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ON A CLASS OF ELLIPTIC CURVES WITH RANK AT MOST TWO

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ABSTRACT. In this note we consider the elliptic curves $y^2 = x^3 + px$ defined over $\mathbb{Q}$ for primes p satisfying $p \equiv 1 \pmod 8$, and review some of their properties. We then compute and list (in the supplement) their ranks, and give, when the rank is positive, the generators of the group of rational points and Mordell-Weil lattice invariant τ for all primes $p < 50000$ of the form $m^2 + 64n^2$.

1. INTRODUCTION

Considerable progress has been made in the study of elliptic curves defined over the rational field $\mathbb{Q}$, but many questions remain unanswered. For example, formulas for, or even estimates of, the rank of many of these curves have not been found. Hence, it is of interest to study properties of particular classes of curves in the hope that some of these questions can be answered in these cases.

In this paper we shall consider the class of elliptic curves

$$C_p : y^2 = x^3 + px,$$

defined over the rational field $\mathbb{Q}$ and where p is a prime. Let $r(C_p)$ denote the rank of this curve, that is the number of independent infinite-order generators of the (Mordell-Weil) group of rational points on this curve. It is known that

$$\text{if } p \equiv 7 \text{ or } 11 \pmod{16}, \quad \text{then } r(C_p) = 0;$$
$$\text{if } p \equiv 3,\ 5,\ 13 \text{ or } 15 \pmod{16}, \quad \text{then } r(C_p) = 0 \text{ or } 1; \text{ and}$$
$$\text{if } p \equiv 1 \text{ or } 9 \pmod{16}, \quad \text{then } r(C_p) = 0,\ 1 \text{ or } 2,$$

see Silverman [8, p. 311]. Bremner and Cassels [1] and Bremner [2] have considered the class of curves C_p for primes $p \equiv 5 \pmod 8$; they showed that $r(C_p) = 1$ for $p < 20000$ and conjectured that this holds for all primes in their class. Here we shall consider the third case above, that is the curves C_p where $p \equiv 1 \pmod 8$. The rank can be zero or two, and it is conjectured that it cannot equal one. We give some evidence in support of this; also this conjecture is a consequence of the full Birch and Swinnerton-Dyer conjecture, in particular it holds when the corresponding Shafarevich-Tate group is finite.

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For primes $p \equiv 1 \pmod 4$, Gauss showed that 2 is a quartic residue modulo p if and only if p can be expressed as a sum of squares $p = x^2 + y^2$ where $8 \mid y$. We shall call a prime with this property a *G-prime* in this paper; it is necessarily congruent to 1 (mod 8). We can easily show that $r(C_p) = 0$ if $p \equiv 1 \pmod 8$ and p is not a G-prime, see §2. The converse is only partially true. There are 625 G-primes less than 50000, in 366 cases the rank of the corresponding curve is two, examples are

$$73, 89, 113, 233, \dots, 49801;$$

whilst in the remaining 259 cases the rank is zero, with examples [1]

$$257, 577, 1097, 1201, \dots, 49633.$$

In §2 we reduce the question of finding rational points on C_p to the problem of solving one or more of three simple quartic equations (numbered (I), (II) and (III)) in rational integers. These three quartic equations correspond to the three 'principal homogeneous spaces' for C_p; see Silverman [8, Chapter 10]. In §3 we give algorithms which generate solutions to one of these equations provided solutions for the remaining two are known; this provides an elementary example of the operation of the Weil-Châtelet group for the curve. This also provides another derivation of the rank estimates quoted above. In §4 we discuss briefly the problem of showing that the rank of our curves cannot be one. Finally, in §5, we describe the computations that have been undertaken to calculate the ranks of our curves for all G-primes less than 50000. The supplement gives the basic data from which the infinite-order generators of the corresponding groups can be constructed, the values of τ for the Mordell-Weil lattice (see final section of this paper), and values of the L-functions when these groups are finite.

In a forthcoming paper further computations will be presented. This will include evaluations of the second derivatives of the L-functions at $s = 1$ for the rank-two curves C_p listed in the supplement (thus giving further data in support of the full Birch and Swinnerton-Dyer conjecture) and evaluations of the L-functions for non G-primes. Also, all of these computations will be extended to primes $p < 100000$. For example, the order of the Shafarevich-Tate group Ш for the curve C_{50177} is 256, and 50177 is the first G-prime larger than 50000 for which the rank of the corresponding elliptic curve C_p is zero.

2. Preliminaries

We shall study the elliptic curves

$$(1) \qquad y^2 = x^3 + px \quad \text{for primes} \quad p \equiv 1 \pmod 8.$$

The group of rational points on (1) will be denoted $C_p(\mathbb{Q})$, which we shall usually abbreviate to C_p. Clearly, the point $(0, 0) \in C_p$ and has order two. Except for this point and the neutral element, a point belonging to C_p has the form $(u/s^2, v/s^3)$, where u, v and s are nonzero integers, and $(u, s) =$

[1] There is a possible connection here with real quadratic fields. If $\mathrm{cl}(p)$ denotes the class number of the field $\mathbb{Q}(\sqrt{p})$, then, for G-primes less than 1097, $r(C_p) = 2$ if and only if $\mathrm{cl}(p) = 1$. It is unclear whether this is a coincidence or not; it is false when $p = 1097$ and for some larger primes.

$(v, s) = 1$. We have

(2) $$(u/s^2, v/s^3) + (0, 0) = (ps^2/u, psv/u^2) \in C_p,$$

and so $u = r^2$, for some r, and $r \mid v$; or $u = pr_1^2$, for some r_1, and $pr_1 \mid v$. *We shall assume from now on that the first case always holds.* Therefore, with the exception of the neutral point and the point $(0, 0)$, a typical point on C_p has the form

(3) $$(r^2/s^2, rt/s^3) \quad \text{with} \quad (r, s) = (t, s) = 1 \quad \text{and} \quad rst \neq 0.$$

[For each such point a second point on C_p is always given by (2), and vice versa.] Using this assumption, we can write equation (1), cancelling r^2/s^6, in the form

(4) $$r^4 + ps^4 = t^2$$

with r, s and t as in (3). Note this implies $(r, t) = 1$. As $p \equiv 1 \pmod 8$, r and s have different parities, and t is odd; we shall consider these cases separately. This equation has been discussed previously in Mordell [5].

Case 1. Equation (4) has the form $16r^4 + ps^4 = t^2$ and s is odd.

Rewriting this, we have $ps^4 = (t - 4r^2)(t + 4r^2)$. Now $t - 4r^2$ and $t + 4r^2$ cannot have a common factor, and so each is a fourth power or p times a fourth power. Eliminating t and renaming the variables ($r \to t$ and $s \to rs$) we obtain two possibilities for equation (4) in this case:

(I) $$r^4 - ps^4 = -8t^2 \quad \text{with} \quad (r, s) = (s, t) = (t, r) = 1 \text{ and } r, s \text{ odd},$$

or

(II) $$r^4 - ps^4 = 8t^2 \quad \text{with} \quad (r, s) = (s, t) = (t, r) = 1 \text{ and } r, s \text{ odd}.$$

We can impose a restriction on p as follows. If either of (I) or (II) is soluble, and if q (a prime) divides t, then $r^4 \equiv ps^4 \pmod q$, and so the Legendre symbol $(p/q) = (q/p) = 1$ by quadratic reciprocity; this gives $(t/p) = 1$. Therefore, we can find an integer u satisfying $u^2 \equiv t \pmod p$, and then we have $\mp 8u^4 \equiv r^4 \pmod p$, with the upper sign for (I) and the lower sign for (II). As $p \equiv 1 \pmod 8$, -1 is a quartic residue modulo p, and so 2 must also be a quartic residue modulo p (that is, p is a G-prime) and both (I) and (II) are insoluble if this is not so.

Case 2. Equation (4) has the form $r^4 + 16ps^4 = t^2$ and r is odd.

Arguing as above, we obtain four subcases, with $s = s_1 s_2$ and $(s_1, s_2) = 1$:

$$\begin{aligned} t - r^2 &= 2s_1^4 & t + r^2 &= 8ps_2^4, \\ t - r^2 &= 2ps_1^4 & t + r^2 &= 8s_2^4, \\ t - r^2 &= 8s_1^4 & t + r^2 &= 2ps_2^4, \\ t - r^2 &= 8ps_1^4 & t + r^2 &= 2s_2^4. \end{aligned}$$

The first two subcases are impossible because r is odd. The fourth subcase gives

$$(5) \qquad r^2 = s_2^4 - 4ps_1^4\,, \qquad t = s_2^4 + 4ps_1^4\,,$$

and so $(s_2^2 - r)(s_2^2 + r) = 4ps_1^4$. As in Case 1, this gives $s_1 = uv$, $(u, v) = 1$, and

$$s_2^2 + r = 2u^4\,, \quad s_2^2 - r = 2pv^4\,, \qquad \text{or}$$
$$s_2^2 + r = 2pv^4\,, \quad s_2^2 - r = 2u^4.$$

If the first pair of equations apply, then $s_2^2 = u^4 + pv^4$ and $r = u^4 - pv^4$, and the corresponding point on C_p, that is $(r^2/4s^2, rt/8s^3) = 2(u^2/v^2, s_2u/v^3)$, is a double point. Similarly, if the second pair apply, then the point is $2(u^2/v^2, -s_2u/v^3)$, another double point. Hence, as we are mainly concerned with the generators of C_p, we may exclude the fourth subcase completely. Eliminating t from the third subcase, we find that s_1 is even and so, relabelling the variables ($r \to t$, $s_1 \to 2r$ and $s_2 \to s$), we obtain the third possibility for equation (4):

(III) $64r^4 - ps^4 = -t^2$ with $(r, s) = (s, t) = (t, r) = 1$ and s, t odd.

Gauss's result mentioned in the introduction states that '2 is a quartic residue modulo p if and only if p can be expressed in the form $x^2 + 64y^2$'. The proof of this classic result can easily be adapted to show that if equation (III) is soluble, then 2 is a quartic residue modulo p, and so p is a G-prime.

Hence, we need to consider the three equations (I), (II) and (III); they are the three principal homogeneous spaces for C_p (see Silverman [8, Chapter 10]). For (I) or (II) the corresponding point on the curve C_p is

$$(6) \qquad (4t^2/r^2s^2, t(r^4 + ps^4)/r^3s^3) \quad \text{with } r \text{ and } s \text{ odd,}$$

and for equation (III) the corresponding point is

$$(7) \qquad (t^2/16r^2s^2, t(64r^4 + ps^4)/64r^3s^3) \quad \text{with } s \text{ and } t \text{ odd.}$$

In each case p must be a G-prime for a solution to exist. Further, by Gauss's result, quadratic reciprocity, and the usual descent arguments, we see that, corresponding to the cases (I), (II) and (III) above, p can be expressed by three distinct quadratic forms as follows:

$$(8) \qquad p = a^2 + 8b^2 = c^2 - 8d^2 = 64e^2 + f^2.$$

Note that these may provide solutions to (I), (II) or (III) directly. If a is a square, then (I) has the solution $r = \sqrt{a}$, $s = 1$, $t = b$; similarly if c is a square, (II) is soluble, and if e is a square, (III) is soluble. We shall see later that generators of the group C_p, when they exist, are given by solutions of (I) and (II) using (6).

