textit{for } \xi \in \widehat{G}$$

and

$$\tag{15} (F * f)(x) = (Sf)(x) = \sum_{n=0}^{N-1} \int_H f(x_n + y) k_n(x - x_n - y) \, dm_H(y)$$

with

$$\tag{16} k_n(z) = \sum_{l=1}^{L} \beta_n^{(l)} (F * \bar{\check{\chi}}_{K_l})(z).$$

Proof. If $\xi \in \widehat{G} \backslash K'$, both sides of (14) are zero. If $\xi \in K'$, then $\xi \in K_{l_0}$ for some $l_0 \in \{1, \dots, L\}$. Hence $\sum_{l=1}^{L} \beta_n^{(l)} \chi_{K_l}(\xi) = \beta_n^{(l_0)}$, and applying Proposition 3.4 with $M_\xi = M_{l_0}$, $m = m_{l_0}$, and $\beta_n = \beta_n^{(l_0)}$ yields (14). Since $\hat{f}$ is bounded with compact support, we have $\hat{f} \in L_2(\widehat{G})$. Taking an inverse Fourier transform and using (2) yields (15). □

Theorem 3.5 permits us to obtain filtered versions of f from knowledge of f on cosets $x_n + H$, $n = 0, \dots, N-1$. Of particular interest is the case $K' = K$, $\widehat{F} = \chi_K$. Then we can compute the function f itself, since (15) reads

$$\tag{17} f(x) = \sum_{n=0}^{N-1} \int_H f(x_n + y) k_n(x - x_n - y) \, dm_H(y)$$

with (16) simplifying to

$$k_n(z) = \sum_{l=1}^{L} \beta_n^{(l)} \hat{\chi}_{K_l}(-z). \tag{18}$$

The case $L = m_1 = 1$ yields the classical sampling theorem [18, 33]. This requires that the sets $K + \eta$, $\eta \in H^\perp$, be mutually disjoint.

It may happen that some of the sets K_l have measure zero. In this case $\hat{\chi}_{K_l}(z) = 0$ for all $z \in G$; hence these sets do not contribute to the sampling series Sf in (15). They may therefore be ignored, and the corresponding coefficients $\beta_n^{(l)}$ need not be computed.

The next theorem gives an estimate for the error committed when $\hat{f}(\xi)$ does not vanish outside K. It is the main result of this section.

Theorem 3.6. *Assume that the hypothesis of Theorem* 3.5 *holds except the condition that* $\hat{f}$ *vanishes outside* K. *Assume instead that* $\hat{f} \in L_1(\widehat{G})$ *and that the Poisson formula* (6) *holds for all* $x \in \{x_0, \dots, x_{N-1}\}$ *and almost all* $\xi \in K'$. *Then*

$$|(Sf)(x) - (F * f)(x)| \le \|\widehat{F}\|_\infty \gamma \int_{\widehat{G}\setminus K} |\hat{f}(\xi)|\, dm_{\widehat{G}}(\xi) \tag{19}$$

for all $x \in G$, *where*

$$\gamma = \overline{m}\left(\max_{l=1,\dots,L} \sum_{n=0}^{N-1} |\beta_n^{(l)}|\right), \qquad \overline{m} = \max_{l=1,\dots,L} m_l.$$

Proof. By (2) it is clear that $|(Sf)(x) - (F * f)(x)| \le \|(Sf)^\frown - \widehat{F}\hat{f}\|_1$. Since $\widehat{F}(\xi)$ vanishes outside K', we obtain from (13) and (6) that

$$\begin{aligned}\|(Sf)^\frown - \widehat{F}\hat{f}\|_1 \\ = \int_{K'} \left| \widehat{F}(\xi) \left(\sum_{n=0}^{N-1}\sum_{l=1}^{L} \beta_n^{(l)} \chi_{K_l}(\xi) \sum_{\eta\in H^\perp} \hat{f}(\xi+\eta) e^{2\pi i \langle x_n, \eta\rangle} - \hat{f}(\xi)\right)\right| dm_{\widehat{G}}(\xi).\end{aligned}$$

For $\xi \in K'$ we have $\sum_{n=0}^{N-1}\sum_{l=1}^{L} \beta_n^{(l)} \chi_{K_l}(\xi) = 1$; hence the term with $\eta = 0$ in the summation is equal to $\hat{f}(\xi)$. This gives

$$\|(Sf)^\frown - \widehat{F}\hat{f}\|_1 = \int_{K'} \left| \widehat{F}(\xi) \sum_{n=0}^{N-1}\sum_{l=1}^{L} \beta_n^{(l)} \chi_{K_l}(\xi) \sum_{\eta\in H_0^\perp} \hat{f}(\xi+\eta) e^{2\pi i \langle x_n, \eta\rangle}\right| dm_{\widehat{G}}(\xi),$$

where we have written $H_0^\perp$ for $H^\perp\setminus\{0\}$. Let $P^l(\eta) = \sum_{n=0}^{N-1} \beta_n^{(l)} e^{2\pi i\langle x_n,\eta\rangle}$. We have $P^l(0) = 1$, $P^l(\eta) = 0$ for $\eta \in M_l$, and $\sup_{\eta\in H^\perp} |P^l(\eta)| \le \sum_{n=0}^{N-1} |\beta_n^{(l)}|$.

