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INTERIOR MAXIMUM-NORM ESTIMATES FOR FINITE ELEMENT METHODS, PART II

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ABSTRACT. We consider bilinear forms $A(\cdot, \cdot)$ connected with second-order elliptic problems and assume that for u_h in a finite element space S_h, we have $A(u - u_h, \chi) = F(\chi)$ for χ in S_h with local compact support. We give local estimates for $u - u_h$ in L_∞ and W^1_∞ of the type "local best approximation plus weak outside influences plus the local size of F".

1. INTRODUCTION

This is the second in a series of papers on local estimates for finite element methods. Our main aim here is to extend the maximum-norm interior estimates given in the first part, Schatz and Wahlbin [10], and to give some new interior estimates in W^1_∞. As a by-product of our proofs we also obtain an extension of the global W^1_∞ stability results of Rannacher and Scott [9] from two to an arbitrary number of space dimensions. In order to describe the results, we shall first need some notation. Some familiarity with the first part of this paper would be helpful to the reader.

Let then $\mathscr{D}$ be a bounded domain in R^N, $N \geq 1$, and let $S^h = S^h(\mathscr{D}) \subseteq W^1_\infty(\mathscr{D})$, $0 < h < 1/2$, be a one-parameter family of finite element spaces (the "h-method"). We shall use standard terminology for W^m_q, $\overset{\circ}{W}{}^m_q$ and their associated norms and seminorms. For a domain $\Omega \subseteq \mathscr{D}$ we let $S^h(\Omega)$ denote the restrictions of functions in S^h to Ω, and we let $\overset{\circ}{S}{}^h(\Omega)$ denote the set of those functions in $S^h(\mathscr{D})$ with compact support in the interior of Ω. We consider a basic domain Ω_0 and also Ω_d with $\Omega_0 \subset\subset \Omega_d \subseteq \mathscr{D}$, where $d = \operatorname{dist}(\partial\Omega_0, \partial\Omega_d)$. We shall assume that the meshes are locally quasi-uniform of size h; we shall then require

$$(1.1) \qquad d \geq c_0 h \quad \text{for} \quad c_0 > 0 \quad \text{large enough.}$$

Let now u be a function on Ω_d and $u_h \in S^h(\Omega_d)$ be such that

$$(1.2) \qquad A(u - u_h, \chi) = F(\chi) \quad \text{for} \quad \chi \in \overset{\circ}{S}{}^h(\Omega_d).$$

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Here,

$$(1.3)\quad A(v, w) = \int \Big(\sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} + \sum_{i=1}^{N} a_i(x) \frac{\partial v}{\partial x_i} w + a(x) v w \Big) dx,$$

where the coefficients a_{ij}, a_i and a are sufficiently smooth in Ω_d and the a_{ij} satisfy the uniform ellipticity condition

$$(1.4)\qquad \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq c_{\mathrm{ell}}|\xi|^2 \text{ for } \quad \xi \in R^n,$$

with $c_{\mathrm{ell}} > 0$ independent of x in Ω_d. No coercivity condition, local or global, will be assumed for our main results. Further, $F(\varphi)$ is a bounded linear functional on $\overset{\circ}{W}{}^1_1(\Omega_d)$.

In the first part of this paper, [10], we gave local maximum-norm estimates for the error $e = u - u_h$ satisfying (1.2) in the case of $F \equiv 0$. Here we shall extend those results to the case of nontrivial F, and also give estimates for the gradient of the error.

Nonvanishing functionals F as in (1.2) arise in a variety of situations. Typically, they represent a perturbation term for quantities which do not quite satisfy the original Ritz-Galerkin equations: For example, in Nitsche and Schatz [8] and in Cayco, Schatz and Wahlbin [2], they occur naturally in proving superconvergence estimates for difference quotients. Again, in [2], they will be necessary in analyzing the behavior of finite element methods on meshes which are locally isoparametric approximations of smooth mappings of translation invariant grids. In Schatz, Sloan and Wahlbin [12] they arise in investigating superconvergence on meshes which are locally symmetric with respect to a point.

In this paper (§5) we shall give an application to local maximum-norm estimates for gradients when numerical integration is taken into account.

We proceed to state our two main results. The technical assumptions A.0–A.5 referred to below are given in detail in an appendix. We first give a local maximum-norm error estimate. As in [10], $\bar{r} = 0$ or 1 according to whether the optimal order r of approximation in L_p is greater than or equal to two, respectively.

Theorem 1.1. *Given $1 \leq q \leq \infty$ and s a nonnegative integer, there exists a constant C depending only on q, s, N, the constant c_0 in* (1.1), *the ellipticity constant c_{ell} in* (1.4) *on Ω_d, the maximum norm of the coefficients of A and a sufficient number of their derivatives in Ω_d, and also the constants involved in* A.0–A.5 *over Ω_d, such that if $e = u - u_h$ satisfies* (1.2), *then*

$$(1.5)\quad \begin{aligned} \|e\|_{L_\infty(\Omega_0)} \leq{} & C(\ln d/h)^{\bar{r}} \min_{\chi \in S^h} \|u - \chi\|_{L_\infty(\Omega_d)} \\ & + C d^{-s-N/q} \|e\|_{W_q^{-s}(\Omega_d)} + C h (\ln d/h)^{\bar{r}} |||F|||_{-1,\infty,\Omega_d} \\ & + C(\ln d/h) |||F|||_{-2,\infty,\Omega_d}. \end{aligned}$$

Here,

$$(1.6)\qquad |||F|||_{-1,\infty,\Omega_d} = \sup_{\substack{\varphi \in \overset{\circ}{W}{}^1_1(\Omega_d) \\ |\varphi|_{W_1^1(\Omega_d)} = 1}} F(\varphi),$$

and

$$(1.7)\qquad |||F|||_{-2,\infty,\Omega_d} = \sup_{\substack{\varphi\in \overset{\circ}{W}{}^2_1(\Omega_d)\\ |\varphi|_{W^2_1(\Omega_d)}=1}} F(\varphi).$$

Our second main result is a corresponding estimate for gradients.

Theorem 1.2. *Under the conditions of Theorem* 1.1,

$$(1.8)\qquad \begin{aligned} &|e|_{W^1_\infty(\Omega_0)} + d^{-1}\|e\|_{L_\infty(\Omega_0)} \\ &\quad \le C\min_{\chi\in S^h}\left(|u-\chi|_{W^1_\infty(\Omega_d)} + d^{-1}\|u-\chi\|_{L_\infty(\Omega_d)}\right) \\ &\qquad + Cd^{-1-s-N/q}\|e\|_{W_q^{-s}(\Omega_d)} + C\ln(d/h)|||F|||_{-1,\infty,\Omega_d}. \end{aligned}$$

Remark 1.1. In some applications, $F(\varphi)$ (for $\varphi \in \overset{\circ}{W}{}^1_1(\Omega_d)$) is "naturally" given from $F(\chi)$ (for $\chi \in \overset{\circ}{S}{}^h(\Omega_d)$), see, e.g., [2], [12]. In other applications, e.g., our present one to numerical quadrature in §5, useful estimation of F involves steps which require that $\chi \in S^h$. In (1.2) the functional is seen only by how it acts on $\overset{\circ}{S}{}^h(\Omega_d)$. Since $\overset{\circ}{S}{}^h(\Omega_d) \subseteq \overset{\circ}{W}{}^1_1(\Omega_d)$, we may then assume by the Hahn-Banach extension theorem that we actually have

$$|||F|||_{-1,\infty,\Omega_d} = \sup_{\substack{\chi\in \overset{\circ}{S}{}^h(\Omega_d)\\ |\chi|_{W^1_1(\Omega_d)}=1}} F(\chi)$$

in Theorem 1.1. □

We shall next discuss the relationship of these results with earlier work. In the analogous cases of local error estimates in L_2 and W_2^1, respectively, they were given in [8]. As already remarked, the case of Theorem 1.1 for $F \equiv 0$ is contained in [10]. To the best of our knowledge, a complete proof of Theorem 1.2, even in the case $F \equiv 0$, has not been published. (In Chen [3] the author assumes that the global two-dimensional results of [9] generalize to arbitrary space dimension N. He also makes an intuitively reasonable claim concerning suitable mesh perturbations [3, p.3, following Eq.(3.1)] which, however, appears hard to substantiate in a rigorous manner.) Our proofs are based on the techniques of [10] and the idea of a regularized Green's function from [9]. Without using a regularized Green's function, a straightforward application of the techniques of [10] would introduce an unnecessary factor $(\ln d/h)^{\bar{r}}$ in Theorem 1.2, cf. Remark 3.1 below. For the necessity of the factor $(\ln d/h)^{\bar{r}}$ in Theorem 1.1 we refer the reader to Haverkamp [6].

As a by-product of our proof we also obtain an extension of the global W^1_∞-stability results of [9] in two space dimensions to arbitrary space dimension. This is briefly discussed in §4.

An outline of the paper is as follows. In §2 we collect various preliminary results, which will be used in the proofs of Theorems 1.1 and 1.2, given in §3. As already mentioned, in §4 we will show how global results follow from our techniques and, in §5, we give an application to numerical quadrature. We conclude with an appendix in which we state our basic assumptions A.0–A.5.

2. Preliminaries

For the convenience of the reader we shall first collect some technical results which will be used in our proofs of Theorems 1.1 and 1.2. As we will see at the beginning of §3, we shall only need these technical lemmas in the case that the domains are balls of unit size. Let thus B_r denote a ball of radius r around a point x_0.