For later use we have the following consequences of the equations (8):

$$(9) \qquad (a/p) = (b/p) = (c/p) = (d/p) = (e/p) = (f/p) = 1\,,$$

$$(10) \qquad a \equiv 1 \text{ or } 7 \pmod 8 \quad \text{and} \quad c \equiv 1 \text{ or } 3 \pmod 8).$$

For (9), we have by (8), $a^2 \equiv p \pmod{b}$, and so, by quadratic reciprocity, $(b/p) = 1$. Further, as both -1 and 2 are quartic residues modulo p, we can find an integer n to satisfy $n^4 \equiv -8 \pmod{p}$ which, using (8) again, gives $(nb)^4 \equiv a^2b^2 \pmod{p}$ and $(\pm ab/p) = 1$ for some choice of the sign. But $p \equiv 1 \pmod{8}$, and so $(a/p) = 1$ follows by the first result. For (10), we have by (8) and as a is odd, $(2p/a) = 1$, and so $(2/a) = 1$ by (9). Therefore, $a \equiv \pm 1 \pmod{8}$ follows using the properties of the Jacobi symbol. The remaining parts of (9) and (10) are proved similarly.

3. Two points known

In this section we show that if two of the equations (I), (II) or (III) are soluble, then the third is also soluble. We shall also give a characterization of the general solutions of each of these three equations. These provide an illustration of the operation of the Weil-Châtelet group of C_p, see Silverman [8, Chapter 10].

First we consider the case when a solution $\{r_1, s_1, t_1\}$ of equation (I) corresponding to the point P_1 on the curve C_p, and a solution $\{r_2, s_2, t_2\}$ of equation (II) corresponding to P_2 on C_p, are known. In this case we shall describe an algorithm which gives *two* solutions to equation (III); these solutions correspond to the points $P_1 + P_2$ and $P_1 - P_2$ on C_p. We may assume that $(r_1, s_1, t_1) = (r_2, s_2, t_2) = 1$.

We shall work with the following expressions:

$$\begin{aligned} &A = r_1s_1t_2 + r_2s_2t_1, \qquad B = r_1s_1t_2 - r_2s_2t_1, \\ (11)\qquad &C = (r_1^2r_2^2 - ps_1^2s_2^2)/8, \qquad D = r_1^2s_2^2 + r_2^2s_1^2, \\ &K = r_2s_2t_2(r_1^4 + ps_1^4) - r_1s_1t_1(r_2^4 + ps_2^4), \\ &L = r_2s_2t_2(r_1^4 + ps_1^4) + r_1s_1t_1(r_2^4 + ps_2^4). \end{aligned}$$

A number of identities exist between these expressions; they are given in the following lemmas. The most important is

Lemma 1. *The equation $AB = CD$ holds.*

Proof. We have

$$\begin{aligned} 8AB &= 8t_2^2r_1^2s_1^2 - 8t_1^2r_2^2s_2^2 \\ &= (r_2^4 - ps_2^4)r_1^2s_1^2 + (r_1^4 - ps_1^4)r_2^2s_2^2 \qquad \text{by (I) and (II)} \\ &= r_1^2r_2^2(r_1^2s_2^2 + r_2^2s_1^2) - ps_1^2s_2^2(r_1^2s_2^2 + r_2^2s_1^2) = 8CD. \qquad \square \end{aligned}$$

Lemma 2. *Let $U = r_1^2r_2^2 + ps_1^2s_2^2$ and $V = r_2^2s_1^2 - r_1^2s_2^2$. Then*

(i) $KL = AB(U^2 - pV^2)$,
(ii) $4(A^2 + B^2) = UV$,
(iii) $L^2 - K^2 = (A^2 - B^2)(64C^2 + pD^2)$.

Proof. (i) We have, using the identity

$$x^3 + 3x^2y - 3xy^2 - y^3 = (x - y)(x^2 + 4xy + y^2)$$

in the fourth line,

$$\begin{aligned}
8KL &= 8t_2^2r_2^2s_2^2(r_1^4 + ps_1^4)^2 - 8t_1^2r_1^2s_1^2(r_2^4 + ps_2^4)^2 \\
&= (r_2^4 - ps_2^4)r_2^2s_2^2(r_1^4 + ps_1^4)^2 + (r_1^4 - ps_1^4)r_1^2s_1^2(r_2^4 + ps_2^4)^2 \\
&= D[r_1^6r_2^6 + 3pr_1^4r_2^4s_1^2s_2^2 - 3p^2r_1^2r_2^2s_1^4s_2^4 \\
&\qquad\qquad - p^3s_1^6s_2^6 - p(r_1^2r_2^2 - ps_1^2s_2^2)(r_2^4s_1^4 + r_1^4s_2^4)] \\
&= 8DC[r_1^4r_2^4 + 4pr_1^2r_2^2s_1^2s_2^2 + p^2s_1^4s_2^4 - p(r_2^4s_1^4 + r_1^4s_2^4)] \\
&= 8AB(U^2 - pV^2)
\end{aligned}$$

by Lemma 1. Propositions (ii) and (iii) follow in a similar manner. □

Lemma 3. *The integers* $C, D, r_1, \dots, t_2$ *satisfy the following congruence properties:*

(i) r_1, r_2, s_1, s_2 *are odd and* $C \in \mathbb{Z}$,
(ii) $p \nmid r_1r_2s_1s_2$ *and* $p \nmid C$,
(iii) $2 \parallel D$ *and* $t_1 \equiv t_2 \pmod 2$,
(iv) $2 \mid A$, $2 \mid B$, *and* $2 \mid C$.

Proof. Parts (i) and (ii) follow from our assumptions that $(r_1, s_1, t_1) = (r_2, s_2, t_2) = 1$. For (iii) and (iv), D is even by (i) but, as D is a sum of squares, $4 \mid D$ would contradict (i). Secondly, if t_1 and t_2 have different parities, then both A and B are odd, but this conflicts with the evenness of D by Lemma 1, and so (iii) follows. Consequently, both A and B are even, and the evenness of C follows by Lemma 1. □

Definition. Let the coordinates (see (6) and (7)) of the points P_1, P_2, $P_1 + P_2$ and $P_1 - P_2$ be denoted by (x_1, y_1), (x_2, y_2), (x_{12}, y_{12}) and (x_{21}, y_{21}), respectively.

The next two lemmas give expressions for $x_{12}, \dots, y_{21}$.

Lemma 4. *We have* $x_{12} = K^2/16A^2B^2$, $x_{21} = L^2/16A^2B^2$.

Proof. The line through the points (x_1, y_1) and (x_2, y_2) has equation

$$(x_2 - x_1)y = (y_2 - y_1)x + y_1x_2 - x_1y_2.$$

If we let $r_1r_2s_1s_2 = Z$, then

$$\begin{aligned}
x_2 - x_1 &= 4AB/Z^2, \\
y_1x_2 - x_1y_2 &= 4t_1t_2K/Z^3, \\
y_2 - y_1 &= [r_1^3s_1^3t_2(r_2^4 + ps_2^4) - r_2^3s_2^3t_1(r_1^4 + ps_1^4)]/Z^3,
\end{aligned}$$

and our equation for the line becomes

$$4ABZy = [r_1^3s_1^3t_2(r_2^4 + ps_2^4) - r_2^3s_2^3t_1(r_1^4 + ps_1^4)]x + 4t_1t_2K. \tag{12}$$

Squaring both sides of this equation and replacing y^2 by x^3+px, we obtain a cubic in x whose roots are x_1, x_2 and x_{12}, viz:

$$(r_1^2s_1^2x-4t_1^2)(r_2^2s_2^2x-4t_2^2)(16A^2B^2x-K^2)=0,$$

and the result follows. An exactly similar argument gives the value of x_{21}. □

Lemma 5. *We have*

$$\text{(i)}\qquad y_{12}=[K(64A^2C^2+pB^2D^2)]/64A^3B^3,$$

$$\text{(ii)}\qquad y_{21}=-[L(64B^2C^2+pA^2D^2)]/64A^3B^3.$$

Proof. By Lemma 4, (i) follows by substituting the value of x_{12} in (12) and collecting terms, and (ii) follows similarly. □

Theorem 1. *We have*

$$\text{(i)}\qquad -K^2=64A^2C^2-pB^2D^2,$$

$$\text{(ii)}\qquad -L^2=64B^2C^2-pA^2D^2.$$

Proof. (i) As (x_{12}, y_{12}) is a point on C_p, we have, by Lemmas 4 and 5, and dividing by $K^2/(4AB)^6$,

$$(64A^2C^2+pB^2D^2)^2=K^4+256pA^4B^4=K^4+256pA^2B^2C^2D^2$$

by Lemma 1. Hence,

$$\pm K^2=64A^2C^2-pB^2D^2. \tag{13}$$

To evaluate the sign, suppose $2^t \,\|\, B$; then by Lemmas 1 and 3 we have $2^{2t+2} \,\|\, pB^2D^2$ and $2^{2t+6} \mid 64A^2C^2$. Hence, $2^{2t+2} \,\|\, K^2$, which shows that $\pm K^2/2^{2t+2}$ and $-pB^2D^2/2^{2t+2}$ are odd integers congruent modulo 8. Therefore, the only possible sign in (13) is minus, and (i) follows. The proof of (ii) is similar. □

This theorem provides an algorithm for solving equation (III) in §2 as follows: In (i) of Theorem 1 cancel the common factors of K, AC and BD (or of L, BC and AD in part (ii)); then AC and BD become squares (and similarly for BC and AD), thus providing the required solutions. To justify this, we consider first the case when A, B, C and D have a common factor.

Lemma 6. *If q divides A, B, C and D, then q^2 divides both K and L.*

Proof. By Lemmas 1 and 3 we note that 2 divides A, B, C and D, no higher power of 2 has this property, and, by definition, 4 divides both K and L. Hence, as $p \nmid q$, we may assume that q is coprime to both 2 and p.