Hence,

$$
\begin{aligned}
&\|(Sf)^{\frown} - \widehat{F}\hat{f}\|_1 \\
&= \int_{K'} \left| \sum_{l=1}^{L} \chi_{K_l}(\xi)\widehat{F}(\xi) \sum_{\eta \in H_0^{\perp}\setminus M_l} \hat{f}(\xi+\eta)P^l(\eta) \right| dm_{\widehat{G}}(\xi) \\
&\le \sum_{l=1}^{L} \int_{K_l} \sum_{\eta \in H_0^{\perp}\setminus M_l} |\widehat{F}(\xi)\hat{f}(\xi+\eta)P^l(\eta)|\, dm_{\widehat{G}}(\xi) \\
&\le \|\widehat{F}\|_\infty \left(\max_{l=1,\dots,L} \sum_{n=0}^{N-1} |\beta_n^{(l)}| \right) \sum_{l=1}^{L} \sum_{\eta \in H_0^{\perp}\setminus M_l} \int_{K_l+\eta} |\hat{f}(\xi)|\, dm_{\widehat{G}}(\xi).
\end{aligned}
\tag{20}
$$

Since the sets K_l are disjoint and $(K_l+\eta)\cap K = \varnothing$ for $\eta \in H_0^{\perp}\setminus M_l$, it follows that

$$
\begin{aligned}
\sum_{l=1}^{L} \sum_{\eta \in H_0^{\perp}\setminus M_l} \int_{K_l+\eta} |\hat{f}(\xi)|\, dm_{\widehat{G}}(\xi) &\le \sum_{\eta \in H_0^{\perp}} \sum_{l=1}^{L} \int_{(K_l+\eta)\setminus K} |\hat{f}(\xi)|\, dm_{\widehat{G}}(\xi) \\
&= \sum_{\eta \in H_0^{\perp}} \int_{(K'+\eta)\setminus K} |\hat{f}(\xi)|\, dm_{\widehat{G}}(\xi).
\end{aligned}
\tag{21}
$$

It remains to clarify how many of the translates $K'+\eta$ may contain a given point ξ. If $\xi \in K'+\eta$, then there is l_0 such that $\xi-\eta \in K_{l_0}$. If also $\xi \in K'+\eta'$, then $(\xi-\eta)+(\eta-\eta') = \xi-\eta' \in K' \subseteq K$, which implies that $\eta-\eta' \in M_{l_0}\cup\{0\}$. Since $M_{l_0}\cup\{0\}$ has only $m_{l_0} \le \overline{m}$ elements, there are at most $\overline{m}$ different η with $\xi \in K'+\eta$. Therefore,

$$
\sum_{\eta \in H_0^{\perp}} \int_{(K'+\eta)\setminus K} |\hat{f}(\xi)|\, dm_{\widehat{G}}(\xi) \le \overline{m} \int_{\widehat{G}\setminus K} |\hat{f}(\xi)|\, dm_{\widehat{G}}(\xi).
\tag{22}
$$

The proof is now completed by combining (20), (21), and (22). □

For the case $K' = K$, $\widehat{F} = \chi_K$, we obtain

Corollary 3.7. *Under the hypothesis of Theorem* 3.6 *we have for* $K' = K$ *and* $\widehat{F} = \chi_K$

$$
|(Sf)(x) - f(x)| \le (1+\gamma) \int_{\widehat{G}\setminus K} |\hat{f}(\xi)|\, dm_{\widehat{G}}(\xi).
\tag{23}
$$

Proof. We have $|(Sf)(x)-f(x)| \le |(Sf)(x)-(F*f)(x)|+|(F*f)(x)-f(x)|$. The assertion now follows from (19) and

$$
|(F*f)(x)-f(x)| \le \|\widehat{F}\hat{f}-\hat{f}\|_1 = \|\chi_K\hat{f}-\hat{f}\|_1 = \int_{\widehat{G}\setminus K} |\hat{f}(\xi)|\, dm_{\widehat{G}}(\xi). \quad \square
$$

We proceed by giving examples and discussing some special cases, where the application of the theorem is relatively simple. In order to clarify the basic concepts of the theory, we begin by working out a one-dimensional example, which leads to the well-known case of nonuniform but periodic sampling on the real line. Let $G = \mathbf{R}$, $H = \mathbf{Z}$, $H^\perp = \mathbf{Z}$, $K = K' = [-1, 1]$, and $f \in L_1(\mathbf{R})$ be such that the hypothesis of Theorem 3.6 holds. We find that for $\xi \in K'$

$$(H^\perp \backslash \{0\}) \cap (K - \xi) = \begin{cases} \{1\}, & -1 < \xi < 0, \\ \{-1\}, & 0 < \xi < 1, \\ \{1, 2\}, & \xi = -1, \\ \{-1, 1\}, & \xi = 0, \\ \{-2, -1\}, & \xi = 1. \end{cases}$$

It follows that K_1, K_2 are the open intervals $(-1, 0)$ and $(0, 1)$, respectively. The other sets, namely, $K_3 = \{-1\}$, $K_4 = \{0\}$, and $K_5 = \{1\}$, have measure zero and may be ignored. Hence, we have $L = 2$. Since $M_1 = \{1\}$ and $M_2 = \{-1\}$, it follows that $m_1 = m_2 = \overline{m} = 2$, and that $\eta_1^{(1)} = 1$ and $\eta_1^{(2)} = -1$. In general we have to choose $N \geq \overline{m}$ in order for the systems (12) to be solvable. Letting $N = \overline{m} = 2$, we obtain shifts $x_0, x_1 \in \mathbf{R}$ and choose without loss of generality $x_0 = 0$. For $l = 1, 2$ the equations (12) read

$$\beta_0^{(l)} + \beta_1^{(l)} = 1,$$
$$\beta_0^{(l)} + \beta_1^{(l)} e^{2\pi i \langle x_1, \eta_1^{(l)} \rangle} = 0$$

with $\langle x_1, \eta_1^{(l)} \rangle = [(-1)^{l-1} x_1]$. This yields the coefficients

$$\beta_1^{(l)} = (1 - e^{2\pi i \langle x_1, \eta_1^{(l)} \rangle})^{-1} = \frac{1}{2} + i(-1)^{l+1} \frac{\sin(2\pi x_1)}{2 - 2\cos(2\pi x_1)},$$
$$\beta_0^{(l)} = \overline{\beta_1^{(l)}}, \qquad l = 1, 2.$$

Furthermore, we obtain $\hat{\chi}_{K_l}(-z) = \exp((-1)^l \pi i z) \sin(\pi z)/(\pi z)$. Together with the choice $\widehat{F} = \chi_K$ this yields for $n = 0, 1$ the functions

$$k_n(z) = \frac{\sin(\pi z)}{\pi z} \left(\cos(\pi z) + (-1)^{n+1} \frac{\sin(2\pi x_1)}{1 - \cos(2\pi x_1)} \sin(\pi z) \right).$$

The sampling series now reads

$$(Sf)(x) = \sum_{n=0}^{1} \sum_{j \in \mathbf{Z}} f(x_n + j) k_n(x - x_n - j).$$

Similar examples have been discussed in [19, 39, 16, 8].