Our first result pertains to the following conormal Neumann problem in $B_3 \subset\subset \mathscr{D}$:

$$Lv = f \text{ in } B_3, \quad \frac{\partial v}{\partial n_L} = 0 \quad \text{on} \quad \partial B_3. \tag{2.1}$$

Here, L is the second-order differential operator naturally associated with the form A in (1.3), which is in this context assumed to be coercive over $W_2^1(B_3)$, i.e.,

$$A(v, v) \geq c_{\text{co}} \|v\|^2_{W_2^1(B_3)} \text{ with } \quad c_{\text{co}} > 0. \tag{2.2}$$

From Krasovskiĭ [7], e.g., we have the following:

Lemma 2.1. *Let the form A be coercive over $W_2^1(B_3)$. There exists a constant C such that for $G(x, y)$, the Green's function for the problem* (2.1),

$$|D_x^\alpha D_y^\beta G(x, y)| \leq C|x - y|^{-N+2-|\alpha+\beta|} \quad \textit{for} \quad |\alpha + \beta| > 0, \; x, y \in B_3.$$

The constant C depends on α, β, c_{ell}, various norms of the coefficients and their derivatives, and on the coercivity constant c_{co}.

We shall also need a priori estimates in L_q-based norms for the problem (2.1), as well as for the problem

$$Lv = D_i f \text{ in } B_3, \quad \frac{\partial v}{\partial n_L} = 0 \quad \text{on} \quad \partial B_3, \quad i = 1, \ldots, N. \tag{2.3}$$

Lemma 2.2. *Let the form A be coercive over $W_2^1(B_3)$. There exists a constant C independent of q, $1 < q \leq 2$, such that for v satisfying* (2.1),

$$\|v\|_{W_q^2(B_3)} \leq \frac{C}{q-1} \|f\|_{L_q(B_3)}.$$

Similarly, for v in (2.3),

$$\|v\|_{W_q^1(B_3)} \leq \frac{C}{q-1} \|f\|_{L_q(B_3)}.$$

Essentially, these results can be found, e.g., in Schechter [13]. In the lemma above we require a rather exact dependence on q. For this, one needs to trace the constants through a proof (and we have found Gilbarg and Trudinger [5, Chapter 9] a convenient place for doing so, with appropriate modifications in that they treat a Dirichlet problem rather than a Neumann problem). Let us remark that Lemmas 2.1 and 2.2 also hold for the adjoint operator L^* and, in §3, will be applied to this case without explicit mention.

Our next lemma has to do with cutting down functions in S^h to have compact support. It is an easy, indeed trivial, consequence of our superapproximation hypothesis A.3, cf. [10, Prop. 2.2].

Lemma 2.3. *Let $D_1 \subset\subset D_2 \subset\subset D_3$. There exists a constant C such that given $\chi \in S^h(D_3)$, there exists $\eta \in \overset{\circ}{S}{}^h(D_3)$ with $\eta \equiv \chi$ on D_2 such that*

$$\|\chi - \eta\|_{W_2^1(D_3\setminus D_2)} \le C\|\chi\|_{W_2^1(D_3\setminus D_1)}$$

and

$$\|\eta\|_{L_q(D_3)} \le C\|\chi\|_{L_q(D_3)} \quad \textit{for} \quad 1 \le q \le \infty.$$

We now state a well-known error estimate for Galerkin approximations in the problem (2.1).

Lemma 2.4. *Let the form A be coercive over $W_2^1(B_3)$. There exists a constant C such that if v satisfies* (2.1) *and $v_h \in S^h(B_3)$ satisfies*

$$A(v - v_h, \chi) = 0 \quad \textit{for} \quad \chi \in S^h(B_3), \tag{2.4}$$

then

$$\|v - v_h\|_{L_2(B_3)} + h\|v - v_h\|_{W_2^1(B_3)} \le Ch^2\|f\|_{L_2(B_3)}.$$

The standard proof uses extensions of functions beyond B_3 and relies on the fact that the mesh on B_3 is actually formed by intersecting B_3 with a "quasi-uniform" mesh having the appropriate approximation properties for extended functions.

Our final technical preliminaries are concerned with local estimates in W_2^1.

Lemma 2.5. *Let the form A be coercive over $W_2^1(B_3)$. Let $x_0 \in B_3$, $d_j = 2^{-j}$ and $\Omega_j = \{x \in B_3 : d_j \le |x - x_0| \le d_{j-1}\}$, for $d_j \ge c_0 h$, $c_0 > 0$ large enough* (*cf.* (1.1)). *Further let*

$$\Omega_j^1 = \begin{cases} \Omega_{j+1} \cup \Omega_j \cup \Omega_{j-1} & \textit{if } \Omega_{j+1} \subset\subset B_3, \\ \Omega_{j+1} \cup \Omega_j \cup \Omega_{j-1} \cup \{x \in B_3 : \operatorname{dist}(x, \partial B_3) \le d_j\} & \\ \qquad\qquad \textit{if } \Omega_{j+1} \cup \Omega_j \cup \Omega_{j-1} \textit{ meets } \partial B_3. \end{cases} \tag{2.5}$$

There exists a constant C independent of j such that for $v - v_h$ satisfying (2.4),

$$\|v - v_h\|_{W_2^1(\Omega_j)} \le C\Big(h^{r-1}\|v\|_{W_2^r(\Omega_j^1)} + d_j^{-N/2-1}\|v - v_h\|_{L_1(\Omega_j^1)}\Big).$$

If $\Omega_j \subset\subset B_3$, this result is contained in [8]; it was extended up to boundaries in [10, Lemma 4.4].

Remark 2.1. The reason for the change in Ω_j^1 in (2.5) to include a "collar" around the boundary if we are close to ∂B_3 is as follows. The proof involves use of cutoff functions ω and superapproximation, Assumption A.3. The proof of superapproximation typically involves inverse properties. If we are at the boundary, and if the cutoff function ω is not identically constant near the boundary, superapproximation may fail since, for elements τ meeting the boundary, $\tau \cap B_3$ may not satisfy inverse properties. □

Our last result in this section is a special case of [8, (5.6)]. Here we do not assume that the form A is coercive over $W_2^1(B_3)$. Instead it is merely assumed to be coercive on $\overset{\circ}{W}{}_2^1(\Omega)$ for Ω sufficiently small (cf. [8, R1, p. 940]). Such is obviously the case in our present situation.

Lemma 2.6. *Let $w_h \in S^h(B_3)$ satisfy $A(w_h, \chi) = F(\chi)$ for $\chi \in \overset{\circ}{S}{}^h(B_3)$, with $F(\cdot)$ a linear functional on $\overset{\circ}{W}{}^1_2(B_3)$. For any $1 \le q \le \infty$ and $s \ge 0$ there exists a constant C independent of w_h and F such that*

$$\|w_h\|_{W_2^1(B_1)} \le C\|w_h\|_{W_q^{-s}(B_3)} + CK_F,$$

where

$$K_F = \sup_{\substack{\varphi \in \overset{\circ}{W}{}^1_2(B_3) \\ |\varphi|_{W_2^1(B_3)}=1}} F(\varphi).$$

We finally remark that we shall also use two other results from Part I, viz., [10, Lemma A.1 and Lemma 5.3]. These will not be stated here but will be referred to at the appropriate places in §3.

3. Proofs of Theorems 1.1 and 1.2

It turns out to be convenient to first prove Theorem 1.2.

3.A. **Proof of Theorem 1.2.** We start with some preliminary reductions. First, since the norms W_q^{-s} and $|||F|||_{-s,\infty,\Omega_d}$ are based on duality with respect to $\overset{\circ}{W}{}^s_{q'}$ spaces, it suffices to prove our results with Ω_0 replaced by a ball B_d and Ω_d by a concentric ball B_{3d} (say), cf. [8, Lemma 1.1]. Secondly, we shall now show that it suffices to consider the case $d = 1$. Let us thus assume that we have proven the following form of the theorem in the case $d = 1$: If $A(e, \chi) = F(\chi)$ for $\chi \in \overset{\circ}{S}{}^h(B_3)$, then

$$\|e\|_{W^1_\infty(B_1)} \le C\|u\|_{W^1_\infty(B_3)} + C\|e\|_{W_q^{-s}(B_3)} + C(\ln 1/h)|||F|||_{-1,\infty,B_3}. \tag{3.1}$$

We claim that then Theorem 1.2 would follow for general $c_0 h \le d \le 1$. For, if we scale the situation from B_{3d} to B_3 by introducing a new variable $y = x/d$, we have with $\tilde{e}(y) = \tilde{u}(y) - \tilde{u}_h(y) = u(yd) - u_h(yd)$ and with

(3.2)
$$\tilde{A}(v, w) \equiv \int_{B_3} \Big(\sum_{i,j=1}^{N} a_{ij}(yd)\frac{\partial v}{\partial y_i}\frac{\partial w}{\partial y_j} + d\sum_{i=1}^{N} a_i(yd) v \frac{\partial w}{\partial y_i} + d^2 a(yd) v w\Big) dy$$

that $A(e, \chi) = d^{N-2}\tilde{A}(\tilde{e}, \tilde{\chi})$. Hence (1.2) becomes

$$\tilde{A}(\tilde{e}, \tilde{\chi}) = d^{2-N}F(\chi) \equiv \tilde{F}(\tilde{\chi}) \quad \text{for} \quad \tilde{\chi} \in \overset{\circ}{S}{}^{h/d}(B_3). \tag{3.3}$$

The parameter h is now replaced by h/d. In this we appeal to our scaling hypotheses A.4. (Note that $h/d \le 1/c_0$ is assumed sufficiently small. This means that the difference domain $B_3 \setminus B_1$ contains sufficiently many (scaled) elements to allow operations such as "cutting down to local support", "local approximation", ...) Note also that $\tilde{A}$ has the same modulus of ellipticity as A and that the norms of coefficients and their derivatives have certainly not increased. Thus, from (3.1) we have

$$\|\tilde{e}\|_{W^1_\infty(B_1)} \le C\|\tilde{u}\|_{W^1_\infty(B_3)} + C\|\tilde{e}\|_{W_q^{-s}(B_3)} + C(\ln d/h)|||\tilde{F}|||_{-1,\infty,B_3}. \tag{3.4}$$