Secondly, with U and V as given in Lemma 2, we have $U^2+pV^2=64C^2+pD^2$, and so $q^2 \mid U^2+pV^2$ and, by Lemma 2, $q^2 \mid UV$. Together, these show that $q \mid U$ and $q \mid V$. Hence, by Lemma 2 again, we see that $q^4 \mid KL$ and $q^4 \mid L^2-K^2$, and the lemma follows. □

Now let $q^{u_1} \,\|\, A$, $q^{u_2} \,\|\, B$, $q^{u_3} \,\|\, C$ and $q^{u_4} \,\|\, D$. By Lemma 7 we may assume that one of u_1, u_2, u_3 or u_4 is zero. So, for the first case, suppose u_1 is zero, and then (by Lemma 1) $u_2=u_3+u_4$. This gives $q^{u_3} \,\|\, AC$, $q^{u_3+2u_4} \,\|\, BD$, $q^{2u_3+u_4} \,\|\, BC$, and $q^{u_4} \,\|\, AD$. Hence, the factor q^{2u_3} can be cancelled from both sides of equation (i) in Theorem 1. Now the only occurrence of

q in this equation is: q^{4u_4} in the prime factorization of B^2D^2. Similarly, in equation (ii), q^{2u_4} can be cancelled throughout, leaving the factor q^{4u_3} in B^2C^2. The cases when u_2, u_3 or u_4 are zero can be dealt with similarly. If this process is carried out on all primes dividing AB, then, via Theorem 1, two solutions of equation (III) are given by this algorithm.

We shall illustrate this algorithm with the prime 11969. We have (see (8))

$$11969 = 81^2 + 8 \times 26^2 = 113^2 - 8 \times 10^2 = 65^2 + 64 \times 11^2.$$

Now 81 is a square, and so we have a solution to equation (I) given by: $r_1 = 9, s_1 = 1, t_1 = 26$. Secondly, although 113 is not a square, we have $(113 + 10\sqrt{8})(3 + \sqrt{8})^2 = 2401 + 848\sqrt{8}$ (using the identity $3^2 - 8 \times 1^2 = 1$), and so equation (II) has the solution: $r_2 = 49$, $s_2 = 1$, $t_2 = 848$. Substituting these values in (11), we obtain

$$\begin{aligned} A &= 8906 = 2 \times 61 \times 73, & B &= 6358 = 2 \times 11 \times 17^2, \\ C &= 22814 = 2 \times 11 \times 17 \times 61, & D &= 2842 = 2 \times 17 \times 73, \\ K &= 4 \times 5 \times 11 \times 17 \times 73 \times 2131, & L &= 4 \times 5 \times 17 \times 61 \times 102301. \end{aligned}$$

Now $AC = 61^2 \times 4 \times 11 \times 17 \times 73$ and $BD = 17^2 \times 4 \times 11 \times 17 \times 73$. Hence, we can cancel the factor $(4 \times 11 \times 17 \times 73)^2$ from (11) and we obtain the following solution of (III):

$$-10655^2 = 64 \times 61^4 - 11969 \times 17^4.$$

Similarly, $BC = 11^2 \times 17^2 \times 4 \times 17 \times 61$ and $AD = 73^2 \times 4 \times 17 \times 61$, and so the second solution is

$$-511505^2 = 64 \times 187^4 - 11969 \times 73^4.$$

Further algorithms. An exactly similar algorithm to the above exists when solutions of (I) and (III) are known, or when solutions to (II) and (III) are known. Suppose $\{r_1, s_1, t_1\}$ is a solution to (III) corresponding to the point $Q_1 = (x'_1, y'_1)$ on the curve C_p with $(r_1, s_1, t_1) = 1$, and $\{r_2, s_2, t_2\}$ is a solution to (I) [or (II)] corresponding to the point $Q_2 = (x'_2, y'_2)$ on the curve C_p with $(r_2, s_2, t_2) = 1$. Following the procedure above, we define

$$(14) \qquad \begin{aligned} A' &= 8r_1s_1t_2 + r_2s_2t_1, & B' &= 8r_1s_1t_2 - r_2s_2t_1, \\ C' &= 8r_1^2r_2^2 \pm ps_1^2s_2^2, & D' &= 8r_1^2s_2^2 \mp r_2^2s_1^2, \end{aligned}$$

where the upper signs apply when $\{r_2, s_2, t_2\}$ is a solution to (I), and the lower signs apply when equation (II) is the given one. As in Lemma 1, it is a simple matter to show that $A'B' = C'D'$. Also, we define K' and L' by

$$\begin{aligned} K' &= r_1s_1t_1(r_2^4 + ps_2^4) - r_2s_2t_2(64r_1^4 + ps_1^4), \\ L' &= r_1s_1t_1(r_2^4 + ps_2^4) + r_2s_2t_2(64r_1^4 + ps_1^4), \end{aligned}$$

and if $Q_1 + Q_2 = (x'_{12}, y'_{12})$ and $Q_1 - Q_2 = (x'_{21}, y'_{21})$, then

$$x'_{12} = 4K'^2/A'^2B'^2, \qquad y'_{12} = K'(A'^2C'^2 + pB'^2D'^2)/A'^3B'^3,$$
$$x'_{21} = 4L'^2/A'^2B'^2, \qquad y'_{21} = -L'(B'^2C'^2 + pA'^2D'^2)/A'^3B'^3.$$

Corresponding to Theorem 1 we have

Theorem 2. *There holds*

$$\text{(i)} \qquad \pm 8K'^2 = A'^2C'^2 - pB'^2D'^2,$$
$$\text{(ii)} \qquad \pm 8L'^2 = B'^2C'^2 - pA'^2D'^2,$$

where the upper signs apply if $\{r_2, s_2, t_2\}$ *is a solution to equation* (I), *and the lower signs apply when a solution to equation* (II) *is given.*

Proof. See the proof of Theorem 1. □

The algorithm described above also applies here. In (i) of Theorem 2 we cancel the common factors of K', $A'C'$ and $B'D'$, and the resulting expressions provide solutions to (II) [or (I)] as $A'C'$ and $B'C'$ are then squares. Note that we obtain *two* solutions corresponding to the points $Q_1 + Q_2$ and $Q_1 - Q_2$. Therefore, if $Q_1 = P_1 + P_2$ and $Q_2 = P_1$, our new solutions to equation (II) [or (I)] given by Theorem 2 correspond to the points P_2 and $2P_1 + P_2$ on C_p.

We shall show now that this is always the case. If we have a solution to one of our equations $(*)$ (where $(*)$ is (I), (II) or (III)), with corresponding point $P \in C_p$, then, for all points $R \in C_p$, there is another solution to $(*)$ corresponding to the point $P + 2R$, and all solutions of $(*)$ are generated in this way.

Theorem 3. *Suppose we are given a nontrivial solution to equation* (I), (II) *or* (III) *corresponding to the point* $P \in C_p$; *then this equation has infinitely many solutions, and the corresponding points on* C_p *have the form* $P + 2R$, *where* R *is an arbitrary point on* C_p.

Note. We are not assuming that the points P and R are of the same type.

Proof. We use the same method as in the previous two cases. We shall give the proof for equations (I) and (II); an exactly similar argument applies in the remaining cases. Suppose the point P has coordinates $(4t^2/r^2s^2)$, $t(r^4 + ps^4)/r^3s^3$, where $\mp 8t^2 = r^4 - ps^4$, and R has coordinates $(a^2/c^2$, $ab/c^3)$, where $b^2 = a^4 + pc^4$ (see (3) and (7)). The coordinates of $2R$ are

$$((a^4 - pc^4)^2/4a^2b^2c^2, (a^4 - pc^4)(a^8 + 6pa^4c^4 + p^2c^8)/8a^3b^3c^3),$$

and we may assume that r and s are odd, and a and c have different parities. Following the procedures above, we define

$$A'' = 4tabc + rs(a^4 - pc^4), \qquad B'' = 4tabc - rs(a^4 - pc^4),$$
$$C'' = \mp r^2b^2 + 2ps^2a^2c^2, \qquad D'' = 2r^2a^2c^2 \pm s^2b^2,$$
$$K'' = rst(a^8 + 6pa^4c^4 + p^2c^8) - abc(r^4 + ps^4)(a^4 - pc^4)/2,$$

where the upper [lower] signs apply when equation (I) [(II)] is being used.

Lemma 7. *We have* $A''B'' = C''D''$.

Proof. Using the equations $\mp 8t^2 = r^4 - ps^4$ and $b^2 = a^4 + pc^4$, we obtain

$$\begin{aligned} C''D'' &= \mp 2r^4a^2b^2c^2 - r^2s^2b^4 + 4pr^2s^2a^4c^4 \pm 2ps^4a^2b^2c^2 \\ &= \mp 2a^2b^2c^2(r^4 - ps^4) - r^2s^2(b^4 - 4pa^4c^4) = A''B''. \qquad \square \end{aligned}$$

Continuing the main proof, we see that the coordinates of the point $P + 2R$ are

$$(4K''^2/A''^2B''^2,\ K''(A''^2C''^2 + pB''^2D''^2)/A''^3B''^3),$$

and we have

$$\mp 8K''^2 = A''^2C''^2 - pB''^2D''^2.$$

We now cancel the common factors of K'', $A''C''$ and $B''D''$ in this equation, and the result is a new solution to equation (I) [or (II)]; the details follow exactly those given above for Lemmas 2 and 6. $\square$

Example. Let $p = 73$ and let P and R be the generators of the group C_p corresponding to equations (I) and (II), respectively. Hence, using the supplement table, (I), (3) and (4), we have $r = s = 1$, $t = 3$, $a = 2$, $b = 77$ and $c = 3$, and substituting these values in the above, we have

$$A'' = -353, \quad B'' = 17 \times 673, \quad C'' = -673, \quad D'' = 17 \times 353,$$
$$K'' = 353 \times 673 \times 873.$$

These values now give a new solution to equation (I) corresponding to the point $P + 2R$ [as $(353, 673) = 1$]:

$$1^4 - 73 \times 17^4 = -8 \times 873^2.$$

Finally, we prove the converse of Theorem 3; again the method of proof is very similar to that used in the above proofs.

Theorem 4. *If* $\{r_1, s_1, t_1\}$ *and* $\{r_2, s_2, t_2\}$ *are two solutions to one of the equations* (I), (II) *or* (III) *with corresponding points* P_1 *and* P_2, *then there is a point* $R \in C_p$ *with the property* $P_2 = P_1 + 2R$.