The quantity γ in (19) is a measure for the stability of the reconstruction of f from its sampled values. In the example above we see that $|\beta_1^{(l)}| \to \infty$, and therefore $\gamma \to \infty$ if x_1 approaches an integer. This has to be expected, since for x_1 an integer, the two shifted copies of H would coincide, and we would lose necessary information. For $x_1 \notin \mathbf{Z}$ the reconstruction is possible, but becomes increasingly unstable if x_1 approaches an integer.

We have already investigated the solvability of the systems (12) in the case of $K' = \{0\}$. Now we turn to the important case $K' = K$. Let us write $\widetilde{M}_l$ for $M_l \cup \{0\}$ and observe that the equations (12) may be written in the equivalent form

$$\sum_{n=0}^{N-1} \beta_n^{(l)} e^{2\pi i \langle x_n, \eta \rangle} = \delta(\eta), \qquad \eta \in \widetilde{M}_l, \tag{24}$$

with $\delta(\eta) = 1$ for $\eta = 0$, and $\delta(\eta) = 0$ otherwise. If $K' = K$, we have that $(K_l + \eta) \subset K'$ if $\eta \in \widetilde{M}_l$, and $(K_l + \eta) \cap K = \varnothing$ for $\eta \in H^\perp \backslash \widetilde{M}_l$. It follows that for each $\eta \in \widetilde{M}_l$ there must be an $l' \in \{1, \dots, L\}$ such that $K_{l'} = K_l + \eta$, and $\widetilde{M}_{l'} = \widetilde{M}_l - \eta$. We may thus define an equivalence relation on $\{1, \dots, L\}$ by letting $l \equiv l'$ if and only if there is $\eta \in \widetilde{M}_l$ such that $K_{l'} = K_l + \eta$. Clearly, the equivalence class of any l consists of $|\widetilde{M}_l| = m_l$ elements. The following proposition is a consequence of the fact that the equations (12) for l, l' with $l \equiv l'$ are closely related.

Proposition 3.8. *Let $A^{(l)}$, $l = 1, \dots, L$, denote the matrices of the linear systems* (12). *If $K' = K$ and all L systems* (12) *admit a solution, then* $\operatorname{rank}(A^{(l)}) = m_l$, $l = 1, \dots, L$, *and therefore* $N \geq \max_{l=1,\dots,L} m_l$.

Proof. Fix $l \in \{1, \dots, L\}$, and let $\widetilde{M}_l = \{0 = \eta_0^{(l)}, \eta_1^{(l)}, \dots, \eta_{m_l-1}^{(l)}\}$. The system (12) reads

$$\sum_{n=0}^{N-1} A_{jn}^{(l)} \beta_n^{(l)} = \delta_{j0}, \qquad j = 0, \dots, m_l - 1, \tag{25}$$

where $A_{jn}^{(l)} = e^{2\pi i \langle x_n, \eta_j^{(l)} \rangle}$ are the entries of the $m_l \times N$ matrix $A^{(l)}$, and δ_{jk} is the Kronecker delta. Since $K' = K$, there are $l_1, \dots, l_{m_l-1}$ such that $K_{l_k} = K_l + \eta_k^{(l)}$, $\widetilde{M}_{l_k} = \widetilde{M}_l - \eta_k^{(l)}$. The equations (12), written in the form (24), read for $l = l_k$, $k = 1, \dots, m_l - 1$, as follows:

$$\sum_{n=0}^{N-1} \beta_n^{(l_k)} e^{2\pi i \langle x_n, \eta_j^{(l)} - \eta_k^{(l)} \rangle} = \delta(\eta_j^{(l)} - \eta_k^{(l)}) = \delta_{jk}, \qquad j = 0, \dots, m_l - 1.$$

With $\tilde{\beta}_n^{(l_k)} = \beta_n^{(l_k)} e^{-2\pi i \langle x_n, \eta_k^{(l)} \rangle}$ this may be written as

$$\sum_{n=0}^{N-1} A_{jn}^{(l)} \tilde{\beta}_n^{(l_k)} = \delta_{jk}, \qquad j = 0, \dots, m_l - 1. \tag{26}$$

Note that the right-hand sides of the systems (25), (26) are the canonical unit vectors in $\mathbf{C}^{m_l}$. Since all these systems are solvable, we have that $A^{(l)}\mathbf{C}^N = \mathbf{C}^{m_l}$. $\square$

In general, the functions $k_n(z)$ defined in (16) are quite complicated. We consider two special cases where they simplify considerably. The first one occurs when the coefficients $\beta_n^{(l)}$ are independent of l and leads to the sampling theorem of Gaarder [11]. The following corollary is a generalization.

Corollary 3.9. *Let the function f be as in Theorem* 3.6 *and let $\eta_1, \dots, \eta_{m-1}$ denote the elements of the finite set $(H^\perp \backslash \{0\}) \cap (K - K)$. If there are $x_n + H \in G/H$, $n = 0, \dots, N-1$, such that the system of equations* (10) *has a solution $(\beta_0, \dots, \beta_{N-1}) \in \mathbf{C}^N$, then the estimate* (23) *holds with*

$$(Sf)(x) = \sum_{n=0}^{N-1} \beta_n \int_H f(x_n + y) \hat{\chi}_K(x_n + y - x) \, dm_H(y)$$

and $\gamma = m \sum_{n=0}^{N-1} |\beta_n|$.