It is elementary to check that

$$|e|_{W^1_\infty(B_d)} + d^{-1}\|e\|_{L_\infty(B_d)} = d^{-1}\|\tilde{e}\|_{W^1_\infty(B_1)}, \tag{3.5}$$

$$\|\tilde{u}\|_{W^1_\infty(B_3)} = d|u|_{W^1_\infty(B_{3d})} + \|u\|_{L_\infty(B_{3d})} \tag{3.6}$$

and

$$\|\tilde{e}\|_{W_q^{-s}(B_3)} = d^{-s-N/q}\|e\|_{W_q^{-s}(B_{3d})}. \tag{3.7}$$

Finally,

$$|||\widetilde{F}|||_{-1,\infty,B_3} = \sup_{\substack{\tilde{\varphi}\in \overset{\circ}{W}{}^1_1(B_3)\\ |\tilde{\varphi}|_{W^1_1}=1}} \widetilde{F}(\tilde{\varphi}),$$

where

$$\begin{aligned}\widetilde{F}(\tilde{\varphi}) &= d^{2-N}F(\varphi)\\ &\le d^{2-N}|||F|||_{-1,\infty,B_{3d}}|\varphi|_{W_1^1(B_{3d})}\\ &= d^{2-N}|||F|||_{-1,\infty,B_{3d}}d^{N-1}|\tilde{\varphi}|_{W_1^1(B_3)}.\end{aligned}$$

Hence, $|||\widetilde{F}|||_{-1,\infty,B_3} \le d|||F|||_{-1,\infty,B_{3d}}$. Using this and (3.5)–(3.7) in (3.4), we obtain

$$\begin{aligned}&|e|_{W^1_\infty(B_d)} + d^{-1}\|e\|_{L_\infty(B_d)}\\ &\quad\le C\left(|u|_{W^1_\infty(B_{3d})} + d^{-1}\|u\|_{L_\infty(B_{3d})}\right)\\ &\qquad + Cd^{-1-s-N/q}\|e\|_{W_q^{-s}(B_{3d})} + C(\ln d/h)|||F|||_{-1,\infty,B_{3d}}.\end{aligned} \tag{3.8}$$

Writing here $e = u - u_h = (u-\chi) - (u_h - \chi)$ for a general $\chi \in S^h$, we obtain (1.8) of Theorem 1.2.

In the rest of §3.A we shall thus give a proof of the inequality (3.1). This will be accomplished through a sequence of lemmas.

In our first lemma we take $\tilde{\delta}$, as in A.5, to be a $\mathscr{C}^1$ function supported in an element $\tau_h^0 \subseteq B_2$ and satisfying

$$\|\tilde{\delta}\|_{L_q} + h\|\nabla\tilde{\delta}\|_{L_q} \le Ch^{-N(1-\frac{1}{q})} \quad \text{for} \quad 1 \le q \le \infty. \tag{3.9}$$

We assume that the form A is coercive over $W_2^1(B_3)$ and let $v \in W_2^1(B_3)$ be defined by

$$L^*v = D_i\tilde{\delta}, \quad \frac{\partial v}{\partial n_L^*} = 0 \quad \text{on} \quad \partial B_3 \quad (i = 1, \dots, N). \tag{3.10}$$

Further, we let $v_h \in S^h(B_3)$ be given by

$$A(\chi, v - v_h) = 0 \quad \text{for} \quad \chi \in S^h(B_3). \tag{3.11}$$

Lemma 3.1. *With the form A coercive over $W_2^1(B_3)$, with $\tau_h^0 \in B_2$, and with further notation as above, we have*

$$\|v - v_h\|_{W_1^1(B_3)} \le C, \tag{3.12}$$

where C depends on the quantities specified in Theorem 1.1 *and also on the coercivity constant of A (cf.* (2.2)).

Proof. Let $x_0 \in \tau_h^0$ and set $d_j = 2^{-j}$. Introduce

$$\Omega_j = \{x : d_j \le |x - x_0| \le d_{j-1}\} \cap B_3.$$

With C_* to be chosen (sufficiently large) and $J_* = \left[-\frac{\ln(C_*h)}{\ln 2}\right]$ so that $2^{-J_*} \simeq C_*h$, we let

$$\Omega_h^* = B_3 \setminus \Big(\bigcup_{j=-3}^{J_*} \Omega_j \Big).$$

Further, we set $\Omega_j^\ell = \Omega_{j-\ell} \cup \cdots \cup \Omega_{j+\ell}$ for $\ell = 1, 2, 3$, with the modification as in (2.5) to include an annulus at ∂B_3 if $\Omega_{j+\ell}$ meets ∂B_3. Note that this modification occurs only if $d_j = 0(1)$, since $\tau_h^0 \in B_2$.

With $e = v - v_h$ we then have

$$\|e\|_{W_1^1(B_3)} = \|e\|_{W_1^1(\Omega_h^*)} + \Big(\sum_{j=-3}^{J_*} \|e\|_{W_1^1(\Omega_j)} \Big). \tag{3.13}$$

Using Hölder's inequality, the standard error estimates for the Neumann problem of Lemma 2.4, and (3.9), we obtain

$$\begin{aligned} \|e\|_{W_1^1(\Omega_h^*)} &\le (C_*h)^{N/2} \|e\|_{W_2^1(B_3)} \\ &\le C C_*^{N/2} h^{N/2+1} \|D_i \widehat{\delta}\|_{L_2(\tau_h^0)} \le C C_*^{N/2}. \end{aligned} \tag{3.14}$$

Further, again using Hölder's inequality and then the local error estimates of Lemma 2.5, we get

$$\begin{aligned} \|e\|_{W_1^1(\Omega_j)} &\le d_j^{N/2} \|e\|_{W_2^1(\Omega_j)} \\ &\le C d_j^{N/2} \Big[h^{r-1} \|v\|_{W_2^r(\Omega_j^1)} + d_j^{-N/2-1} \|e\|_{L_1(\Omega_j^1)} \Big]. \end{aligned} \tag{3.15}$$

With G the Green's function for the problem (3.10) we have

$$D^\alpha v(x) = \int_{\tau_h^0} D_x^\alpha G(x, y) D_i \widetilde{\delta}(y) dy = -\int_{\tau_h^0} (D_{i,y} D_x^\alpha G(x, y)) \widetilde{\delta}(y) dy. \tag{3.16}$$

Since $x \in \Omega_j^1$ satisfies $|x - y| \ge C d_j$ for $y \in \tau_h^0$, the estimates of Lemma 2.1 give, with $|\alpha| \le r$ in (3.16),

$$|D^\alpha v(x)| \le C d_j^{-N+2-(r+1)} \|\widetilde{\delta}\|_{L_1(\tau_h^0)} \le C d_j^{-N+2-(r+1)},$$

where we used (3.9) in the last step. Thus,

$$\|v\|_{W_2^r(\Omega_j^1)} \le C d_j^{-N/2-r+1}. \tag{3.17}$$

Let us pause here to give credit to Rannacher and Scott.

Remark 3.1. If we had followed a straightforward adaption of [10], our final result would have ended up with an unnecessary factor $(\ln 1/h)^{\bar{r}}$. The integration by parts performed in (3.16) (following [9]) is precisely the reason why this logarithmic loss is now avoided. □

Continuing now with the proof of Lemma 3.1, from (3.15) we have

$$\begin{aligned} \sum_{j=-3}^{J_*} \|e\|_{W_1^1(\Omega_j)} &\le C \Big(\sum_{j=-3}^{J_*} d_j^{1-r} \Big) h^{r-1} + C \sum_{j=-3}^{J_*} d_j^{-1} \|e\|_{L_1(\Omega_j^1)} \\ &\le C(C_*) + C \sum_{j=-3}^{J_*} d_j^{-1} \|e\|_{L_1(\Omega_j^1)}. \end{aligned} \tag{3.18}$$

Let us introduce

$$I := \frac{1}{C_* h}\|e\|_{L_1(\Omega_h^*)} + \sum_{j=-3}^{J_*} d_j^{-1}\|e\|_{L_1(\Omega_j)}. \tag{3.19}$$

From (3.13), (3.14) and (3.18) we thus obtain (since modifications of Ω_j^ℓ at the boundary are done only if $d_j = 0(1)$)

$$\|e\|_{W_1^1(B_3)} \le C(C_*) + CI. \tag{3.20}$$

Now, again using the L_2 error estimates for the Neumann problem (Lemma 2.4) and (3.9), we have

$$\begin{aligned}\frac{1}{C_* h}\|e\|_{L_1(\Omega_h^*)} &\le C_*^{N/2-1} h^{N/2-1}\|e\|_{L_2(B_3)} \\ &\le C(C_*)^{N/2-1} h^{N/2+1}\|D_i\widetilde{\delta}\|_{L_2(\tau_h^0)} \\ &\le C(C_*)^{N/2-1}.\end{aligned} \tag{3.21}$$

In the following estimations we take $C_* \ge C_0$, C_0 large enough. It is then easily seen that the constants appearing below can be taken independent of C_*. We shall perform a duality argument to estimate $\|e\|_{L_1(\Omega_j)}$. Thus, with $(e, \eta) = \int e\eta dx$,

$$\|e\|_{L_1(\Omega_j)} = \sup_{\substack{\eta\in\mathscr{C}_0^\infty(\Omega_j) \\ \|\eta\|_{L_\infty(\Omega_j)}=1}} (e, \eta). \tag{3.22}$$

Let now $L^*\psi = \eta$ in B_3 with homogeneous conormal Neumann boundary conditions. Then, for $\chi \in S^h(B_3)$,

$$\begin{aligned}(e, \eta) = A(e, \psi) &= A(e, \psi - \chi) \\ &\le C\|e\|_{W_1^1(B_3\setminus\Omega_j^2)}\|\psi - \chi\|_{W_\infty^1(B_3\setminus\Omega_j^2)} \\ &\quad + C\|e\|_{W_2^1(\Omega_j^2)}\|\psi - \chi\|_{W_2^1(\Omega_j^2)}.\end{aligned} \tag{3.23}$$