Proof. We give the proof for equation (I); the same argument applies in the remaining cases. As above, we define

$$A^* = r_1s_1t_2 + r_2s_2t_1, \qquad B^* = r_1s_1t_2 - r_2s_2t_1,$$
$$C^* = r_1^2r_2^2 + ps_1^2s_2^2, \qquad D^* = (r_1^2s_2^2 - r_2^2s_1^2)/8,$$

$$K^* = r_2s_2t_2(r_1^4 + ps_1^4) - r_1s_1t_1(r_2^4 + ps_2^4),$$
$$L^* = r_2s_2t_2(r_1^4 + ps_1^4) + r_1s_1t_1(r_2^4 + ps_2^4).$$

Repeating the arguments of Lemmas 1 to 6, we have

$$A^*B^* = C^*D^*,$$

the coordinates of $P_2 - P_1$ are

$$(L^{*2}/16A^{*2}B^{*2}, -L^*(B^{*2}C^{*2} - 64pA^{*2}D^{*2})/64A^{*3}B^{*3}),$$

and

$$L^{*2} = B^{*2}C^{*2} - 64pA^{*2}D^{*2}.$$

Note that we have a plus sign on the left-hand side of this last equation, as $2 \parallel C^*$ in this case; see the proof of Theorem 1. The result now follows using (5) of §2. □

4. One point known

In view of the results above a natural question to ask is: suppose we are given a solution to just one of our equations (I), (II) or (III); is there an algorithm which will generate solutions to the remaining two equations? This is a much harder problem; it is not definitely known that solutions exist, but we have the

Conjecture. If p is a G-prime and the rank of the curve C_p is not zero, then it equals 2.

Silverman [8], and others, have shown that this Conjecture is a consequence of the Shafarevich-Tate Conjecture, which states that the Shafarevich-Tate group Ш for C_p is finite. Although some progress has been made on this second conjecture recently, it remains open at this time. The numerical evidence presented below shows that our conjecture is valid for all primes $p < 50000$. Also, this Conjecture can be replaced by the following apparently simpler question.

Suppose we have a solution $\{r, s, t\}$ to equation (I) [the argument is similar in the other two cases]. Then we can find solutions to equations (II) and (III) provided we can find a nontrivial simultaneous integer solution $\{x, y, z, w\}$ to the pair of equations

$$x^2 + 16txy - 8r^4y^2 = 8s^4z^2 + pw^2,$$
$$xy = zw.$$

Using (9) and (10), we can easily show that this pair of equations has common local solutions for all primes q. But this does not necessarily lead to simultaneous integer (global) solutions. We note that the second equation above is identical to that in Lemma 1; there it was the main link in the algorithm, here it seems to be the main stumbling block to progress; for further details see Rose [6].

5. Numerical data

Extensive computer searches have been undertaken to find the ranks and generators of the curves (1) for all G-primes $p < 50000$; the results are presented in the supplement. After some preliminary trials using a HP 28s calculator, the main searches were made using the package PARI/GP (developed by Cohen and his collaborators in Bordeaux, France) on a Sun 4. First, attempts were made to solve one or more of the equations (I), (II) and (III). If these failed to give solutions, then the value of the L-function for the curve at $s = 1$ was calculated in order to prove that the rank was indeed zero, that is, (I), (II) and (III) are insoluble; see below.

The method used to attempt to solve our equations is as follows. First consider equation (I). Note, by (10), we can choose the sign of a so that $a \equiv 1 \pmod 8$. Rewriting (I) and using (8), we look for integers x and y to satisfy

$$(a^2+8b^2)(x^2+8y^2) = (ax \pm 8by)^2 + 8(ay \mp bx)^2 = r^4 + 8t^2 = ps^4;$$

that is, we look for s, x and y to satisfy

$$x^2 + 8y^2 = s^4, \qquad ax \pm 8by \text{ is a square, } r^2, \tag{15}$$

and then $t = ay \mp bx$. The equation in (15) has the parametric solution

$$x = (m^2-2n^2)^2 - 8m^2n^2, \quad y = 2mn(m^2-2n^2), \quad s = m^2+2n^2.$$

Hence, using (15), we try various integers m and n until we find a pair such that

$$a((m^2-2n^2)^2 - 8m^2n^2) + 16bmn(m^2-2n^2) \text{ is a square, } r^2, \tag{16}$$

and then the values of s and t are determined using the above. For equation (II) the left-hand side of (16) is replaced by $c((m^2+2n^2)^2 + 8m^2n^2) + 16dmn(m^2+2n^2)$ [if $c \equiv 3 \pmod 8$, put $3c \pm 8d$ for c and $c \pm 3d$ for d, see (8) and (9)], and for equation (III) the left-hand side of (16) is replaced by $\pm e((m^2-4n^2)^2 - 16m^2n^2) + fmn(m^2-4n^2)$, where both signs must be considered.

The method can be extended in the following way. Multiplying (16) by a and rewriting, we have

$$w^2 = 8pu^2 + av^2, \tag{17}$$

where

$$u = mn, \quad v = r \quad \text{and} \quad w = a(m^2-2n^2) + 8bmn. \tag{18}$$

Using (9) and (10), we can easily see that equation (17) is soluble by Legendre's Theorem. Hence, one way to solve (I) is to look for general solutions to (17) subject to the conditions (18). In practice we found the most efficient method was to use (16) directly. First we tried all values of m and n satisfying $0 < m < 500$ and odd, and $-250 < n < 250$. If this failed, using the first few primes q, we sieved out those values of m and n for which (16) is impossible modulo q, and then tried larger values of m and n to solve (16). For example, if $p \equiv 2 \pmod 5$ (generally the most intractable case), then $5 \mid r$ and so the left-hand side of (16) must be congruent to 25 modulo 100.

It is worth pointing out that, for each G-prime p under consideration, in all cases where solutions to one of the equations (I), (II) or (III) were found, solutions to the remaining two equations were also found—the prime 41521 was by far the most refractory—that is, in all 366 cases where the rank of the corresponding curve is positive, it does, in fact, equal two. Also in all of these cases, at least one of the three equations has a 'small' solution; that is, one with m and n (in (16) or its replacements for equations (II) or (III)) less than 20, the 'worst' case (for primes less than 50000) being $p = 47497$, where, for equation (II), the smallest solution is given by $m = 7$ and $n = 18$. Note that the smallest solutions of the two remaining equations can be very 'large', for example with the prime $p = 41521$. In this example the smallest solutions for equations (I) and (II) are given in the supplement (for (I) the solution is generated by $m = 156347$, $n = 41668$), but note that equation (III) has the solution 5, 1, 39 and the corresponding values of m and n in this case are 1 and 0, respectively.

Rubin [7], developing some work of Kolyvagin and others, has proved some parts of the Birch and Swinnerton-Dyer Conjecture. In particular he has shown that, if an elliptic curve has complex multiplication (our curves C_p have complex multiplication in the field $\mathbb{Q}(i)$ of Gaussian numbers), and if the value of the L-function for the curve at the point $s = 1$ is nonzero, then the curve has only finitely many points defined over $\mathbb{Q}(i)$; and so the rank of the curve over $\mathbb{Q}$ is zero. We applied this to our numerical work. For those curves C_p where we were unable to find solutions to equations (I), (II) or (III) after fairly short trials, we calculated the values at $s = 1$ of the corresponding L-functions. In all cases we found these values to be positive, and hence, by the result quoted above, the ranks are zero and no further trials were required. Following Buhler, Gross and Zagier [3], we used the following formula to calculate the L-function:

$$L(C_p, 1) = 2 \sum_{n=1}^{\infty} \frac{a_p(n)}{n} \exp\left(\frac{-\pi n}{4p}\right),$$

where $a_p(q)$ is the trace of Frobenius for primes q, and it is extended to all positive integers in the usual way (see, for example, Cohen [4, p. 406]). [Note that the curve C_p has conductor $64p^2$, and the factor 2 occurs because the sign of the functional equation is positive for all of our curves; this was calculated using the method given in Cohen [4, p. 406].] To keep the computations within reasonable time bounds, we replaced ∞ in the above sum by $16p$ and took the sum over those n satisfying $n \equiv 1 \pmod 4$, because for all curves under consideration the coefficients $a_p(n)$ are zero otherwise. If we then divided the result by the product of the real period, the Tamagawa numbers (in all cases $c_p = 2$ and the remainder are all equal to 1), and the inverse of the square of the order of the torsion subgroup of C_p ($= 1/4$ in all cases, see (2)) as required by the Birch and Swinnerton-Dyer Conjecture, we obtained in all cases a square integer to at least five decimal places. Hence, as a by-product of these calculations we obtained (assuming the validity of this conjecture) the values S of the orders of the Shafarevich-Tate groups Ш. For all G-primes p for which $r(C_p) = 0$ we found that $S = 16$ except in the following cases: $S = 64$ when p is one of the following 28 primes:

4937	12161	15017	25601	31337	33937	44497
10657	12697	18257	26497	31817	34297	47161
10937	13417	23857	28697	32297	35897	47657
11777	14897	25057	29761	33377	36857	47777

and $S = 144$ when p is

21577, 28537, 30937.

Note the connection between the fact that in all cases $16 \mid S$ and the 4-descent described above, and that the structure of Ш is determined by the corresponding 2-descent and the Cassels pairing $\mathbb{Z}/\sqrt{S}\mathbb{Z} \times \mathbb{Z}/\sqrt{S}\mathbb{Z}$. Some authors have suggested that the value of S increases, if only slowly, as the value of the conductor increases. In particular, if S_p denotes the order of Ш for the elliptic curve C_p, then it is conjectured that for large p the approximate value of S_p is $p^{1/4 \pm o(1)}$. Our data shows a fairly uniform spread through the range 1—50000 for the higher values of S, and so no conclusions can be drawn from our calculations. Owing to the computer time required, we did not calculate the orders S of the groups Ш for those primes p where $r(C_p) = 2$; this will be undertaken in the sequel.

Our calculations have established the values of $r(C_p)$ for all primes p less than 50000 and congruent to 1 modulo 8. For the 629 non G-primes q, $r(C_q) = 0$ (see §2), and for the 625 G-primes p the table in the supplement gives either the value of the L-function at $s = 1$ when $r(C_p) = 0$ (in 259 cases), or the values of r and s for equations (I) and (II) when $r(C_p) = 2$ from which the coordinates of the generators of the group C_p can easily be calculated using (6) (366 cases in all). In some cases equations (I) or (II) have two distinct solutions where both r and s are roughly similar in size; in these cases the solution with the smaller value of s was chosen. This was not checked in all cases.

Finally, at the referee's suggestion and after some discussions with John Cremona, we have included some data on the Mordell-Weil lattices of the rank-two curves. Suppose P_1 and P_2 generate C_p modulo torsion (that is, C_p is generated by P_1, P_2 and $(0, 0)$, see (2)). Let $\langle P_i, P_j\rangle$, for $i, j = 1$ or 2, denote the Néron-Tate height pairing and let R_{C_p} denote the elliptic regulator of C_p, see Silverman [8, p. 232]. Over the complex field $\mathbb{C}$, the generators of the Mordell-Weil lattice Λ for C_p can be taken to be

$$\begin{aligned}\omega_1 &= \sqrt[+]{\langle P_1, P_1\rangle},\\ \omega_2 &= (\langle P_1, P_2\rangle + i\sqrt{R_{C_p}})/\sqrt[+]{\langle P_1, P_1\rangle}.\end{aligned}$$

Then the invariant τ, a complex number in the upper half-plane, is defined by

$$\tau = \omega_2/\omega_1 = (\langle P_1, P_2\rangle + i\sqrt{R_{C_p}})/\langle P_1, P_1\rangle$$

modulo transformations by elements of $SL(2, \mathbb{Z})$. Once τ has been moved to the fundamental region of the group $SL(2, \mathbb{Z})$, it is independent of the choice of the generators P_1 and P_2 of C_p, and so provides information about the shape of the Mordell-Weil lattice Λ.