Proof. Let $K' = K$. Since all sets M_l are subsets of $(H^\perp \backslash \{0\}) \cap (K - K)$, it follows that $\overline{m} = \max_{1 \leq l \leq L} m_l \leq m$ and that the systems of equations (12) have the solution $\beta_n^{(l)} = \beta_n$, $l = 1, \dots, L$, $n = 0, \dots, N-1$. Now we apply Corollary 3.7 and note that (16) simplifies to $k_n(z) = \beta_n \hat{\chi}_K(-z)$ because the $\beta_n^{(l)}$ do not depend on l. □

Note that in this case the sets K_l and M_l need not be determined. Gaarder used the following approach to construct suitable sampling sets. Let $U = \{x_0 + H, \dots, x_{N'-1} + H\}$ be a finite subgroup of G/H such that the system (10) has a solution $(\beta_0, \dots, \beta_{N'-1})$ where some of the β_n are zero. Hence, the corresponding cosets $x_n + H$ can be dropped and the union of the remaining cosets is a suitable sampling set.

The second case allowing for simplification occurs when the sets K_l are translates of some set K_0.

Corollary 3.10. *Assume the hypothesis of Theorem* 3.6 *holds with* $K' = K$ *and* $F = \chi_K$, *and that in addition there is a set* $K_0 \subset \widehat{G}$ *such that all sets* K_l *of positive measure may be written as* $K_l = K_0 + r_l$, $r_l \in \widehat{G}$. *Then the estimate* (23) *holds with* $(Sf)(x)$ *given by* (15) *and*

$$k_n(z) = \hat{\chi}_{K_0}(-z) \sum_{l=1}^{L} \beta_n^{(l)} e^{2\pi i \langle r_l, z \rangle}.$$

Proof. Ignoring sets of measure zero, we have $\chi_{K_l}(\xi) = \chi_{K_0}(\xi - r_l)$; hence $\hat{\chi}_{K_l}(-z) = \hat{\chi}_{K_0}(-z) \exp(2\pi i \langle r_l, z \rangle)$. Inserting this into (18) and applying Corollary 3.7 yield the desired result. □

The main difficulty in applying this result to a concrete case is the determination of the quantities K_0, L, r_l, m_l, and M_l, $l = 1, \dots, L$. After this is accomplished, the coefficients $\beta_n^{(l)}$ may be found from (12). In the following example we describe a class of sampling sets in $\mathbf{R}^s$ where Corollary 3.10 may be applied and the above quantities can easily be computed. This is possible because K is assumed to be a rectangle and the subgroup H is chosen appropriately.

Example 3.11. Let $G = \mathbf{R}^s$, and

$$K = K' = [-d_1/2, d_1/2] \times \cdots \times [-d_s/2, d_s/2]$$

with $d_1, \dots, d_s > 0$. Let $P_1, \dots, P_s$ be positive integers, W the diagonal matrix with entries $W_{ii} = P_i/d_i$, $i = 1, \dots, s$, and $H = W\mathbf{Z}^s$. Define the sets $J \subset \mathbf{Z}^s$ and $K_0 \subset \mathbf{R}^s$ by

$$J = \{k \in \mathbf{Z}^s : 0 \le k_i \le P_i - 1, \; i = 1, \dots, s\},$$
$$K_0 = \left(-\frac{d_1}{2}, -\frac{d_1}{2} + \frac{d_1}{P_1}\right) \times \cdots \times \left(-\frac{d_s}{2}, -\frac{d_s}{2} + \frac{d_s}{P_s}\right).$$

Then $L = \prod_{i=1}^{s} P_i = |J|$, all m_l are equal to $\overline{m} = L$, and the sets K_l and $\widetilde{M}_l = M_l \cup \{0\}$ are given by

$$K_l = K_0 + r_l,$$
$$\widetilde{M}_l = W^{-1}J - r_l \quad \text{with } r_l = W^{-1}k(l),$$

where $k(l) = (k_1, \dots, k_s)^T$ is the uniquely determined element of J such that $l = 1 + k_1 + P_1 k_2 + P_1 P_2 k_3 + \cdots + P_1 \cdots P_{s-1} k_s$. The shifts x_n may be chosen from the set $W[0, 1)^s$. If the systems (12) admit a solution, the sampling series is given by Corollary 3.10.

Proof. We have $H^\perp = W^{-T}\mathbf{Z}^s = W^{-1}\mathbf{Z}^s$. Consider the center point of K_0, $\xi_0 = (-d_1/2 + d_1/2P_1, \dots, -d_s/2 + d_s/2P_s)^T \in K$. For $\eta \in H^\perp$ we find that $\xi_0 + \eta \in K$ if and only if $\eta \in W^{-1}J$, i.e., $\widetilde{M}_{\xi_0} = M_{\xi_0} \cup \{0\} = W^{-1}J$. An elementary computation shows that the set $K_1 = \{\xi \in K,\ M_\xi = M_{\xi_0}\}$ equals K_0. Hence, $K_1 = K_0$, $\widetilde{M}_1 = W^{-1}J$, and $m_1 = |\widetilde{M}_1| = |J|$. According to the discussion preceding Proposition 3.8 we obtain further sets K_l by setting $K_l = K_1 + W^{-1}k(l)$, $l = 2, \dots, |J|$, with $\widetilde{M}_l = \widetilde{M}_1 - W^{-1}k(l)$. Since $K \setminus \bigcup_{l=1}^{|J|} K_l$ has measure zero, all other sets K_l must have measure zero. □