By our approximation assumption A.1, and using also the estimate for the Green's function, Lemma 2.1, we have for a suitable χ

$$\begin{aligned}\|\psi - \chi\|_{W_\infty^1(B_3\setminus\Omega_j^2)} &\le Ch\|\psi\|_{W_\infty^2(B_3\setminus\Omega_j^1)} \\ &\le Ch\int_{\Omega_j} d_j^{-N}|\eta(y)|dy \le Ch\|\eta\|_{L_\infty(\Omega_j)} = Ch.\end{aligned} \tag{3.24}$$

Also,

$$\begin{aligned}\|\psi - \chi\|_{W_2^1(\Omega_j^2)} &\le Ch\|\psi\|_{W_2^2(B_3)} \\ &\le Ch\|\eta\|_{L_2(\Omega_j)} \le Chd_j^{N/2}.\end{aligned} \tag{3.25}$$

Thus, from (3.23)–(3.25),

$$|(e, \eta)| \le Ch\|e\|_{W_1^1(B_3\setminus\Omega_j^2)} + Chd_j^{N/2}\|e\|_{W_2^1(\Omega_j^2)}. \tag{3.26}$$

Here, again using the local W_2^1 estimates of Lemma 2.5, essentially as in (3.15), we get

$$\|e\|_{W_2^1(\Omega_j^2)} \le Ch\|v\|_{W_2^2(\Omega_j^3)} + Cd_j^{-N/2-1}\|e\|_{L_1(\Omega_j^3)}, \tag{3.27}$$

where, by the Green's function estimates of Lemma 2.1, $\|v\|_{W_2^2(\Omega_j^3)} \le C d_j^{-N/2-1}$ (for C_* large enough). Hence, by (3.26), (3.27) and (3.22),

$$\begin{aligned}\|e\|_{L_1(\Omega_j)} \le\; & Ch\|e\|_{W_1^1(B_3\setminus\Omega_j^2)} \\ & + Chd_j^{N/2}[hd_j^{-N/2-1} + d_j^{-N/2-1}\|e\|_{L_1(\Omega_j^3)}] \\ \le\; & Ch^2d_j^{-1} + Ch\|e\|_{W_1^1(B_3)} + Chd_j^{-1}\|e\|_{L_1(\Omega_j^3)}.\end{aligned} \tag{3.28}$$

From (3.19), (3.21) and (3.28) we now have

$$\begin{aligned}I \le\; & C(C_*)^{N/2-1} + Ch^2\sum_{j=-3}^{J_*} d_j^{-2} + C\|e\|_{W_1^1(B_3)} h \sum_{j=-3}^{J_*} d_j^{-1} \\ & + Ch\sum_{j=-3}^{J_*} d_j^{-2}\|e\|_{L_1(\Omega_j^3)}.\end{aligned} \tag{3.29}$$

As already remarked, the constants C occurring may be taken independent of C_* provided that constant is large enough. Since now

$$h^\ell \sum_{j=-3}^{J_*} d_j^{-\ell} \le \frac{C}{(C_*)^\ell}, \quad \ell = 1, 2,$$

and

$$h\sum_{j=-3}^{J_*} d_j^{-2}\|e\|_{L_1(\Omega_j^3)} = \sum_{j=-3}^{J_*}\left(\frac{h}{d_j}\right) d_j^{-1}\|e\|_{L_1(\Omega_j^3)} \le \max_{j=-3,\dots,J}\left(\frac{h}{d_j}\right)3I = \frac{3}{C_*}I,$$

we have from (3.29) that

$$I \le C(C_*)^{N/2-1} + \frac{C}{C_*^2} + \frac{C}{C_*}\|e\|_{W_1^1(B_3)} + \frac{C}{C_*}I.$$

Hence, for C_* large enough,

$$I \le C(C_*)^{N/2-1} + \frac{C}{C_*}\|e\|_{W_1^1(B_3)}.$$

Recalling now (3.20), i.e., $\|e\|_{W_1^1(B_3)} \le C(C_*) + CI$, we obtain the lemma by choosing, if needed, C_* even larger. □

In our next result we do not assume that the form A is necessarily coercive over $W_2^1(B_3)$.

Lemma 3.2. *Assume that $w_h \in S^h$ satisfies $A(w_h, \chi) = (f, \chi)$ for $\chi \in \mathring{S}_h(B_3)$. Then*

$$\|w_h\|_{W_\infty^1(B_1)} \le C\|w_h\|_{W_q^{-s}(B_3)} + C\|f\|_{L_\infty(B_3)}.$$

Remark 3.2. This result is not optimal with respect to the norm of f involved, but it suffices for our present purposes. A better result is obtained by taking $u \equiv 0$ in Theorem 1.2. □

Proof of Lemma 3.2. Let k be such that the bilinear form

$$A_k(v, w) = A(v, w) + k(v, w) \tag{3.30}$$

is coercive over $W_2^1(B_3)$. We note that this can be accomplished with k bounded in terms of c_{ell} and quantities involving the coefficients and their derivatives. We shall also let $\widetilde{\delta} = \widetilde{\delta}_{j,x_0}$ be such that, for $x_0 \in B_1$,

$$D_j\chi(x_0) = (D_j\chi, \widetilde{\delta}), \quad \text{all} \quad \chi \in S^h.$$

By Assumption A.5, $\widetilde{\delta}$ can be taken to satisfy the hypotheses of Lemma 3.1. We let v and v_h be as in Lemma 3.1 (see (3.10), (3.11)), but now based on the operator $L_k^* = L^* + kI$, which thus satisfies the coercivity hypothesis of that lemma.

We shall need to cut down w_h to have compact support in B_3. Our purpose is to estimate $D_j w_h(x_0)$, for $x_0 \in B_1$. Using Lemma 2.3 (with $D_1 = \varnothing$, $D_2 = B_{1.4}$ and $D_3 = B_{1.5}$), we have $\eta_h \in \overset{\circ}{S}{}^h(B_{1.5})$ with $\eta_h \equiv w_h$ in $B_{1.4}$ such that

$$\|\eta_h\|_{W_2^1(B_{1.5})} \le C\|w_h\|_{W_2^1(B_{1.5})}, \tag{3.31}$$

and

$$\|\eta_h\|_{L_\infty(B_{1.5})} \le C\|w_h\|_{L_\infty(B_{1.5})}. \tag{3.32}$$

We then obtain, for $x_0 \in B_1$,

$$\begin{aligned} D_j w_h(x_0) &= D_j\eta_h(x_0) = (D_j\eta_h, \widetilde{\delta}) \\ &= -(\eta_h, D_j\widetilde{\delta}) = A_k(\eta_h, v) \\ &= A_k(\eta_h, v_h) = A(\eta_h, v_h) + k(\eta_h, v_h). \end{aligned} \tag{3.33}$$

Let now $\chi_h \in \overset{\circ}{S}{}^h(B_{1.4})$ (recall that $\eta_h = w_h$ on $B_{1.4}$) with $\chi_h \equiv v_h$ in $B_{1.3}$ such that

$$\|v_h - \chi_h\|_{W_2^1(B_{1.5}\setminus B_{1.3})} \le C\|v_h\|_{W_2^1(B_{1.5}\setminus B_{1.2})} \tag{3.34}$$

and

$$\|\chi_h\|_{L_1(B_{1.5})} \le C\|v_h\|_{L_1(B_{1.5})}. \tag{3.35}$$

Again, such a χ_h is found from Lemma 2.3. From (3.33) we then have, using our basic assumption about w_h (i.e., that $A(w_h, \chi) = (f, \chi)$ for $\chi \in \overset{\circ}{S}{}^h(B_3)$), and the fact that $\eta_h = w_h$ on the support of χ_h,

$$D_j w_h(x_0) = A(\eta_h, v_h - \chi_h) + k(\eta_h, v_h) + (f, \chi_h). \tag{3.36}$$

Here, by (3.31) and (3.34),

$$|A(\eta_h, v_h - \chi_h)| \le C\|w_h\|_{W_2^1(B_{1.5})}\|v_h\|_{W_2^1(B_{1.5}\setminus B_{1.2})}. \tag{3.37}$$

By Lemma 2.6 (with a nonessential change of domains) we have, since now trivially $K_F \le C\|f\|_{L_\infty(B_3)}$,

$$\|w_h\|_{W_2^1(B_{1.5})} \le C\|w_h\|_{W_q^{-s}(B_3)} + C\|f\|_{L_\infty(B_3)}. \tag{3.38}$$

Further, since $A(\chi, v_h) = 0$ for $\chi \in \overset{\circ}{S}{}^h(B_2\setminus B_{1.1})$, again from Lemma 2.6 (now with $F \equiv 0$), we obtain

$$\|v_h\|_{W_2^1(B_{1.5}\setminus B_{1.2})} \le C\|v_h\|_{W_1^1(B_3\setminus B_{1.1})}. \tag{3.39}$$

Using Lemma 2.1 and Lemma 3.1, we conclude that

$$\|v_h\|_{W_1^1(B_3\setminus B_{1.1})} \le \|v\|_{W_1^1(B_3\setminus B_{1.1})} + C\|v - v_h\|_{W_1^1(B_3)} \le C. \tag{3.40}$$

Hence, from (3.37)–(3.40),

$$|A(\eta_h, v_h - \chi_h)| \le C\|w_h\|_{W_q^{-s}(B_3)} + C\|f\|_{L_\infty(B_3)}. \tag{3.41}$$