For each of the rank-2 curves discussed in this paper we computed (using PARI/GP) the value of τ, and these values are given in the table in the supplement. In some cases, to obtain a value of τ in the fundamental region of $SL(2, \mathbb{Z})$, ± 1 was added to the computed complex number; no other $SL(2, \mathbb{Z})$ transformation was required. In this region the values of τ for approximately half of the curves under consideration lie in a segment of the annular region bounded by the ellipses $375x^2 + y^2 = 100$ and $480x^2 + y^2 = 200$, and the lines $2x \pm 1 = 0$, where x denotes the real part and y the imaginary part. Only six values of τ lie below this region, and the remainder above. Also the proportion of values of τ with positive or negative real part is approximately equal, and those with larger imaginary part tend to correspond with the larger values of the prime p.

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Supplement to
ON A CLASS OF ELLIPTIC CURVES WITH RANK AT MOST 2

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We list below data for the elliptic curve

$$C_p : y^2 = x^3 + px$$

for each 'G-prime' p less than 50000. We give, for each curve in this range, the value of the L–function at 1 and the order S of the Shafarevich-Tate group Ш as predicted by the Birch and Swinnerton-Dyer conjecture, if the rank is zero. If the rank = 2, we give the two values of r and s from which the generators of the Mordell-Weil group can easily be constructed using (6), and the value of $\tau = \omega_2/\omega_1$ of the Mordell-Weil lattice, where $a + ib$ is written a, b (see last paragraph).

Prime	*Rank*	I *point, rank* 2 *only*	II *point, rank* 2 *only*	τ
73	2	1, 1	3, 1	-0.42136, 3.70975
89	2	3, 1	7, 1	0.25820, 4.79915
113	2	3, 1	5, 1	0.41331, 4.48636
233	2	11, 3	5, 1	0.48210, 3.58534
257	0	$L(1) = 7.40907, S = 16$		
281	2	3, 1	47, 7	-0.22809, 11.81452
337	2	35, 9	5, 1	-0.05956, 6.23399
353	2	13, 3	7, 1	-0.44182, 4.20334
577	0	$L(1) = 6.05276, S = 16$		
593	2	3, 1	5, 1	0.48071, 5.03903
601	2	83, 17	7, 1	-0.33842, 5.45261
617	2	195, 43	5, 1	-0.19632, 5.61055
881	2	3, 1	157, 17	-0.48756, 18.15133
937	2	5, 3	45, 7	-0.42973, 11.62307
1033	2	13, 3	85, 7	0.06569, 7.58315
1049	2	3, 1	7, 1	-0.49569, 5.99058
1097	0	$L(1) = 5.15461, S = 16$		
1153	2	1, 1	9, 1	-0.40743, 6.18935
1193	2	47, 9	11, 1	0.45560, 4.46677
1201	0	$L(1) = 5.03920, S = 16$		
1217	0	$L(1) = 5.02255, S = 16$		
1249	2	11, 3	7, 1	-0.49338, 4.91196

Prime	Rank	1 point, rank 2 only	11 point, rank 2 only	r
1289	2	41, 9	19, 1	-0.37828, 5.10380
1433	2	3, 1	7, 1	-0.48472, 5.85549
1481	0	$L(1) = 4.78199, S = 16$		
1553	2	15, 11	59, 7	0.25107, 6.90827
1601	2	117, 19	7, 1	0.15524, 7.21936
1609	2	19, 3	29, 1	0.00547, 12.81853
1721	0	$L(1) = 4.60577, S = 16$		
1753	2	5, 9	7, 1	0.48954, 3.97054
1777	2	5, 1	525, 73	0.47957, 22.98789
1801	2	1, 1	547, 79	0.19453, 24.89577
1889	2	51, 11	7, 1	0.31093, 6.09387
1913	2	57, 11	17, 1	-0.40965, 4.87389
2089	2	11, 3	53, 1	-0.18042, 7.41783
2113	2	5, 3	7, 1	0.48143, 5.10477
2129	2	3, 1	157, 7	-0.13885, 19.47539
2273	2	19, 3	7, 1	-0.49565, 4.81242
2281	2	299, 67	21, 1	-0.43435, 7.01621
2393	2	3, 1	7, 1	0.46625, 6.20771
2441	0	$L(1) = 4.22042, S = 16$		
2473	2	7, 1	11, 1	-0.15226, 9.04684
2593	2	1, 1	73, 1	0.35189, 13.45925
2657	0	$L(1) = 4.13190, S = 16$		
2689	2	7, 1	9, 1	-0.11393, 8.47626
2833	0	$L(1) = 4.06617, S = 16$		
2857	0	$L(1) = 4.05760, S = 16$		
2969	2	3, 1	1543, 193	0.23718, 24.95098
3049	2	7, 1	27, 1	-0.24301, 7.99620
3089	2	7, 3	11, 1	-0.45345, 5.72969
3121	0	$L(1) = 3.96893, S = 16$		
3137	2	5, 3	815, 103	-0.48132, 23.58055
3217	2	5, 1	1942845, 57097	-0.01031, 49.37582
3257	2	125, 57	815, 1	0.25650, 13.05028
3313	2	17, 3	19, 1	-0.37758, 6.48209
3361	2	121, 17	9, 1	0.34645, 6.29887
3449	0	$L(1) = 3.87101, S = 16$		
3529	2	1, 1	2683, 343	0.40631, 30.32619
3673	2	23, 3	9, 1	0.06481, 8.59214
3761	0	$L(1) = 3.78810, S = 16$		
3833	2	1, 3	73, 7	-0.15485, 8.36706
4001	0	$L(1) = 3.72997, S = 16$		
4049	2	93, 41	61, 7	-0.33022, 6.24137
4057	0	$L(1) = 3.71703, S = 16$		
4153	2	5, 1	31, 1	0.07976, 13.05205
4177	0	$L(1) = 3.69004, S = 16$		
4201	2	7, 1	1587, 41	0.25118, 28.30657
4217	0	$L(1) = 3.68126, S = 16$		
4273	2	1, 3	11, 1	-0.47627, 6.00761

Prime	Rank	1 point, rank 2 only	11 point, rank 2 only	r
4289	2	71, 9	193, 23	0.13383, 8.16403
4297	0	$L(1) = 3.66400, S = 16$		
4409	0	$L(1) = 3.64051, S = 16$		
4457	2	265, 33	4075, 479	-0.08955, 14.66498
4481	0	$L(1) = 3.62580, S = 16$		
4513	2	257, 33	9, 1	0.27654, 6.98405
4657	0	$L(1) = 3.59104, S = 16$		
4721	0	$L(1) = 3.57881, S = 16$		
4801	2	67, 19	1951, 49	-0.41475, 22.89767
4817	0	$L(1) = 3.56085, S = 16$		
4937	0	$L(1) = 14.15603, S = 64$		
4993	2	7, 1	9, 1	-0.49061, 6.71078
5081	2	3, 1	1367, 161	0.10623, 22.52430
5113	2	25, 11	17, 1	0.27147, 9.55796
5209	2	7, 3	9, 1	-0.47884, 5.90205
5233	2	5, 1	13, 1	0.45984, 7.52959
5297	0	$L(1) = 3.47728, S = 16$		
5393	2	25, 3	11, 1	-0.47410, 5.77620
5569	0	$L(1) = 3.43402, S = 16$		
5689	2	67, 41	63, 1	0.34332, 6.04763
5737	0	$L(1) = 3.40860, S = 16$		
5881	2	77, 9	991, 79	-0.16976, 26.78138
6089	2	13, 3	13, 1	-0.11486, 9.76499
6121	0	$L(1) = 3.35383, S = 16$		
6353	2	3, 1	29, 1	0.26843, 8.49113
6361	2	1471, 331	9, 1	0.14577, 8.11163
6449	2	457, 51	11, 1	0.37794, 5.89638
6481	0	$L(1) = 3.30626, S = 16$		
6521	0	$L(1) = 3.30118, S = 16$		
6529	2	1, 9	9, 1	-0.14390, 8.43804
6553	2	7, 3	9, 1	-0.47486, 5.95883
6569	2	9, 1	443, 49	0.17493, 24.10072
6689	2	9, 1	17, 1	0.23056, 10.40718
6761	2	9, 1	491, 1	0.48516, 22.01904
6793	0	$L(1) = 3.26762, S = 16$		
6841	0	$L(1) = 3.26188, S = 16$		
6857	0	$L(1) = 3.25997, S = 16$		
7121	0	$L(1) = 3.22933, S = 16$		
7129	0	$L(1) = 3.22842, S = 16$		
7393	2	19, 3	207, 7	0.06794, 9.45031
7481	2	43, 33	71, 7	0.37980, 13.75473
7489	2	19, 3	71, 1	0.47916, 12.84974
7529	2	9, 1	43, 1	0.41539, 12.85986
7577	2	25, 3	985, 31	-0.07503, 15.64933
7753	2	245, 27	13, 1	-0.25735, 8.50790
7793	0	$L(1) = 3.15731, S = 16$		
7817	0	$L(1) = 3.15491, S = 16$		