The class of sampling sets described by Example 3.11 is quite large. For $s = 1$, $P_1 = d_1 = 2$, we obtain the one-dimensional example discussed above. For two-dimensional examples we refer the reader to pp. 77–82 of the paper by Butzer and Hinsen [3]. Their approach is different from ours, but the sampling sets of all four applications given there fall under the description of Example 3.11. For example, let $s = 2$, $d_1 = 1$, $d_2 = 1/2$, $P_1 = 4$, $P_2 = 2$. This gives $K = [-1/2, 1/2] \times [-1/4, 1/4]$, $W_{11} = W_{22} = 4$; hence $H = 4\mathbf{Z}^2$, $H^\perp = \frac{1}{4}\mathbf{Z}^2$, and $L = \overline{m} = m_l = P_1P_2 = 8$. The shifts x_n, $n = 0, \dots, N-1$, may be chosen from the set $[0, 4)^2$, and we need $N \geq 8$ for the systems (12) to be solvable. The sampling set investigated in [3, p. 81] is now obtained by choosing x_n, $n = 0, \dots, 7$, as the column vectors of the matrix

$$\begin{pmatrix} a & a & b & b & 4-a & 4-a & 4-b & 4-b \\ b & 4-b & a & 4-a & b & 4-b & a & 4-a \end{pmatrix}$$

with $a = \sqrt{2}/(1+\sqrt{2})$ and $b = \sqrt{2}$. While this choice gives a very aesthetic sampling set, it is not more difficult to apply Corollary 3.10 to a different choice of the x_n. The sampling sets of Example 3.11 can also be obtained using the results of [8], as well as with the approach of Cheung [6].

4. Implementation

For certain applications, e.g., computerized tomography [7, 8, 27, 28, 34], it is important to take advantage of the particular shape of the set K in order to find efficient sampling sets. If K is not a rectangle, finding the sets K_l, M_l may be considerably more complicated than in Example 3.11. It is therefore desirable to develop a suitable computer implementation which reduces the demands on the user to the absolute minimum. In this section we will discuss one such method. After formulating the general algorithm, we give a detailed discussion of the case $G = T^s$. It is hoped that this will assist the reader in implementing the algorithm without too much difficulty.

Our approach is to compute $(Sf)^\wedge(\xi)$ according to (13) for ξ in a finite set $K'' \subseteq K'$. The function $(Sf)(x)$ can then be found by inverting the Fourier transform numerically, which can usually be done by FFT techniques. In general, this will cause an additional discretization error, but this error can be avoided if the group $\widehat{G}$ is discrete, as in the case of $G = T^s$. The advantage of this approach is that it leads to fast algorithms, is relatively easy to implement, and does not require the user to perform tedious computations.

$(Sf)^\wedge$ may be computed as follows. For each $\xi \in K''$ we have to determine the index l for which $\xi \in K_l$, and to evaluate $\int_H f(x_n+y)e^{-2\pi i\langle x_n+y, \xi\rangle}\, dm_H(y)$.

Let $\widetilde{H} \subset \widehat{G}$ consist of exactly one representative of each coset $\xi + H^{\perp} \in \widehat{G}/H^{\perp} = \widehat{H}$. For each $\xi \in K''$ there is exactly one $\xi_0 \in \widetilde{H}$ such that $\xi \in \xi_0 + H^{\perp}$. For $y \in H$ we have $\langle y, \xi\rangle = \langle y, \xi_0\rangle$. Let $g_n : H \to \mathbf{C}$ be given by $g_n(y) = f(x_n + y)$, $y \in H$. Then

$$\begin{aligned}\int_H f(x_n + y) e^{-2\pi i \langle x_n + y, \xi\rangle}\, dm_H(y) \\ &= e^{-2\pi i \langle x_n, \xi\rangle} \int_H g_n(y) e^{-2\pi i \langle y, \xi_0\rangle}\, dm_H(y) \\ &= e^{-2\pi i \langle x_n, \xi\rangle} \hat{g}_n(\xi_0 + H^{\perp}),\end{aligned}$$

where the Fourier transform is taken with respect to H. We obtain the following general algorithm. Let A denote a suitable array used for computing $(Sf)\hat{}(\xi)$, $\xi \in K''$, and assume the coefficients $\beta_n^{(l)}$ are already computed.

Algorithm 4.1

For all $\xi \in K''$ set $A(\xi) = 0$;
For $n = 0, \dots, N-1$ do;
  Compute $\hat{g}_n(\xi_0 + H^{\perp})$ for all $\xi_0 \in \widetilde{H}$;
  For all $\xi \in K''$ do;
    Find $1 \le l \le L$ and $\xi_0 \in \widetilde{H}$ such that $\xi \in K_l$ and $\xi \in \xi_0 + H^{\perp}$;
    Add $\beta_n^{(l)} e^{-2\pi i \langle x_n, \xi\rangle} \hat{g}_n(\xi_0 + H^{\perp})$ to the current value of $A(\xi)$;
  end;
end;
For all $\xi \in K''$ multiply $A(\xi)$ by $\widehat{F}(\xi)$;

We will now give a detailed description of the implementation of Algorithm 4.1 in the case of H being a finite subgroup of the s-dimensional torus group $G = T^s$. Since $\widehat{G} = \mathbf{Z}^s$ is discrete, K is a finite set. This is an optimal situation for using Algorithm 4.1 because we can choose $K'' = K'$ and the inverse Fourier transform can be carried out without discretization errors. In order to be able to implement the algorithm for given H, K, K', and x_n, $n = 0, \dots, N-1$, we face the following tasks: Finding a suitable mathematical representation of H; determining the set $\widetilde{H}$; finding the sets M_l, $l = 1, \dots, L$, and the coefficients $\beta_n^{(l)}$; for $\xi \in K''$ finding l with $\xi \in K_l$ and $\xi_0 \in \widetilde{H}$ with $\xi \in \xi_0 + H^{\perp}$; computing $\hat{g}_n(\xi_0 + H^{\perp})$ for $\xi_0 \in \widetilde{H}$.