Further, again using Lemmas 2.1 and 3.1, and (3.35),

$$\begin{aligned} |(f, \chi_h)| &\le \|f\|_{L_\infty(B_3)}\|\chi_h\|_{L_1(B_{1.4})} \\ &\le C\|f\|_{L_\infty(B_3)}\|v_h\|_{L_1(B_3)} \\ &\le C\|f\|_{L_\infty(B_3)}(1 + \|v\|_{L_1(B_3)}) \\ &\le C\|f\|_{L_\infty(B_3)}. \end{aligned} \tag{3.42}$$

From (3.36), (3.41) and (3.42) we now have

$$|D_j w_h(x_0)| \le C\|w_h\|_{W_q^{-s}(B_3)} + C\|f\|_{L_\infty(B_3)} + |k(\eta_h, v_h)|. \tag{3.43}$$

It remains to estimate the last term in (3.43), $|k(\eta_h, v_h)|$. We note that if the basic form A had been (uniformly) coercive, we could have taken $k = 0$. Recalling that k may be bounded in terms of c_{ell} and the coefficients of A, we have from (3.32) and Lemma 3.1 (since $\|v\|_{L_1(B_3)} \le C$) that

$$|k(\eta_h, v_h)| \le C\|w_h\|_{L_\infty(B_{1.5})}\|v_h\|_{L_1(B_3)} \le C\|w_h\|_{L_\infty(B_{1.5})}. \tag{3.44}$$

It is, thus, a "lower-order" term. A technique for treating a similar situation was given in Appendix 1 of [10]. We shall not repeat the full arguments here but merely apply a result from that appendix. Let thus $L\psi = f$ in B_3, with now ψ satisfying homogeneous Dirichlet boundary conditions. (Shrinking our domains, if necessary, we may assume that this problem has a solution.) Then

$$A(\psi - w_h, \chi) = 0 \quad \text{for} \quad \chi \in \overset{\circ}{S}{}^h(B_3).$$

Lemma A.1 in [10] now applies exactly to this situation and says that

$$\|\psi - w_h\|_{L_\infty(B_{1.5})} \le C(\|\psi\|_{W_\infty^1(B_3)} + \|\psi - w_h\|_{W_q^{-s}(B_3)}).$$

Hence,

$$\|w_h\|_{L_\infty(B_{1.5})} \le C\|\psi\|_{W_\infty^1(B_3)} + C\|w_h\|_{W_q^{-s}(B_3)}.$$

Since, as is easily seen, $\|\psi\|_{W_\infty^1(B_3)} \le C\|f\|_{L_\infty(B_3)}$, we thus have from (3.44) that

$$|k(\eta_h, w_h)| \le C\|w_h\|_{L_\infty(B_{1.5})} \le C\|f\|_{L_\infty(B_3)} + C\|w_h\|_{W_q^{-s}(B_3)}.$$

Together with (3.43) this completes the proof of the lemma. □

We now come to our final lemma in this subsection.

Lemma 3.3. *Assume that the form* A *is coercive over* $W_2^1(B_3)$ *and let* $\omega \in \mathscr{C}_0^\infty(B_3)$. *Let* w *and* $w_h \in S^h(B_3)$ *satisfy* $A(w - w_h, \chi) = F(\omega\chi)$ *for* $\chi \in S^h(B_3)$. *Then*

$$\|w - w_h\|_{W_\infty^1(B_2)} \le C\|w\|_{W_\infty^1(B_3)} + C(\ln 1/h)|||F|||_{-1,\infty,B_3}.$$

Proof. With $x_0 \in \tau_0^h \subseteq B_2$ and $\widetilde{\delta}$ the delta function for the ith first derivative, see (A.5), we have

$$D_i w_h(x_0) = (D_i w_h, \widetilde{\delta}) = (D_i w, \widetilde{\delta}) + (w - w_h, D_i\widetilde{\delta}). \tag{3.45}$$

With v and v_h as in (3.10), (3.11) we then have

$$\begin{aligned}(w - w_h, D_i\widetilde{\delta}) &= A(w - w_h, v) = A(w - w_h, v - v_h) + F(\omega v_h) \\ &= A(w, v - v_h) + F(\omega v_h) \equiv I_1 + I_2.\end{aligned} \tag{3.46}$$

Here, by Lemma 3.1,

$$|I_1| \le C\|w\|_{W^1_\infty(B_3)}\|v - v_h\|_{W^1_1(B_3)} \le C\|w\|_{W^1_\infty(B_3)}. \tag{3.47}$$

Further, since $\omega v_h \in \overset{\circ}{W}{}^1_1(B_3)$,

$$\begin{aligned}|I_2| &\le |||F|||_{-1,\infty,B_3}\|\omega v_h\|_{W^1_1(B_3)} \\ &\le C|||F|||_{-1,\infty,B_3}(\|v - v_h\|_{W^1_1(B_3)} + \|v\|_{W^1_1(B_3)}) \\ &\le C|||F|||_{-1,\infty,B_3}(1 + \|v\|_{W^1_1(B_3)}),\end{aligned} \tag{3.48}$$

where we used Lemma 3.1 in the last step.

By Lemma 2.2, with $1 < q \le 2$, and (3.9),

$$\|v\|_{W^1_1(B_3)} \le C\|v\|_{W^1_q(B_3)} \le \frac{C}{q-1}\|\widetilde{\delta}\|_{L_q(\tau_0^h)} \le \frac{C}{q-1}h^{-N(1-\frac{1}{q})} \le C(\ln 1/h),$$

if we choose $q = 1 + 1/(\ln 1/h)$. Hence from (3.48)

$$|I_2| \le C(\ln 1/h)|||F|||_{-1,\infty,B_3}. \tag{3.49}$$

From (3.46), (3.47) and (3.49) it follows that

$$|(w - w_h, D_i\widetilde{\delta})| \le C\|w\|_{W^1_\infty(B_3)} + C(\ln 1/h)|||F|||_{-1,\infty,B_3}.$$

Since also $\|\widetilde{\delta}\|_{L_1} \le C$ the desired result now follows from (3.45). $\square$

We are now set to prove Theorem 1.2, or, in light of our preliminary reductions, the estimate (3.1). Recall that $A(u - u_h, \chi) = F(\chi)$ for $\chi \in \overset{\circ}{S}{}^h(B_3)$. Let $\omega \in \mathscr{C}_0^\infty(B_3)$ be such that $\omega \equiv 1$ on B_2. Then with $\widetilde{u} = \omega u$,

$$A(\widetilde{u} - u_h, \chi) = F(\omega\chi) \quad \text{for} \quad \chi \in \overset{\circ}{S}{}^h(B_2). \tag{3.50}$$

Let k be such that $A_k(v, w) = A(v, w) + k(v, w)$ is coercive over $W^1_2(B_3)$. Note again that k may be assumed bounded in terms of c_{ell} and the coefficients of A. Define then $(\widetilde{u})_h \in S^h(B_3)$ by

$$A_k(\widetilde{u} - (\widetilde{u})_h, \chi) = F(\omega\chi) \quad \text{for} \quad \chi \in S^h(B_3). \tag{3.51}$$

Hence, from Lemma 3.3,

$$\begin{aligned}\|\widetilde{u} - (\widetilde{u})_h\|_{W^1_\infty(B_2)} &\le C\|\widetilde{u}\|_{W^1_\infty(B_3)} + C(\ln 1/h)|||F|||_{-1,\infty,B_3} \\ &\le C\|u\|_{W^1_\infty(B_3)} + C(\ln 1/h)|||F|||_{-1,\infty,B_3}.\end{aligned} \tag{3.52}$$

We now have for $\chi \in \overset{\circ}{S}^h(B_2)$

$$\begin{aligned} F(\omega\chi) &= A(\widetilde{u} - u_h\,,\, \chi) = A(\widetilde{u} - (\widetilde{u})_h\,,\, \chi) + A((\widetilde{u})_h - u_h\,,\, \chi) \\ &= A_k(\widetilde{u} - (\widetilde{u})_h\,,\, \chi) - k(\widetilde{u} - (\widetilde{u})_h\,,\, \chi) + A((\widetilde{u})_h - u_h\,,\, \chi)\,, \end{aligned}$$

i.e., by (3.51),

$$A((\widetilde{u})_h - u_h\,,\, \chi) = k(\widetilde{u} - (\widetilde{u})_h\,,\, \chi) \quad \text{for} \quad \chi \in \overset{\circ}{S}^h(B_2).$$

Thus, from Lemma 3.2 (after an inconsequential change of the domains involved) we have, using the triangle inequality, the fact that $\omega \equiv 1$ on B_2, and (3.52),

$$\begin{aligned} \|(\widetilde{u})_h - u_h\|_{W^1_\infty(B_1)} &\le C\|(\widetilde{u})_h - u_h\|_{W_q^{-s}(B_2)} + C\|\widetilde{u} - (\widetilde{u})_h\|_{L_\infty(B_2)} \\ &= C\|((\widetilde{u})_h - \widetilde{u}) + (u - u_h)\|_{W_q^{-s}(B_2)} + C\|\widetilde{u} - (\widetilde{u})_h\|_{L_\infty(B_2)} \\ &\le C\|\widetilde{u} - (\widetilde{u})_h\|_{L_\infty(B_2)} + C\|u - u_h\|_{W_q^{-s}(B_2)} \\ &\le C\|u\|_{W^1_\infty(B_3)} + C(\ln 1/h)|||F|||_{-1,\infty,B_3} + C\|e\|_{W_q^{-s}(B_2)}. \end{aligned} \tag{3.53}$$

Hence, by (3.52) and (3.53),

$$\begin{aligned} \|u - u_h\|_{W^1_\infty(B_1)} &\le \|\widetilde{u} - (\widetilde{u})_h\|_{W^1_\infty(B_1)} + \|(\widetilde{u})_h - u_h\|_{W^1_\infty(B_1)} \\ &\le C\|u\|_{W^1_\infty(B_3)} + C(\ln 1/h)|||F|||_{-1,\infty,B_3} + C\|e\|_{W_q^{-s}(B_3)} \end{aligned}$$

which is the desired estimate (3.1).