Prime	Rank	I point, rank 2 only	II point, rank 2 only	τ
7841	0	$L(1) = 3.15249, S = 16$		
7993	2	29, 9	15, 1	0.47413, 5.70663
8081	0	$L(1) = 3.12882, S = 16$		
8161	0	$L(1) = 3.12113, S = 16$		
8209	2	13, 9	69, 7	-0.28665, 8.18246
8233	2	7, 1	67, 7	0.34802, 15.75490
8273	2	3, 1	5485, 529	0.21601, 33.96022
8369	2	2089, 219	11, 1	-0.24472, 7.62063
8537	2	75, 41	55, 1	0.15569, 14.80030
8609	2	9, 1	1409, 103	-0.45781, 26.81332
8713	2	1, 1	51, 1	0.14729, 8.96657
8761	0	$L(1) = 3.06626, S = 16$		
8969	2	29, 3	781, 47	-0.25712, 18.78534
9001	0	$L(1) = 3.04561, S = 16$		
9137	0	$L(1) = 3.03421, S = 16$		
9209	0	$L(1) = 3.02827, S = 16$		
9241	0	$L(1) = 3.02564, S = 16$		
9281	0	$L(1) = 3.02238, S = 16$		
9337	2	5, 1	1528135, 113207	-0.19915, 51.11870
9377	2	5, 3	18061895, 678647	-0.08241, 55.42424
9473	2	87, 17	89, 7	-0.28796, 7.56907
9521	2	7, 3	1291, 89	0.42997, 25.93856
9601	2	7, 1	14617, 967	0.40415, 34.86028
9649	2	557, 89	21, 1	0.33445, 8.68085
9697	0	$L(1) = 2.98913, S = 16$		
9721	2	41, 27	1193, 103	-0.22015, 14.99177
9769	0	$L(1) = 2.98390, S = 16$		
10177	2	3505, 353	1225, 113	-0.30612, 12.62882
10337	0	$L(1) = 2.94204, S = 16$		
10369	2	1, 1	3609, 337	-0.29214, 31.97170
10433	2	9, 1	25, 1	-0.34607, 8.41691
10457	2	13135, 1611	339215, 7849	0.06085, 16.18982
10513	2	59, 33	333, 31	-0.20354, 8.84614
10657	0	$L(1) = 11.67881, S = 64$		
10889	2	189, 19	13, 1	0.24609, 8.76032
10937	0	$L(1) = 11.60333, S = 64$		
10993	2	5, 1	693, 41	0.24569, 20.75436
11057	2	1995, 233	3805, 329	-0.32768, 18.86964
11113	2	7, 1	11, 1	-0.48301, 7.42255
11161	0	$L(1) = 2.88617, S = 16$		
11177	0	$L(1) = 2.88513, S = 16$		
11257	0	$L(1) = 2.87999, S = 16$		
11321	2	63, 227	17, 1	-0.19310, 9.80701
11329	0	$L(1) = 2.87511, S = 16$		
11617	0	$L(1) = 2.85742, S = 16$		
11633	2	3, 1	13, 1	-0.47480, 7.63653
11657	2	95, 33	1915, 71	-0.19519, 14.94638

Prime	Rank	I point, rank 2 only	II point, rank 2 only	τ
11777	0	$L(1) = 11.39065, S = 64$		
11801	0	$L(1) = 2.84621, S = 16$		
11969	2	9, 1	49, 1	-0.15001, 14.70714
12041	2	19, 3	61, 1	-0.48151, 12.67156
12049	2	17, 3	11, 1	0.49631, 6.25408
12073	2	11, 3	21, 1	0.43172, 7.42925
12161	0	$L(1) = 11.29965, S = 64$		
12241	2	541, 233	277, 7	0.40484, 11.03837
12329	2	729, 97	11, 1	-0.10609, 9.23748
12377	0	$L(1) = 2.81251, S = 16$		
12409	2	31, 3	33, 1	-0.37424, 7.98017
12457	2	95, 9	8115, 553	-0.20691, 34.93605
12473	2	1667, 249	247, 23	-0.30163, 6.83721
12497	0	$L(1) = 2.80573, S = 16$		
12553	0	$L(1) = 2.80259, S = 16$		
12569	2	39, 19	41, 1	0.28861, 12.50831
12577	0	$L(1) = 2.80126, S = 16$		
12641	0	$L(1) = 2.79770, S = 16$		
12697	0	$L(1) = 11.17846, S = 64$		
12713	0	$L(1) = 2.79373, S = 16$		
12841	2	18374591, 2465433	11, 1	0.01487, 9.54164
13049	0	$L(1) = 2.77557, S = 16$		
13121	0	$L(1) = 2.77176, S = 16$		
13217	0	$L(1) = 2.76671, S = 16$		
13337	0	$L(1) = 2.76047, S = 16$		
13417	0	$L(1) = 11.02537, S = 64$		
13441	2	93581, 11379	591, 17	0.42382, 16.54517
13633	2	355, 43	15, 1	0.38861, 6.87030
13841	2	63, 19	11, 1	0.33301, 7.28064
13913	2	63, 11	25, 1	-0.25622, 11.11838
13921	0	$L(1) = 2.73105, S = 16$		
14249	2	9, 1	11, 1	-0.48911, 7.34076
14321	2	33, 43	113947, 5663	0.07423, 25.62016
14369	0	$L(1) = 2.70951, S = 16$		
14401	0	$L(1) = 2.70800, S = 16$		
14149	2	4501, 569	13, 1	0.09411, 9.78645
14537	2	3215, 417	35, 1	0.29224, 10.76173
14593	2	11, 3	351, 31	-0.00070, 10.04268
14633	2	183, 17	11, 1	-0.25403, 8.24270
14713	2	11, 1	15, 1	-0.46149, 7.77371
14737	2	5, 1	3839965, 57199	-0.39222, 56.62083
14753	2	9, 1	193, 17	0.45445, 18.21982
14897	0	$L(1) = 10.74070, S = 64$		
14929	2	11, 1	269, 23	0.17793, 15.82804
14969	2	123, 17	31, 1	-0.06635, 13.06344
15017	0	$L(1) = 10.71918, S = 64$		
15073	2	259, 27	39, 1	-0.33222, 11.53172

Prime	Rank	I point, rank 2 only	II point, rank 2 only	τ
15121	2	3113, 339	19, 1	-0.07637, 11.17215
15193	2	179, 81	199, 1	0.29667, 7.34989
15217	0	$L(1) = 2.67094, S = 16$		
15241	0	$L(1) = 2.66989, S = 16$		
15289	2	11, 1	1527, 119	-0.10587, 29.07412
15313	2	937, 187	27, 1	0.37915, 8.31812
15361	0	$L(1) = 2.66466, S = 16$		
15569	2	3, 1	37, 1	-0.14849, 13.63221
15761	2	273507, 75889	13, 1	0.10725, 9.49145
15809	2	9, 1	17, 1	-0.19513, 10.62580
15881	2	117, 11	5957, 233	0.04489, 16.71908
16193	2	717, 67	79, 7	0.35393, 6.43451
16249	0	$L(1) = 2.62749, S = 16$		
16361	2	9, 1	792979, 45367	-0.24652, 53.39924
16417	0	$L(1) = 2.62074, S = 16$		
16433	2	1113, 107	3667, 103	0.11455, 10.02792
16553	2	213, 67	1997, 167	0.11525, 10.06051
16633	2	695, 123	14001, 337	0.01880, 10.13888
16649	2	5839, 657	3283, 289	0.18319, 8.73931
16673	2	1585, 177	353, 17	0.29124, 7.84812
16729	0	$L(1) = 2.60843, S = 16$		
17033	2	25, 9	19, 1	0.01803, 11.37264
17137	0	$L(1) = 2.59277, S = 16$		
17209	0	$L(1) = 2.59005, S = 16$		
17321	2	72127, 6681	6443, 97	-0.30303, 12.59383
17377	2	15305, 2097	25, 1	0.30990, 8.52137
17417	0	$L(1) = 2.58228, S = 16$		
17497	0	$L(1) = 2.57933, S = 16$		
17609	2	27, 11	13, 1	-0.49167, 5.61382
17729	2	19, 3	5807, 257	0.00783, 20.67332
17737	0	$L(1) = 2.57056, S = 16$		
17881	2	12731, 1161	2031, 89	-0.37259, 15.04321
17929	0	$L(1) = 2.56365, S = 16$		
18011	2	17, 3	2023537, 132481	-0.29180, 56.66268
18121	0	$L(1) = 2.55683, S = 16$		
18169	2	11, 1	87, 7	0.32830, 16.74173
18233	2	31, 3	65, 1	0.30274, 9.18452
18257	0	$L(1) = 10.20822, S = \mathbf{64}$		
18289	2	229, 33	101, 7	-0.32073, 7.46027
18353	0	$L(1) = 2.54871, S = 16$		
18433	2	1, 1	233, 1	0.21795, 20.67932
18481	2	299, 33	613, 47	0.41949, 18.41465
18521	0	$L(1) = 2.54291, S = 16$		
18593	2	35, 3	79, 1	-0.27804, 9.39044
18617	0	$L(1) = 2.53963, S = 16$		
18713	2	231, 67	2873, 47	0.09374, 9.80574
19121	2	1, 3	83, 7	-0.10726, 17.51039

Prime	Rank	I point, rank 2 only	II point, rank 2 only	τ
19249	2	11, 1	21, 1	-0.41403, 8.38133
19289	2	3, 1	24023, 527	-0.34354, 35.73510
19417	0	$L(1) = 2.51305, S = 16$		
19433	0	$L(1) = 2.51254, S = 16$		
19441	0	$L(1) = 2.51228, S = 16$		
19457	0	$L(1) = 2.51176, S = 16$		
19577	0	$L(1) = 2.50790, S = 16$		
19777	0	$L(1) = 2.50154, S = 16$		
19793	2	29, 9	277, 17	-0.16412, 9.59929
19841	0	$L(1) = 2.49952, S = 16$		
19993	2	529, 59	409, 17	0.37413, 13.39975
20089	2	1, 3	1393, 97	-0.09648, 28.64906
20113	2	107, 9	12653, 967	0.03435, 20.25243
20177	2	266115, 32059	22355, 1457	-0.27039, 12.62211
20297	0	$L(1) = 2.48536, S = 16$		
20353	2	35, 3	31, 1	0.40852, 8.17227
20369	2	83, 33	13, 1	0.14198, 9.70393
20521	2	4561, 569	1059, 73	-0.33876, 12.38631
20641	2	1517, 179	23, 1	0.18819, 10.99953
20809	2	1, 1	19, 1	0.44207, 8.52455
20857	2	328293635, 103905609	15, 1	-0.07305, 10.17905
20873	2	69, 11	13, 1	-0.48958, 5.60729
20897	0	$L(1) = 2.46732, S = 16$		
20921	0	$L(1) = 2.46662, S = 16$		
21017	0	$L(1) = 2.46379, S = 16$		
21089	2	219, 19	103, 7	-0.30768, 7.95277
21193	0	$L(1) = 2.45866, S = 16$		
21313	2	163, 19	1351, 1	-0.08527, 10.19746
21433	2	5, 1	9615, 577	0.40224, 34.87518
21481	0	$L(1) = 2.45038, S = 16$		
21521	0	$L(1) = 2.44924, S = 16$		
21529	0	$L(1) = 2.44901, S = 16$		
21569	2	39537, 3101	17, 1	0.20574, 9.23885
21577	0	$L(1) = 22.02885, S = \mathbf{144}$		
21601	0	$L(1) = 2.44697, S = 16$		
21673	2	197, 19	309, 17	-0.20932, 9.08583
21713	2	3, 1	85, 7	-0.11675, 10.47821
21841	2	11, 1	7389, 257	0.17961, 34.77247
21961	0	$L(1) = 2.43688, S = 16$		
21977	0	$L(1) = 2.43644, S = 16$		
22441	0	$L(1) = 2.42371, S = 16$		
22697	0	$L(1) = 2.41688, S = 16$		
22721	0	$L(1) = 2.41624, S = 16$		
22777	2	565, 81	15, 1	-0.01217, 10.63682
23017	0	$L(1) = 2.40841, S = 16$		
23041	0	$L(1) = 2.40781, S = 16$		
23057	2	925, 81	463925, 7849	0.01085, 23.38997