We will discuss each of these steps and begin with giving a representation of H suitable for our purpose. The finite subgroups of T^s are precisely the abscissa sets of integration lattices as defined in [37], which have been studied in considerable detail; see, e.g., [29, 22, 23, 38] and the references given there. In the next proposition we collect the needed results from this theory. In choosing the Hermite normal form for the generating matrix of $H^{\perp}$, we are following an approach taken in [23].

Proposition 4.2. *Let H be a finite subgroup of the s-dimensional torus group T^s. Then there exists a unique nonsingular lower triangular $s \times s$ matrix W*

such that

(i) *as a set,* $H = [WZ^s]$, *and* $H^\perp = W^{-T}\mathbf{Z}^s$;

(ii) $|H| = |\det W|^{-1}$;

(iii) *the matrix* $B = W^{-T}$ *has Hermite normal form*; *i.e.,* B *is an upper triangular integer matrix satisfying*

$$\begin{aligned} B_{ii} > 0, \qquad & i = 1, \dots, s, \\ 1 - B_{ii} \le B_{ij} \le 0, \qquad & i < j \le s. \end{aligned} \tag{27}$$

Proof. See [29, pp. 125–126, 131–132] and [31, Theorems II.2, II.3]. □

In practice, the subgroup H is specified by choosing a generator matrix $\widetilde{W}$ such that $H = [\widetilde{W}\mathbf{Z}^s]$. If $\widetilde{W}^{-T}$ is not in Hermite normal form, one can use one of the algorithms published in [2, 17] to obtain the uniquely determined matrix W of Proposition 4.2. The description of H provided by Proposition 4.2 is not yet fully satisfactory, since for each $y \in H$ there exist infinitely many $z \in \mathbf{Z}^s$ such that $y = [Wz]$. In the following proposition we remove this ambiguity and obtain at the same time a suitable set $\widetilde{H}$ consisting of exactly one representative of each coset in $\widehat{G}/H^\perp$. The proof is given at the end of this section.

Proposition 4.3. *Let* H *be a finite subgroup of* $G = T^s$ *and* W *its generator matrix as given by Proposition* 4.2. *Let* $\widetilde{H}$ *be the set*

$$\widetilde{H} = \{z \in \mathbf{Z}^s : 0 \le z_i \le N_i - 1,\ i = 1, \dots, s\},$$

where the N_i *are the positive integers* $N_i = 1/W_{ii}$, $i = 1, \dots, n$. *Then the mappings* $z \to [Wz]$ *and* $z \to z + H^\perp$ *are bijections from* $\widetilde{H}$ *onto* H *and* $\widehat{G}/H^\perp = \widehat{H}$, *respectively.*

This means that we have $H = [W\widetilde{H}]$, and $\widehat{H} = \{z + H^\perp,\ z \in \widetilde{H}\}$, and that we can use $\widetilde{H}$ as a convenient index set to label both the elements of H and of $\widehat{H}$. Using Lemma 4.7 below, one can show that the set $\{x \in \mathbf{R}^s : 0 \le x_i < W_{ii},\ i = 1, \dots, s\}$ contains exactly one representative of each coset $x + H \in G/H$. Hence, the shifts x_n may be chosen from this set.

With regard to finding the sets M_l, we observe that $\widehat{G}$ is discrete and K' is therefore a finite set. We may therefore assume that $K'' = K'$. The sets M_l can be determined by inspection. The same is true for finding for each $\xi \in K''$ the index l such that $M_\xi = M_l$, i.e., $\xi \in K_l$. In one or two dimensions, the sets M_l and K_l can often be found by simply drawing the translated sets $K + \eta$, $\eta \in H^\perp$; see, e.g., [8, p. 74]. In higher dimensions, however, the task is more formidable. We will therefore give an outline of a general algorithm. It requires a user-supplied subroutine to test if a given $\xi \in \widehat{G}$ is an element of K, and as a priori information a finite set $M_{\max} \subset H^\perp \backslash \{0\}$ such that $(H^\perp \backslash \{0\}) \cap (K - K') \subseteq M_{\max}$. For each $\xi \in K'$ the algorithm will determine the set $M_\xi = (H^\perp \backslash \{0\}) \cap (K - \xi)$ by testing for all $\eta \in M_{\max}$ if $\xi + \eta \in K$. If M_ξ is different from all previously found sets $M_{\xi'}$, it is stored as a new set M_l. In this way the parameter L, as well as the m_l, M_l, and K_l, $l = 1, \dots, L$, are found. Let I denote a suitable integer array used to store for each $\xi \in K'$ the index l with $\xi \in K_l$.

Algorithm 4.4

Determine $M_{\max} \subset H^\perp \backslash \{0\}$ such that $(H^\perp \backslash \{0\}) \cap (K - K') \subseteq M_{\max}$;
$L = 0$;
For all $\xi \in K'$ do;
  $M = \varnothing$;
  For all $\eta \in M_{\max}$ do;
    If $(\xi + \eta \in K)$ then $M = M \cup \{\eta\}$;
  end;
  If $((L \geq 1)$ and $(M = M_l$ for some $l \leq L))$
    then $I(\xi) = l$;
    else do; $L = L + 1$; $M_L = M$; $m_L = |M_L| + 1$; $I(\xi) = L$; end;
end;

Having found the sets M_l, we can solve the linear systems (12) for the coefficients $\beta_n^{(l)}$. In general, we will have to choose $N \geq \overline{m} = \max_{1 \leq l \leq L} m_l$ in order to avoid an overdetermined system, which might have no solution.

For $\xi \in K'$ we can now find the index l with $\xi \in K_l$ by letting $l = I(\xi)$. The sets K_l are thus implicitly determined by the array I, since $K_l = \{\xi \in K', I(\xi) = l\}$. There is no need for an explicit representation.