This completes the proof of Theorem 1.2. □

3.B. **Proof of Theorem 1.1.** As in the previous subsection (cf. (3.1)–(3.8)) it suffices to show, taking $d = 1$, that

$$\begin{aligned} \|e\|_{L_\infty(B_1)} &\le C(\ln 1/h)^{\bar{r}}\|u\|_{L_\infty(B_3)} + C\|e\|_{W_q^{-s}(B_3)} \\ &\quad + Ch(\ln 1/h)^{\bar{r}}|||F|||_{-1,\infty,B_3} + C(\ln 1/h)|||F|||_{-2,\infty,B_3}. \end{aligned} \tag{3.54}$$

Following the last part of §3.A, we let $\omega \in \mathscr{C}_0^\infty(B_3)$ with $\omega \equiv 1$ on B_2, $\widetilde{u} = \omega u$, and we let $(\widetilde{u})_h$ be defined by (3.51). As in (the first parts of) (3.53) we then have

$$\begin{aligned} \|(\widetilde{u})_h - u_h\|_{L_\infty(B_1)} &\le \|(\widetilde{u})_h - u_h\|_{W^1_\infty(B_1)} \\ &\le C\|\widetilde{u} - (\widetilde{u})_h\|_{L_\infty(B_2)} + C\|e\|_{W_q^{-s}(B_2)}. \end{aligned} \tag{3.55}$$

It thus remains to estimate $\|\widetilde{u} - (\widetilde{u})_h\|_{L_\infty(B_2)}$, which we recall satisfies (3.51),

$$A_k(\widetilde{u} - (\widetilde{u})_h\,,\, \chi) = F(\omega\chi) \quad \text{for} \quad \chi \in S^h(B_3).$$

In §3.A we relied on Lemma 3.3 for this. Below we shall describe how a suitably modified version of Lemma 3.3 ((3.62) below) follows from the results above and those of [10]. With $\widetilde{\delta}$ now such that

$$\chi(x_0) = (\chi\,,\, \widetilde{\delta}) \quad \text{for} \quad \chi \in S^h(B_3)\,, \;\; x_0 \in B_2\,,$$

we have with $w = \widetilde{u}$, $w_h = (\widetilde{u})_h$,

$$w_h(x_0) = (w_h\,,\, \widetilde{\delta}) = (w\,,\, \widetilde{\delta}) + (w_h - w\,,\, \widetilde{\delta}). \tag{3.56}$$

By A.5 we may again assume that $\widetilde{\delta}$ is supported in the element τ_h^0 $(x_0 \in \overline{\tau_h^0})$ and satisfies the estimate (3.9).

Here, by (3.9),

$$|(w, \widetilde{\delta})| \le C\|w\|_{L_\infty(B_3)}\|\widetilde{\delta}\|_{L_1(\tau_0^h)} \le C\|u\|_{L_\infty(B_3)}. \tag{3.57}$$

We now let v be defined by $L_k^* v = \widetilde{\delta}$ in B_3, with homogeneous conormal Neumann conditions in ∂B_3. Similarly, $v_h \in S^h(B_3)$ is given by $A_k(\chi, v - v_h) = 0$, for $\chi \in S^h(B_3)$. Then

$$\begin{aligned}(w - w_h, \widehat{\delta}) &= A_k(w - w_h, v) = A_k(w - w_h, v - v_h) + A_k(w - w_h, v_h) \\ &= A_k(w, v - v_h) + F(\omega v_h) \\ &= A_k(\omega, v - v_h) + F(\omega(v_h - v)) + F(\omega v).\end{aligned} \tag{3.58}$$

We now further assume, as we may, that ω is supported in $B_{2.5}$. Then so is $w = \omega u$. In $A_k(w, v - v_h)$ we then integrate by parts over each element τ_h^i meeting $B_{2.5}$ to obtain

$$A_k(w, v - v_h) = \sum_{\tau_h^i \cap B_{2.5} \neq \varnothing} \left(\int_{\tau_h^i} w L_k^*(v - v_h)\,dx + \int_{\partial \tau_h^i} w \frac{\partial}{\partial n_{L_k}}(v - v_h)\,d\sigma \right).$$

With $\|\cdot\|_{W_1^{2,h}(B_{2.6})}$ the piecewise W_1^2 norm over elements meeting $B_{2.6}$ we then have, using the trace inequality A.0 (cf. [10, page 430]),

$$|A_k(w, v - v_h)| \le C\|w\|_{L_\infty(B_{2.6})} Q,$$

where $Q = h^{-1}\|v - v_h\|_{W_1^1(B_{2.6})} + \|v - v_h\|_{W_1^{2,h}(B_{2.6})}$. Now [10, Lemma 5.3] says exactly that $Q \le C(\ln 1/h)^{\bar{r}}$ and hence, recalling that $w = \omega u$,

$$|A_k(w, v - v_h)| \le C(\ln 1/h)^{\bar{r}}\|u\|_{L_\infty(B_3)}. \tag{3.59}$$

Furthermore, again from [10, Lemma 5.3, (5.9)],

$$\begin{aligned}|F(\omega(v_h - v))| &\le C|||F|||_{-1,\infty,B_3}\|v_h - v\|_{W_1^1(B_{2.5})} \\ &\le Ch(\ln 1/h)^{\bar{r}}|||F|||_{-1,\infty,B_3}.\end{aligned} \tag{3.60}$$

For the last term in (3.58), we have

$$|F(\omega v)| \le C|||F|||_{-2,\infty,B_3}\|v\|_{W_1^2(B_3)}$$

and, using Lemma 2.2, with $q = 1 + 1/(\ln 1/h)$, and (3.9),

$$\|v\|_{W_1^2(B_3)} \le C\|v\|_{W_q^2(B_3)} \le \frac{C}{q-1}\|\widetilde{\delta}\|_{L_q(\tau_h^0)} \le \frac{C}{q-1} h^{-N(1-1/q)} \le C(\ln 1/h).$$

Thus,

$$|F(\omega v)| \le C(\ln 1/h)|||F|||_{-2,\infty,B_3}. \tag{3.61}$$

From (3.58)–(3.61) we now have that

$$\begin{aligned}|(w - w_h, \widehat{\delta})| \le{}& C(\ln 1/h)^{\bar{r}}\|u\|_{L_\infty(B_3)} + Ch(\ln 1/h)^{\bar{r}}|||F|||_{-1,\infty,B_3} \\ &+ C(\ln 1/h)|||F|||_{-2,\infty,B_3}.\end{aligned}$$

Combining this with (3.57) in (3.56), we obtain

$$\begin{aligned}\|\widetilde{u} - (\widetilde{u})_h\|_{L_\infty(B_2)} \le{}& C(\ln 1/h)^{\bar{r}}\|u\|_{L_\infty(B_3)} + Ch(\ln 1/h)^{\bar{r}}|||F|||_{-1,\infty,B_3} \\ &+ C(\ln 1/h)|||F|||_{-2,\infty,B_3}.\end{aligned} \tag{3.62}$$

(Apart from minor changes in notation, this is the counterpart of Lemma 3.3 referred to above.)

Combining next (3.62) with (3.55) yields the desired estimate (3.54).

This completes the proof of Theorem 1.1. □

4. GLOBAL ESTIMATES

Let $\mathscr{D}$ be a fixed bounded domain in R^N with $\partial\mathscr{D}$ sufficiently smooth, and let A (for simplicity) be coercive on $W_2^1(\mathscr{D})$. Here we shall give global estimates for the Neumann problem with homogeneous boundary conditions. Let thus u be a given function on $\mathscr{D}$ and $u_h \in S^h = S^h(\mathscr{D})$ satisfy

$$A(u - u_h, \chi) = F(\chi) \quad \text{for} \quad \chi \in S^h, \tag{4.1}$$

where F is a linear functional on $W_1^1(\mathscr{D})$.

For our global estimates we shall need some modifications of the assumptions used for our local estimates. In particular, we need assumptions pertaining to those elements which are near or at the boundary. In this regard we shall assume that $\mathscr{D}$ is partitioned into disjoint elements τ_i^h which are globally quasi-uniform of size h. For simplicity we assume that the elements which meet $\partial\mathscr{D}$ are curved to fit $\partial\mathscr{D}$ exactly (an assumption which is not unrealistic in a Neumann problem if one disregards numerical integration). Regarding the assumptions A.0–A.5 we assume the following: A.0, A.4 and A.5 hold without changes. As for A.1 and A.2, we assume that they hold for all domains $G_1 \subseteq G_2 \subseteq \mathscr{D}$ arising as the intersections of $\mathscr{D}$ with two concentric balls $\widetilde{B}_1$ and $\widetilde{B}_2$, $\operatorname{dist}(\widetilde{B}_1, \partial\widetilde{B}_2) \geq c_0 h$; $G_i = \widetilde{B}_i \cap \mathscr{D}$, $i = 1, 2$. Of course, statements such as functions "being in $\overset{\circ}{S}{}^h(G)$" are now suitably modified if we are at the boundary. For A.3 we assume (with the same notation as above) that it holds with $\omega \in \overset{\circ}{\mathscr{C}}{}^\infty(\widetilde{B}_1)$ and η with support in G_2.

Theorem 4.1. *Under the above conditions there exists a constant C independent of u, u_h and h such that for $u - u_h$ as in* (4.1),

$$\|u - u_h\|_{W_\infty^1(\mathscr{D})} \leq C\Big(\min_{\chi\in S^h(\mathscr{D})} \|u - \chi\|_{W_\infty^1(\mathscr{D})} + (\ln 1/h)|||F|||_{-1,\infty,\mathscr{D}}\Big). \tag{4.2}$$

Here,

$$|||F|||_{-1,\infty,\mathscr{D}} = \sup_{\substack{\varphi\in W_1^1(\mathscr{D})\\ \|\varphi\|_{W_1^1(\mathscr{D})}=1}} |F(\varphi)|.$$

Remark 4.1. In the case of $F = 0$, Theorem 4.1 represents an extension of the global two-dimensional results given in [9] to any number of space dimensions.