Prime	*Rank*	*1 point, rank 2 only*	*11 point, rank 2 only*	r
23209	2	7, 1	457731, 89	0.07884, 43.27701
23369	0		$L(1) = 2.39931, S = 16$	
23417	2	541655, 44571	3466985, 141353	0.09608, 16.56422
23473	2	209, 43	75, 1	-0.36779, 6.78966
23537	0		$L(1) = 2.39502, S = 16$	
23561	2	1541, 843	13, 1	-0.23431, 8.21385
23609	2	61, 9	463, 17	0.13706, 10.32291
23633	2	23, 3	331, 7	-0.28864, 15.45131
23761	0		$L(1) = 2.38936, S = 16$	
23801	0		$L(1) = 2.38835, S = 16$	
23833	0		$L(1) = 2.38755, S = 16$	
23857	0		$L(1) = 9.54780, S = 64$	
23873	2	13, 3	151, 7	0.32269, 18.20198
23977	2	24985, 4481	16875, 17	-0.40639, 23.02100
23993	0		$L(1) = 2.38356, S = 16$	
24121	0		$L(1) = 2.38039, S = 16$	
24281	2	3, 1	179807, 14167	-0.35936, 47.15954
24329	2	403, 51	13, 1	0.09566, 9.98439
24473	0		$L(1) = 2.37179, S = 16$	
24841	2	19, 3	28869, 679	-0.03034, 40.45328
24953	2	507, 113	47, 1	0.46839, 8.52895
24977	0		$L(1) = 2.35973, S = 16$	
25033	2	19, 3	13, 1	-0.48260, 6.95425
25057	0		$L(1) = 9.43137, S = 64$	
25121	0		$L(1) = 2.35634, S = 16$	
25169	2	3, 1	29, 1	-0.03242, 12.92769
25409	2	137, 57	8257, 497	0.01081, 10.72171
25457	0		$L(1) = 2.34853, S = 16$	
25537	2	39585715, 4322659	15, 1	0.04343, 10.53134
25601	0		$L(1) = 9.38087, S = 64$	
25609	0		$L(1) = 2.34503, S = 16$	
25633	2	109, 19	55, 1	0.38615, 7.31549
25673	2	35, 3	13, 1	-0.49681, 6.66151
25793	2	783, 97	433, 31	0.26063, 8.82811
25849	2	137, 11	1729, 1	0.32366, 20.10175
25889	2	351, 97	137, 7	0.15636, 18.64191
25913	0		$L(1) = 2.33813, S = 16$	
25969	2	3517, 321	13, 1	0.06145, 10.10934
26041	2	31, 3	89, 7	0.29685, 16.68772
26177	0		$L(1) = 2.33221, S = 16$	
26209	2	253, 27	159, 1	-0.30372, 8.39750
26249	2	723, 67	13, 1	0.03908, 10.19619
26297	0		$L(1) = 2.32954, S = 16$	
26417	2	395, 33	115925, 3521	0.13790, 28.16698
26497	0		$L(1) = 9.30054, S = 64$	
26513	2	13, 33	13, 1	-0.26817, 8.55136
26561	2	9, 1	2273102833, 93117313	-0.05288, 85.68506
26633	2	1441, 177	20195, 673	-0.02696, 10.21174
26777	2	186795, 22777	695, 49	0.35728, 19.69225
26849	0		$L(1) = 2.31748, S = 16$	
26881	2	653, 51	276407, 2281	0.01623, 24.34405
26953	0		$L(1) = 2.31524, S = 16$	
26993	2	3, 1	13, 1	0.47246, 8.11076
27073	2	7, 9	17, 1	0.11868, 10.81894
27281	0		$L(1) = 2.30825, S = 16$	
27329	2	37, 3	2831, 161	-0.39104, 29.28721
27409	2	53, 9	13, 1	-0.48556, 5.88873
27449	2	5529, 1163	137, 7	-0.47669, 10.72351
27457	0		$L(1) = 2.30454, S = 16$	
27481	0		$L(1) = 2.30404, S = 16$	
27673	2	25, 3	25, 1	-0.10133, 12.25848
27809	2	173, 51	11383, 193	-0.10447, 16.39620
27953	2	1, 3	67, 1	0.28917, 9.11557
27961	0		$L(1) = 2.29408, S = 16$	
28001	0		$L(1) = 2.29326, S = 16$	
28201	0		$L(1) = 2.28919, S = 16$	
28297	2	24775, 3177	35, 1	0.19616, 12.02776
28393	2	7, 1	43, 1	-0.16495, 14.16561
28409	2	2319, 419	17, 1	-0.25566, 8.87201
28433	2	75, 17	13, 1	0.03658, 10.23047
28513	0		$L(1) = 2.28290, S = 16$	
28537	0		$L(1) = 20.54176, S = 144$	
28657	0		$L(1) = 2.28003, S = 16$	
28697	0		$L(1) = 9.11692, S = 64$	
28729	0		$L(1) = 2.27859, S = 16$	
28753	2	11, 1	1632077, 125041	0.14893, 49.90214
28817	2	23245, 7401	2605, 199	-0.45698, 18.10836
29017	0		$L(1) = 2.27292, S = 16$	
29033	2	9, 1	2531, 151	-0.25165, 25.53708
29137	0		$L(1) = 2.27058, S = 16$	
29153	2	353, 561	361, 23	0.32028, 7.24964
29201	2	1979, 153	101, 7	0.38785, 13.78767
29209	2	13, 1	23, 1	-0.42723, 8.74317
29569	2	10117, 867	407, 31	-0.40305, 10.76349
29761	0		$L(1) = 9.03432, S = 64$	
29833	2	139, 11	29, 1	-0.44172, 6.76320
30089	0		$L(1) = 2.25240, S = 16$	
30137	0		$L(1) = 2.25150, S = 16$	
30161	2	23, 3	4363, 7	-0.43185, 30.16541
30241	0		$L(1) = 2.24956, S = 16$	
30529	0		$L(1) = 2.24124, S = 16$	
30689	2	1849, 201	353, 23	-0.29273, 7.69648
30697	0		$L(1) = 2.24116, S = 16$	
30817	0		$L(1) = 2.23898, S = 16$	

Prime	*Rank*	I *point, rank 2 only*	II *point, rank 2 only*	r
30841	2	11, 1	44991, 1169	-0.48195, 39.54193
30881	2	37, 3	7319903, 410599	-0.05706, 62.90986
30937	0	$L(1) = 20.13122, S = \mathbf{144}$		
30977	0	$L(1) = 2.23608, S = 16$		
31033	2	169, 43	17, 1	-0.09820, 10.86392
31153	2	13, 1	8349, 311	0.33446, 34.47776
31249	0	$L(1) = 2.23120, S = 16$		
31321	0	$L(1) = 2.22992, S = 16$		
31337	0	$L(1) = 8.91852, S = \mathbf{64}$		
31393	2	55187, 20167	22791, 17	0.20601, 8.29814
31513	2	23, 3	41, 1	0.38266, 8.79248
31601	0	$L(1) = 2.22496, S = 16$		
31649	2	9, 1	97, 7	-0.11408, 10.38648
31721	0	$L(1) = 2.22285, S = 16$		
31769	0	$L(1) = 2.22201, S = 16$		
31817	0	$L(1) = 8.88469, S = \mathbf{64}$		
32009	2	207, 17	131, 1	-0.31689, 8.73299
32089	2	13, 1	2230007, 12113	-0.12170, 44.76521
32233	0	$L(1) = 2.21397, S = 16$		
32297	0	$L(1) = 8.85149, S = \mathbf{64}$		
32321	2	139, 33	17, 1	-0.29891, 8.93427
32353	2	275, 43	111, 7	-0.33680, 7.70973
32377	2	5, 1	31668735, 771169	-0.20996, 68.39849
32497	0	$L(1) = 2.20946, S = 16$		
32713	2	35, 3	789, 41	0.35358, 24.54167
32969	0	$L(1) = 2.20151, S = 16$		
32993	0	$L(1) = 2.20111, S = 16$		
33073	2	11, 1	61, 1	-0.25180, 10.07764
33377	0	$L(1) = 8.77901, S = \mathbf{64}$		
33521	2	1367, 813	19, 1	-0.16467, 10.48562
33569	0	$L(1) = 2.19161, S = 16$		
33577	2	5, 3	1165, 7	-0.12177, 16.65132
33617	0	$L(1) = 2.19083, S = 16$		
33641	0	$L(1) = 2.19013, S = 16$		
33713	0	$L(1) = 2.18926, S = 16$		
33721	0	$L(1) = 2.18913, S = 16$		
33857	2	1118205, 85067	1015, 17	-0.18195, 25.13498
33937	0	$L(1) = 8.74256, S = \mathbf{64}$		
33961	2	203, 19	3109, 1	-0.41165, 27.18649
34129	2	5651, 1009	69, 1	-0.25360, 13.86287
34217	2	905, 129	2755, 199	0.36836, 27.67511
34297	0	$L(1) = 8.71953, S = \mathbf{64}$		
34313	0	$L(1) = 2.17963, S = 16$		
34361	0	$L(1) = 2.17887, S = 16$		
34457	2	52615, 7011	25, 1	-0.11002, 11.87852
34513	2	67, 9	101, 1	-0.30879, 8.80564
34721	0	$L(1) = 2.17320, S = 16$		