The next task is to give an algorithm to find for a given $\xi \in \mathbf{Z}^s$ the uniquely determined $\xi_0 \in \widetilde{H}$ such that $\xi \in \xi_0 + H^\perp$. We need to find integers l_i, m_i, $i = 1, \ldots, s$, such that $\xi - W^{-T}(l_1, \ldots, l_s)^T = (m_1, \ldots, m_s)^T = \xi_0 \in \widetilde{H}$. Since W^{-T} is upper triangular with diagonal elements $N_1, \ldots, N_s$, the algorithm is straightforward. The kth equation reads $\xi_k - N_k l_k - \sum_{i=k+1}^{s} W_{ki}^{-T} l_i = m_k$. With $q_k = \xi_k - \sum_{i=k+1}^{s} W_{ki}^{-T} l_i$ we have $m_k = \operatorname{mod}(q_k, N_k)$, and $l_k = (q_k - m_k)/N_k$. Solving the system by backsubstitution leads to the following algorithm.

Algorithm 4.5

For $k = s$ down to 1 do;
$$q_k = \xi_k - \sum_{i=k+1}^{s} W_{ki}^{-T} l_i;$$
$$m_k = \operatorname{mod}(q_k, N_k);$$
$$l_k = (q_k - m_k)/N_k;$$
end;

Finally, we need an efficient algorithm for computing

$$\hat{g}(\xi_0 + H^\perp) = \int_H g(y) e^{-2\pi i \langle y, \xi_0 \rangle} \, dm_H(y)$$

for all $\xi_0 \in \widetilde{H}$; i.e., we need a Fast Fourier Transform for computing the Fourier transform on H. Making use of Proposition 4.3, we obtain

$$\hat{g}(\xi_0 + H^{\perp}) = \frac{1}{|H|} \sum_{y \in H} g(y) e^{-2\pi i \langle y, \xi_0 \rangle}$$
$$= \frac{1}{N_1 N_2 \cdots N_s} \sum_{j_1=0}^{N_1-1} \cdots \sum_{j_s=0}^{N_s-1} g([WJ]) e^{-2\pi i \langle [WJ], \xi_0 \rangle},$$

where we have written J for $(j_1, \ldots, j_s)^T \in \widetilde{H}$. For $\xi_0 = (m_1, \ldots, m_s)^T \in \widetilde{H}$ we see that

$$\langle [WJ], \xi_0 \rangle = \left[\sum_{k=1}^{s} m_k \left[\sum_{l=1}^{k} W_{kl} j_l \right] \right] = \left[\sum_{k=1}^{s} m_k \left(N_k^{-1} j_k + \sum_{l=1}^{k-1} W_{kl} j_l \right) \right].$$

Hence, computing $\hat{g}(\xi_0)$ for $\xi_0 \in \widetilde{H}$ essentially amounts to performing an s-dimensional FFT, as the following algorithm shows.

Algorithm 4.6

For $J = (j_1, \ldots, j_s)^T \in \widetilde{H}$ let $G(j_1, \ldots, j_s) = \hat{g}([WJ])$.
For $k = s$ down to 1 do;
For all $(j_1, \ldots, j_{k-1}, 0, m_{k+1}, \ldots, m_s)^T \in \widetilde{H}$ compute

$$G(j_1, \ldots, j_{k-1}; m_k, \ldots, m_s)$$
$$= e^{-2\pi i m_k \sum_{l=1}^{k-1} W_{kl} j_l} \frac{1}{N_k} \sum_{j_k=0}^{N_k-1} G(j_1, \ldots, j_k; m_{k+1}, \ldots, m_s) e^{-2\pi i m_k j_k / N_k},$$

$m_k = 0, \ldots, N_k - 1$
end;
end;

Clearly, the result $G(m_1, \ldots, m_s)$ is equal to $\hat{g}((m_1, \ldots, m_s)^T)$. The intermediate results $\widehat{G}(j_1, \ldots, j_{k-1}; m_k, \ldots, m_s)$ can be computed by performing $P_k = \prod_{i \neq k} N_i$ Fast Fourier Transforms of length N_k. Hence, the algorithm requires

$$O\left(\sum_{k=1}^{s} P_k N_k \log N_k \right) = O\left(\prod_{i=1}^{s} N_i \sum_{k=1}^{s} \log N_k \right) = O(|H| \log |H|)$$

operations. This concludes our description of the implementation. It remains to prove Proposition 4.3. We need the following technical lemma.

Lemma 4.7. *Let A be a nonsingular, triangular, real $s \times s$ matrix, and*

$$M = \{ x \in \mathbf{R}^s : |x_i| < |A_{ii}|, \ i = 1, \ldots, s \}.$$

Then $(A\mathbf{Z}^s) \cap M = \{0\}$.

Proof. The assertion, namely, that $z = 0$ is the only integer vector satisfying the triangular system of inequalities $|(Az)_i| < |A_{ii}|$, $i = 1, \ldots, s$, is verified by solving this system using forward substitution or backsubstitution, respectively. □

Proof of Proposition 4.3. It suffices to show that the two mappings are 1-1, since $|\widehat{H}| = |H| = 1/|\det(W)| = \prod_{i=1}^{s} N_i = |\widetilde{H}|$. If $[Wx] = [Wy]$ for $x, y \in \widetilde{H}$, then $x - y \in W^{-1}\mathbf{Z}^s$. On the other hand, if $x + H^{\perp} = y + H^{\perp}$ for $x, y \in \widetilde{H}$, then $x - y \in H^{\perp} = W^{-T}\mathbf{Z}^s$. Since $N_i = (W^{-1})_{ii} = (W^{-T})_{ii}$, and $|x_i - y_i| \leq N_i - 1$, it follows from Lemma 4.7 that $x = y$ in both cases. □