Remark 4.2. The obvious analogue of Remark 1.1 applies.

Proof of Theorem 4.1. We shall give the essential modifications necessary in §3.A. No preliminary scaling arguments need to be performed. We first observe that the principal Lemma 3.1 now holds with $\mathscr{D}$ replacing B_3 and, more importantly, with $\widetilde{\delta}$ in (3.9) supported in any basic element $\tau_h^0 \in \mathscr{D}$. The reason for this is that with our present modified assumptions, the modification in (2.5) when we are close to the boundary, is not needed; cf. Remark 2.1. It is now easy to trace through the proof of Lemma 3.1 to see that

$$\|v - v_h\|_{W_1^1(\mathscr{D})} \le C,$$

where v and v_h are as in (3.9)–(3.11), with $\tau_h^0 \in \mathscr{D}$.

Since we are now in a global setting, Lemma 3.2 will not be needed. Instead, we go directly at (4.2) following the proof of Lemma 3.3. (Again, the concluding arguments for the local case are now superfluous.) In fact, the cutoff function ω there is now not involved and any $\tau_0^h \in \mathscr{D}$ is allowed. The proof now consists of reading through the proof of Lemma 3.3 with the appropriate minor (and simplifying) modifications. □

One may similarly derive maximum-norm estimates for function values for the Neumann problem in (4.1); in the case $F = 0$ and the harder case of the Dirichlet problem, this was done in [11].

5. Application to numerical integration

In this section we shall apply Theorem 1.2 to derive interior estimates in W_∞^1, taking into account the presence of numerical integration. For simplicity, let Ω_1 be a domain of unit size and let u satisfy

$$A(u, \chi) = (f, \chi) \text{ for all } \chi \in \mathring{W}_2^1(\Omega_1). \tag{5.1}$$

We shall assume that our approximate solution $u_h \in S^h(\Omega_1)$ satisfies

$$A_h(u_h, \chi) = (f, \chi)_h \text{ for all } \chi \in \mathring{S}^h(\Omega_1). \tag{5.2}$$

Here, $A_h(\cdot, \cdot)$ is an approximation to the bilinear form $A(\cdot, \cdot)$ and $(\cdot, \cdot)_h$ an approximation to the L_2-inner product $(\cdot, \cdot)$. Note that $e = u - u_h$ satisfies

$$A(e, \chi) = (A_h - A)(u_h, \chi) + (f, \chi) - (f, \chi)_h \text{ for } \chi \in \mathring{S}^h(\Omega_1). \tag{5.3}$$

We shall next state some assumptions on $A_h(\cdot, \cdot)$, $(\cdot, \cdot)_h$ and the subspaces (in addition to those of the Appendix). We shall discuss the first two of these assumptions at the end of the section. Assume then that

$$\begin{aligned} |(A - A_h)(\varphi, \chi)| &\le Ch^{r-1}\|\varphi\|_{W_\infty^{r-1,h}(\Omega_1)}\|\chi\|_{W_1^1(\Omega_1)} \\ &\qquad \text{for } \varphi \in S^h(\Omega_1),\ \chi \in \mathring{S}^h(\Omega_1), \end{aligned} \tag{5.4}$$

and that for f smooth enough,

$$|(f, \chi) - (f, \chi)_h| \le C(f)h^{r-1}\|\chi\|_{W_1^1(\Omega_1)} \text{ for } \chi \in \mathring{S}^h(\Omega_1). \tag{5.5}$$

Furthermore suppose that, given v, there exists $\chi_v \in S^h(\Omega_1)$ such that the following (rather weak) estimates hold:

$$\|v - \chi_v\|_{W_\infty^1(\Omega_1)} \le Ch^{r-2}\|v\|_{W_\infty^r(\Omega_1 + Ch)} \tag{5.6}$$

and

$$\|\chi_v\|_{W_\infty^{r-1,h}(\Omega_1)} \le C\|v\|_{W_\infty^r(\Omega_1 + Ch)}. \tag{5.7}$$

Finally, it will be assumed that the inverse property A.2 (see the Appendix) holds over the wider range

$$0 \le t \le s \le r - 1. \tag{5.8}$$

As an application of Theorem 1.2 we then have

Corollary 5.1. *Let u and u_h satisfy* (5.1) *and* (5.2) *in* Ω_1. *Suppose that the general conditions of the Appendix hold and, in addition,* (5.4)–(5.8). *Then for* $\Omega_0 \subset\subset \Omega_1$,

$$\|e\|_{W^1_\infty(\Omega_0)} \le C\big[h^{r-1}(\ln 1/h)\big(\|u\|_{W^r_\infty(\Omega_1)} + 1\big) + \|e\|_{W^{-s}_q(\Omega_1)}\big]. \tag{5.9}$$

Proof. We shall need to introduce a few subdomains contained between Ω_0 and Ω_1. Abusing notation, we shall call them all Ω_1 although they may be different at each occurrence.

Applying Theorem 1.2 to (5.3) and using (5.4), (5.5) and Remark 1.1 (and approximation theory), we get

$$\begin{aligned} \|e\|_{W^1_\infty(\Omega_0)} &\le Ch^{r-1}\|u\|_{W^r_\infty(\Omega_1)} + C\|e\|_{W^{-s}_q(\Omega_1)} \\ &\quad + C(\ln 1/h)h^{r-1}(\|u_h\|_{W^{r-1,h}_\infty(\Omega_1)} + C(f)). \end{aligned} \tag{5.10}$$

Choosing χ as an approximation to u satisfying (5.6) and (5.7) and then using inverse estimates, we have

$$\begin{aligned} \|u_h\|_{W^{r-1,h}_\infty(\Omega_1)} &\le \|u_h - \chi\|_{W^{r-1,h}_\infty(\Omega_1)} + \|\chi\|_{W^{r-1,h}_\infty(\Omega_1)} \\ &\le Ch^{2-r}\|u_h - \chi\|_{W^1_\infty(\Omega_1)} + C\|u\|_{W^r_\infty(\Omega_1)} \\ &\le Ch^{2-r}\|e\|_{W^1_\infty(\Omega_1)} + C\|u\|_{W^r_\infty(\Omega_1)}. \end{aligned}$$

Thus, from (5.10),

$$\|e\|_{W^1_\infty(\Omega_0)} \le Ch^{r-1}(\ln 1/h)(1+\|u\|_{W^r_\infty(\Omega_1)}) + C\|e\|_{W^{-s}_q(\Omega_1)} + Ch(\ln 1/h)\|e\|_{W^1_\infty(\Omega_1)}.$$

Repeating the argument M times, using that $h \ln 1/h < 1$, we find that

$$\begin{aligned} \|e\|_{W^1_\infty(\Omega_0)} &\le Ch^{r-1}(\ln 1/h)(1 + \|u\|_{W^r_\infty(\Omega_1)}) + C\|e\|_{W^{-s}_q(\Omega_1)} \\ &\quad + (Ch \ln 1/h)^M \|u_h\|_{W^{r-1,h}_\infty(\Omega_1)}, \end{aligned}$$

whereupon an application of an inverse estimate leads to

$$\begin{aligned} \|e\|_{W^1_\infty(\Omega_0)} &\le Ch^{r-1}(\ln 1/h)(1 + \|u\|_{W^1_\infty(\Omega_1)}) + C\|e\|_{W^{-s}_q(\Omega_1)} \\ &\quad + (Ch \ln 1/h)^M h^{-(r-1)-s-N/q}\|u_h\|_{W^{-s}_q(\Omega_1)}. \end{aligned}$$

Writing $\|u_h\|_{W^{-s}_q(\Omega_1)} \le \|e\|_{W^{-s}_q(\Omega_1)} + \|u\|_{W^{-s}_q(\Omega_1)}$ and taking M large enough, we have our desired estimate (5.9). □

We conclude this section with some comments about the two major assumptions (5.4)–(5.5). For (5.4), let us consider only Lagrangian elements on simplices and the highest-order terms in A. Then with $p = \frac{\partial \varphi}{\partial x_i} \in \Pi_{r-2}$ (the polynomials of total degree $\le r-2$), $q = \frac{\partial \chi}{\partial x_j} \in \Pi_{r-2}$, consider $\int_\tau a_{ij}(x)pq\,dx$ over a simplex τ. Let ξ_k, $k = 1, \dots, K$, be quadrature points and $\omega_{k,\tau}$ corresponding weights so that the error over a simplex is

$$E = E_\tau(a_{ij}pq) = \int_\tau a_{ij}pq\,dx - \sum_{k=1}^{K} \omega_{k,\tau}(a_{ij}pq)(\xi_k).$$

Assume that the method is exact on Π_{2r-4}, i.e., $E(\psi) = 0$ for $\psi \in \Pi_{2r-4}$. By the Bramble-Hilbert lemma one then has

$$\begin{aligned} |E| &\le Ch^{N+(2r-3)} \|a_{ij}pq\|_{W_\infty^{2r-3}(\tau)} \\ &\le C(a_{ij})h^{N+(2r-3)} \|p\|_{W_\infty^{r-2}(\tau)} \|q\|_{W_\infty^{r-2}(\tau)} \\ &\le C(a_{ij})h^{r-1} \|p\|_{W_\infty^{r-2}(\tau)} h^{N+(r-2)} \|q\|_{W_\infty^{r-2}(\tau)} \\ &\le C(a_{ij})h^{r-1} \|p\|_{W_\infty^{r-2}(\tau)} \|q\|_{L_1(\tau)}, \end{aligned}$$

where the last step used inverse inequalities. With the lower-order terms of A treated similarly, clearly (5.4) would follow.