Prime	*Rank*	I *point, rank 2 only*	II *point, rank 2 only*	r
34841	2	1483, 153	23, 1	-0.09772, 11.84717
34849	2	1, 1	17, 1	0.18280, 10.79947
34897	0	$L(1) = 2.17045, S = 16$		
34961	2	33022417, 2733291	787, 1	0.37005, 16.44618
35081	2	63, 97	23251, 1567	-0.40855, 35.44704
35089	0	$L(1) = 2.16748, S = 16$		
35153	2	41, 3	35, 1	-0.40968, 8.48262
35201	0	$L(1) = 2.16575, S = 16$		
35401	0	$L(1) = 2.16269, S = 16$		
35449	2	11, 1	14471, 1049	0.11437, 30.26111
35537	0	$L(1) = 2.16061, S = 16$		
35801	0	$L(1) = 2.15662, S = 16$		
35809	2	67997, 7899	146151, 103	-0.10785, 10.25842
35897	0	$L(1) = 8.62070, S = 64$		
35977	0	$L(1) = 2.15398, S = 16$		
35993	2	3, 1	71, 1	0.20759, 10.21247
36017	2	25, 3	9443915, 588769	-0.11598, 63.89831
36097	0	$L(1) = 2.15218, S = 16$		
36137	0	$L(1) = 2.15159, S = 16$		
36209	2	633, 83	667, 1	-0.35311, 13.71344
36457	0	$L(1) = 2.14685, S = 16$		
36529	0	$L(1) = 2.14579, S = 16$		
36721	0	$L(1) = 2.14298, S = 16$		
36793	2	151, 11	33, 1	-0.39821, 10.25788
36833	2	20659, 1707	3740713, 22019	0.10821, 10.54342
36857	0	$L(1) = 8.56401, S = \mathbf{64}$		
36913	0	$L(1) = 2.14019, S = 16$		
37049	0	$L(1) = 2.13822, S = 16$		
37057	0	$L(1) = 2.13811, S = 16$		
37201	0	$L(1) = 2.13604, S = 16$		
37217	0	$L(1) = 2.13581, S = 16$		
37273	2	13, 1	271, 7	-0.36375, 20.44427
37313	2	9, 1	17, 1	-0.47170, 8.30176
37321	0	$L(1) = 2.13432, S = 16$		
37337	2	2985, 307	385, 23	-0.39972, 11.69685
37361	0	$L(1) = 2.13375, S = 16$		
37409	2	117, 19	47, 1	0.41406, 7.27648
37489	2	343, 43	659, 41	-0.41508, 21.48613
37537	0	$L(1) = 2.13124, S = 16$		
37633	2	19, 11	23, 1	0.37075, 9.47321
37897	0	$L(1) = 2.12616, S = 16$		
37993	2	113, 9	907, 49	0.08530, 10.89825
38201	0	$L(1) = 2.12192, S = 16$		
38273	2	283, 51	127, 7	-0.33588, 7.79180
38281	0	$L(1) = 2.12081, S = 16$		
38321	0	$L(1) = 2.12026, S = 16$		
38377	0	$L(1) = 2.11948, S = 16$		

Prime	Rank	1 point, rank 2 only	11 point, rank 2 only	τ
38713	2	5, 1	5287, 199	0.06144, 33.79340
38729	2	2217, 473	19, 1	0.22214, 9.79816
38833	0	$L(1) = 2.11323, S = 16$		
38873	2	553, 99	545, 23	-0.25008, 8.82896
39113	2	2523, 187	29, 1	-0.31436, 9.86615
39161	0	$L(1) = 2.10879, S = 16$		
39233	2	5137, 873	241, 7	0.40629, 15.26249
39409	2	41, 3	27, 1	-0.43840, 8.00433
39521	2	1317, 107	167, 1	0.16839, 18.95900
39761	0	$L(1) = 2.10079, S = 16$		
39769	0	$L(1) = 2.10069, S = 16$		
39841	2	491983, 35729	741817, 29903	0.15632, 16.27168
39857	0	$L(1) = 2.09952, S = 16$		
40009	2	107, 11	981, 41	0.10096, 11.08743
40153	2	17323, 3001	7519, 487	-0.20627, 9.10236
40169	2	33, 1241	385427, 27193	-0.22810, 11.27287
40177	2	2373055, 658151	175108315, 10002433	0.04344, 17.56762
40241	0	$L(1) = 2.09450, S = 16$		
40289	2	57897, 7577	44833, 2231	0.15953, 10.12723
40361	2	9, 1	163, 7	0.07573, 11.23762
40433	2	671, 51	19, 1	0.24103, 9.98820
40529	2	2757, 233	46093, 1559	0.01678, 11.29028
40577	2	245265, 17369	625, 41	-0.14625, 24.61463
40697	0	$L(1) = 2.08861, S = 16$		
40841	2	379, 27	77, 1	0.38633, 7.75475
40961	2	119, 9	41, 1	0.42346, 7.84028
40993	2	45301, 3531	2487, 161	-0.39309, 15.28090
41081	0	$L(1) = 2.08371, S = 16$		
41113	0	$L(1) = 2.08330, S = 16$		
41161	0	$L(1) = 2.08270, S = 16$		
41177	2	222843065, 18552147	25, 1	-0.15881, 10.49694
41201	0	$L(1) = 2.08219, S = 16$		
41233	2	17, 3	68195, 4783	-0.42352, 39.71278
41257	0	$L(1) = 2.08148, S = 16$		
41521	2	395841085619, 279168288857	1323181662709, 1363308377	-0.03370, 11.19566
41681	2	12149, 1881	127021, 7849	0.05852, 11.18193
41729	2	2317, 171	31, 1	-0.05899, 13.11515
41761	0	$L(1) = 2.07517, S = 16$		
41777	0	$L(1) = 2.07497, S = 16$		
41849	2	10971, 1121	6199, 217	0.19896, 9.80246
41969	2	9, 11	1067, 17	0.49786, 22.31552
42089	2	169, 33	2051, 23	-0.10187, 10.43309
42193	2	41, 3	6723, 391	0.14535, 24.12435
42209	2	13, 3	127, 1	-0.25519, 10.34928
42281	0	$L(1) = 2.06876, S = 16$		
42433	2	37, 3	15, 1	-0.49479, 7.13768
42473	2	9, 1	412123, 28513	-0.22993, 51.16545

Prime	Rank	1 point, rank 2 only	11 point, rank 2 only	τ
42737	0	$L(1) = 2.06322, S = 16$		
42793	2	43, 3	1581, 7	0.27622, 27.62514
42937	2	51132585565, 5841408009	15, 1	-0.03448, 10.66944
42953	2	19, 3	101, 7	-0.24812, 10.24128
42961	0	$L(1) = 2.06053, S = 16$		
43049	2	37, 3	637, 23	-0.01192, 10.80160
43313	2	7, 3	15163, 553	-0.18923, 28.57737
43441	2	11, 1	2641997, 23311	-0.46362, 56.22783
43481	2	41, 3	1121, 7	-0.03445, 11.33303
43577	0	$L(1) = 2.05321, S = 16$		
43609	2	16201, 1203	9433, 479	0.17697, 9.92746
43649	0	$L(1) = 2.05236, S = 16$		
43721	2	1493, 1371	229829, 15113	0.01181, 11.34948
43753	2	65, 9	675, 7	-0.28534, 15.68625
43777	2	215, 17	26940824 25, 97621649	-0.09056, 68.29231
43793	2	17, 3	19, 1	0.06048, 11.47529
43889	2	3, 1	229, 7	0.04108, 21.19214
43969	2	47, 9	257, 17	0.21451, 10.48933
44089	0	$L(1) = 2.04722, S = 16$		
44129	0	$L(1) = 2.04676, S = 16$		
44201	0	$L(1) = 2.04592, S = 16$		
44257	0	$L(1) = 2.04527, S = 16$		
44273	2	2403, 257	253, 17	0.33178, 7.57919
44417	2	35, 3	48790375, 3151591	-0.36320, 68.78099
44449	0	$L(1) = 2.04306, S = 16$		
44497	0	$L(1) = 8.17005, S = 64$		
44633	2	349, 33	1447, 47	-0.33816, 25.81336
44777	0	$L(1) = 2.03931, S = 16$		
45137	0	$L(1) = 2.03523, S = 16$		
45161	2	17, 9	214483, 5737	-0.47513, 43.35549
45281	2	420677, 57363	1463, 47	0.46608, 16.90098
45289	0	$L(1) = 2.03352, S = 16$		
45329	2	95269, 6849	8021, 23	0.24036, 8.72405
45433	2	31, 3	17, 1	-0.48567, 7.26291
45481	2	61531, 4227	5101, 167	0.39872, 17.69843
45697	0	$L(1) = 2.02897, S = 16$		
45737	0	$L(1) = 2.02852, S = 16$		
46049	2	31, 9	121, 7	-0.46022, 13.51927
46073	2	1, 3	121, 7	-0.37545, 16.98569
46273	0	$L(1) = 2.02263, S = 16$		
46441	2	1951, 5209	55387, 1583	0.25193, 18.36386
46601	2	4161, 1489	467, 31	-0.41350, 11.03602
46681	2	197, 57	28551, 433	-0.39669, 30.82404
46769	0	$L(1) = 2.01724, S = 16$		
46817	2	113585895, 24183233	570608905, 33204497	0.09971, 17.10727
46889	2	9, 1	211, 7	0.04132, 11.38888
46993	2	13, 1	605, 41	-0.33797, 20.95842

Prime	*Rank*	I *point, rank* 2 *only*	II *point, rank* 2 *only*	r
47017	0	$L(1) = 2.01457, S = 16$		
47057	2	3322715, 408657	15685, 479	0.42748, 17.79909
47129	2	113, 27	257, 7	-0.48694, 15.50315
47161	0	$L(1) = 8.05214, S = \mathbf{64}$		
47297	0	$L(1) = 2.01159, S = 16$		
47497	2	29675, 2011	33685, 599	0.47525, 31.76341
47513	2	3, 1	2479, 23	-0.08679, 30.51014
47569	2	1, 3	107, 7	-0.24995, 10.51318
47609	2	8721, 659	17, 1	0.03515, 11.14778
47657	0	$L(1) = 8.03111, S = \mathbf{64}$		
47713	2	10967, 753	4481, 119	-0.20674, 9.62860
47737	2	82773336995, 9484589161	15, 1	0.04598, 10.60021
47777	0	$L(1) = 8.02606, S = \mathbf{64}$		
47881	0	$L(1) = 2.00542, S = 16$		
48017	2	1005, 97	1405, 17	-0.31105, 14.67154
48281	2	60959289, 5841259	248939657, 6221983	0.05133, 11.23648
48313	2	15373, 4617	15, 1	0.18373, 9.33809
48337	0	$L(1) = 2.00068, S = 16$		
48497	0	$L(1) = 1.99903, S = 16$		
48593	0	$L(1) = 1.99804, S = 16$		
48649	2	271, 97	123, 7	0.36545, 7.63697
48673	2	1, 1	17, 1	-0.17174, 10.90005
48817	2	19089599915, 2486521147	35, 1	-0.11952, 12.29961
48889	2	7, 11	33, 1	-0.27489, 11.97765
48953	0	$L(1) = 1.99435, S = 16$		
49009	0	$L(1) = 1.99378, S = 16$		
49057	2	7315, 1779	15, 1	-0.19793, 9.25227
49121	0	$L(1) = 1.99265, S = 16$		
49193	2	9, 1	43, 1	0.36319, 13.08402
49201	0	$L(1) = 1.99184, S = 16$		
49297	2	5, 1	30004112645, 78324241	0.14259, 86.51850
49369	2	13, 1	21679, 391	-0.36170, 34.96007
49393	2	23, 3	267, 17	0.47895, 18.29497
49409	2	11, 3	343, 23	0.09126, 11.32421
49417	0	$L(1) = 1.98966, S = 16$		
49481	0	$L(1) = 1.98901, S = 16$		
49633	0	$L(1) = 1.98749, S = 16$		
49681	2	59539, 8011	53, 1	0.17170, 13.89372
49801	2	11521, 1929	155771, 5879	0.05731, 11.31807