BIBLIOGRAPHY

1. J. J. Benedetto, *Irregular sampling and frames*, Wavelets—A Tutorial in Theory and Applications (C. K. Chui, ed.), Academic Press, New York, 1992, pp. 445–507.
2. G. H. Bradley, *Algorithms for Hermite and Smith normal matrices and linear Diophantine equations*, Math. Comp. **25** (1971), 897–907.
3. P. L. Butzer and G. Hinsen, *Two-dimensional nonuniform sampling expansions–an iterative approach.* I, II, Appl. Anal. **32** (1989), 53–67, 69–85.
4. P. L. Butzer, W. Splettstößer, and R. L. Stens, *The sampling theorem and linear prediction in signal analysis*, Jahresber. Deutsch. Math.-Verein. **90** (1988), 1–70.
5. K. F. Cheung and R. J. Marks II, *Imaging sampling below the Nyquist density without aliasing*, J. Opt. Soc. Amer. A **7** (1990), 92–105.
6. K. F. Cheung, *A multidimensional extension of Papoulis' generalized sampling expansion with the application in minimum density sampling*, Advanced Topics in Shannon Sampling and Interpolation Theory (R. J. Marks II, ed.), Springer, New York, 1993, pp. 85–119.
7. L. Desbat, *Efficient sampling on coarse grids in tomography*, Inverse Problems **9** (1993), 251–269.
8. A. Faridani, *An application of a multidimensional sampling theorem to computed tomography*, Contemp. Math., vol. 113, Amer. Math. Soc., Providence, RI, 1990, pp. 65–80.
9. H. G. Feichtinger and K. Gröchenig, *Irregular sampling theorems and series expansions of band-limited functions*, J. Math. Anal. Appl. **167** (1992), 530–556.
10. ——, *Theory and practice of irregular sampling*, Wavelets: Mathematics and Applications (J. J. Benedetto and M. W. Frazier, eds.), CRC Press, 1993, pp. 305–363.
11. N. T. Gaarder, *A Note on the multidimensional sampling theorem*, Proc. IEEE **60** (1972), 247–248.
12. K. Gröchenig, *Reconstruction algorithms in irregular sampling*, Math. Comp. **59** (1992), 181–194.
13. E. Hewitt and K. A. Ross, *Abstract harmonic analysis,* Volumes I, II, Springer, New York and Berlin, 1979, 1970.
14. J. R. Higgins, *Five short stories about the cardinal series*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), 45–89.
15. A. J. Jerri, *The Shannon sampling theorem–its various extensions and applications: a tutorial review*, Proc. IEEE **65** (1977), 1565–1596.
16. R. E. Kahn and B. Liu, *Sampling representations and the optimum reconstruction of signals*, IEEE Trans. Inform. Theory **11** (1965), 339–347.
17. R. Kannan and A. Bachem, *Polynomial algorithms for computing the Smith and Hermite Normal Forms of an integer matrix*, SIAM J. Comput. **8** (1979), 499–507.
18. I. Kluvanek, *Sampling theorem in abstract harmonic analysis*, Mat. Časopis Sloven. Akad. Vied **15** (1965), 43–48.
19. A. Kohlenberg, *Exact interpolation of bandlimited functions*, J. Appl. Phys. **24** (1953), 1432–1436.
20. H. J. Landau, Letter to F. Natterer, 1988.
21. ——, *A sparse regular sequence of exponentials closed on large sets*, Bull. Amer. Math. Soc. **70** (1964), 566–569.
22. J. N. Lyness, *An introduction to lattice rules and their generator matrices*, IMA J. Numer. Anal. **9** (1989), 405–419.

23. J. N. Lyness, T. Sørevik, and P. Keast, *Notes on integration and integer sublattices*, Math. Comp. **56** (1991), 243–255.

24. R. J. Marks II, *Introduction to Shannon sampling and interpolation theory*, Springer, New York, 1991.

25. R. J. Marks II (ed.), *Advanced topics in Shannon sampling and interpolation theory*, Springer, New York, 1993.

26. F. Marvasti, *Nonuniform sampling*, Advanced Topics in Shannon Sampling and Interpolation Theory (R. J. Marks II, ed.), Springer, New York, 1993, pp. 121–156.

27. F. Natterer, *The mathematics of computerized tomography*, Wiley, New York, 1986.

28. ———, *Sampling in fan-beam tomography*, SIAM J. Appl. Math. **53** (1993), 358–380.

29. H. Niederreiter, *Random number generation and quasi-Monte Carlo methods*, SIAM, Philadelphia, 1992.

30. ———, *Lattice rules for multiple integration*, Stochastic Optimization: Numerical Methods and Technical Applications (K. Marti, ed.), Lecture Notes in Econom. and Math. Systems, vol. 379, Springer, Berlin, 1992, pp. 15–26.

31. M. Newman, *Integral matrices*, Academic Press, New York, 1972.

32. A. Papoulis, *Signal analysis*, McGraw-Hill, New York, 1977.

33. D. P. Petersen and D. Middleton, *Sampling and reconstruction of wave-number-limited functions in N-dimensional Euclidean space*, Inform. Control **5** (1962), 279–323.

34. P. A. Rattey and A. G. Lindgren, *Sampling the 2-D Radon transform*, IEEE Trans. Acoust. Speech Signal Process. **29** (1981), 994–1002.

35. W. Rudin, *Fourier analysis on groups*, Wiley, New York, 1962.

36. I. H. Sloan, *Numerical integration in high dimensions–The lattice rule approach,* Numerical Integration—Recent Developments, Software and Applications (T. O. Espelid and A. Genz, eds.), Kluwer, Dordrecht, 1992, pp. 55–69.

37. I. H. Sloan and P. J. Kachoyan, *Lattice methods for multiple integration: theory, error analysis and examples*, SIAM J. Numer. Anal. **24** (1987), 116–128.

38. I. H. Sloan and J. N. Lyness, *The representation of lattice quadrature rules as multiple sums*, Math. Comp. **52** (1989), 81–94.

39. J. L. Yen, *On nonuniform sampling of bandwidth-limited signals*, IRE Trans. Circuit Theory **3** (1956), 251–257.

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