We also remark that the case of tensor-product elements is somewhat trickier, unless one uses a quadrature method of sufficiently high accuracy. If not, one has to employ a sharp form of the Bramble-Hilbert lemma (only those derivatives are involved which annihilate the finite elements under consideration). A template for a verification of (5.4) can be found in the stepwise procedure in the proof of Ciarlet [4, Theorem 28.2]; we forego the details.

The estimate (5.5) is similar, for f smooth enough, cf. [4, Theorem 28.3] for ideas for sharper estimates.

Let us also note that (5.4) and (5.5) are rather trivially satisfied if the coefficients a_{ij}, a_j and a, and the function f, are replaced by suitable "interpolants" and then the resulting integrals done exactly by a suitable quadrature rule.

APPENDIX

Here we shall state the basic assumptions, A.0–A.5, on the finite element subspaces which are used in proving Theorems 1.1 and 1.2. Let Ω, $\Omega \subset\subset \mathscr{D} \subset \mathbb{R}^N$, be fixed throughout this discussion. We remark that A.0–A.4 are essentially as in [10]. (Of course, the very minor changes done do not change the results of [10].)

We shall make use of spaces defined relative to partitions of Ω. Let $0 < h < 1/2$ be a parameter, and for each h let τ_i^h, $0 \le i \le I(h)$, be a finite number of disjoint open sets such that $\Omega \subseteq \bigcup_{i=0}^{I(h)} \bar{\tau}_i^h$. The sets $\tau_i^h \cap \Omega$ induce a partition of Ω, and relative to each such partition, we define $W_q^{s,h}(\Omega)$ $(C^{s,h}(\Omega))$ as the space consisting of those functions which belong to $W_q^s(\tau_i^h \cap \Omega)$ $(C^s(\tau_i^h \cap \Omega))$, $0 \le i \le I(h)$. We introduce the seminorms

$$|v|_{W_q^{s,h}(\Omega)} = \begin{cases} \left(\sum_{i=0}^{I(h)} |v|^q_{W_q^s(\tau_i^h \cap \Omega)} \right)^{1/q} & \text{for } 1 \le q < \infty, \\ \max_{i=0,\dots,I(h)} |v|_{W_\infty^s(\tau_i^h \cap \Omega)} & \text{for } q = \infty, \end{cases}$$

and the corresponding norms $\|\cdot\|_{W_q^{s,h}(\Omega)}$. Note that if $v \in W_q^s(\Omega)$, then $\|v\|_{W_q^{s,h}(\Omega)} = \|v\|_{W_q^s(\Omega)}$.

For each $0 < h < 1/2$, $S^h(\Omega)$ will denote a finite-dimensional subspace of $W_\infty^1(\Omega) \cap C^{2,h}(\Omega)$. Our first assumption relates to the geometry of the partitioning sets τ_i^h. We shall assume that a certain trace inequality holds on each of the τ_i^h.

A.0. *Trace.* There exists a constant C such that, for $0 < h < 1/2$, and any $f \in W_1^2(\tau_i^h)$, $i = 0, \dots, I(h)$,

$$\int_{\partial \tau_i^h} |\nabla f| d\sigma \le C\left\{\frac{1}{h}|f|_{W_1^1(\tau_i^h)} + |f|_{W_1^2(\tau_i^h)}\right\}.$$

We remark that the assumption A.0 is satisfied for a large class of partitions of Ω. For example, it holds if the τ_i^h are taken to be N-simplices or N-dimensional parallelepipeds of diameter $c_i h$, $c_i \le C$, provided the ratio of the diameter and the radius of the largest inscribed sphere is uniformly bounded. Briefly, to verify A.0 in these cases one maps each of the τ_i^h onto a standard domain. The inequality can then be proven, with $h = 1$, using integration by parts. The desired inequality is then obtained by mapping back to τ_i^h.

For $G \subseteq \Omega$, $S^h(G)$ is defined as the restriction of $S^h(\Omega)$ to G, and

$$\overset{\circ}{S}{}^h(G) = \{\chi \mid \chi \in S^h(G), \text{supp } \chi \subset\subset G\}.$$

Let $r \ge 2$ be a given integer. We shall assume that there exist positive constants C_1, C_2, C_3, C_4, k_0, γ, and $0 < h_0 < 1/2$ such that the spaces $S^h(\Omega)$ satisfy the following conditions A.1–A.4 for $0 < h \le h_0$.

A.1. *Approximation.* Let $G \subset\subset \Omega$ with $\text{dist}(G, \partial\Omega) \ge k_0 h$. Then for each v there exists a $\chi \in S^h(G)$ such that for $G_1 \subset\subset G_2 \subset G$ with $\text{dist}(G_1, \partial G_2) \ge k_0 h$,

$$\text{(A.1)} \qquad \begin{aligned} &\|v - \chi\|_{W_q^{t,h}(G_1)} \le C_1 h^{\ell - t}\|v\|_{W_q^\ell(G_2)} \\ &\text{for } 0 \le t \le s \le \ell \le r, \ 1 \le q \le \infty, \ t = 0, 1, 2. \end{aligned}$$

Furthermore, if supp $v \subseteq G_1$, then $\chi \in \overset{\circ}{S}{}^h(G_2)$.

Remark A.1. The approximation hypothesis above contains a full norm of v on the right of (A.1) rather than a seminorm. It is thus satisfied for example by certain curved isoparametric elements, cf. [4, (37.27) and discussion, p. 246, 7^- et seq.]. □

A.2. *Inverse properties.* Let $p \ge -1$ be an integer and $G_1 \subset\subset G_2$ with $\text{dist}(G_1, \partial G_2) \ge k_0 h$. Then for $\chi \in S^h(G_2)$,

$$\|\chi\|_{W_2^1(G_1)} \le C_2 h^{-(p+1)} \|\chi\|_{W_2^{-p}(G_2)},$$

and

$$\|\chi\|_{W_q^{s,h}(G_1)} \le C_2 h^{t-s-N(1/q_1 - 1/q)} \|\chi\|_{W_{q_1}^{t,h}(G_2)} \text{ for } 0 \le t \le s \le 2, \ 1 \le q_1 \le q \le \infty.$$

A.3. *Superapproximation.* Let $G_1 \subset\subset G_2$ with $\text{dist}(G_1, \partial G_2) \ge k_0 h$, and let $\omega \in \overset{\circ}{C}{}^\infty(G_1)$. Then for each $\chi \in S^h(G_2)$ there exists an $\eta \in \overset{\circ}{S}{}^h(G_2)$ satisfying

$$\|\omega\chi - \eta\|_{W_2^s(G_2)} \le C_3 h \|\omega\|_{W_\infty^\gamma(G_1)} \|\chi\|_{W_2^s(G_1)}, \quad 1 \le q \le 1, \quad s = 0, 1,$$

and

$$\|\eta\|_{L_q(G_2)} \le C\|\chi\|_{L_q(G_2)} \text{ for } 1 \le q \le \infty.$$

Furthermore, let $G_{-1} \subset\subset G_0 \subset\subset G_1 \subset\subset G_2$ with $\text{dist}(G_{-1}, \partial G_0) \ge k_0 h$ and

$\operatorname{dist}(G_0, \partial G_1) \geq k_0 h$. Then, if $\omega \equiv 1$ on G_1, we have $\eta \equiv \chi$ on G_0 and

$$\|\omega\chi - \eta\|_{W_2^s(G_2)} \leq C_3 h \|\omega\|_{W_\infty^\gamma(G_2)} \|\chi\|_{W_2^s(G_2 \setminus G_{-1})}.$$

Remark A.2. The superapproximation hypothesis above has been discussed in [8] and [10] and, as seen there, is valid for many finite element spaces met in practice. Here we wish to emphasize that, for tensor-product elements, its verification often depends on a *sharp* form of the Bramble-Hilbert lemma (involving only derivatives which annihilate the finite elements under consideration), cf. also Bramble, Nitsche and Schatz [1, Appendix]. Isoparametric cases are, of course, handled by mapping to a reference element. In particular, A.3 is satisfied for the 4-node (isoparametric) quadrilateral, cf. [4, (37.33)]. □

We shall further make the assumption that if a sphere or radius d in Ω is transformed by similarity to a sphere of unit size, then the transformed finite element space satisfies A.1, A.2, and A.3 with h replaced by h/d and with the constants occurring the same as before.

A.4. *Scaling.* Let $B_d \subset\subset \Omega$ be a sphere of radius $d \geq C_4 h$ with center at x_0. The linear transformation $y = (x - x_0)/d$ takes B_d into a sphere B and $S^h(B_d)$ into a new function space $S(B)$. Then $S(B)$ satisfies A.1, A.2, and A.3 with h replaced by h/d. Furthermore, the constants occurring in A.1, A.2, and A.3 remain unchanged, in particular, independent of d.

Our final basic assumption is concerned with the existence of regularized delta functions, cf. [9].

A.5. There exists a constant C_5 such that the following holds.

(i) For any $x_0 \in \Omega$ with $x_0 \in \tau_i^h$ there exists a function $\tilde{\delta}_0 \in \mathscr{C}^1$ with support in τ_i^h such that

$$\chi(x_0) = \int_{\tau_i^h} \chi \tilde{\delta}_0 \text{ for all } \chi \in S^h,$$

and

$$\text{(A.2)} \qquad \|\tilde{\delta}_0\|_{L_q} \leq C_5 h^{-N(1-\frac{1}{q})}, \quad \|\nabla \tilde{\delta}_0\|_{L_q} \leq C_5 h^{-N(1-\frac{1}{q})-1} \text{ for } 1 \leq q \leq \infty.$$

(ii) Similarly, for $j = 1, \ldots, n$, there exists $\tilde{\delta}_{1,j}$ such that

$$\frac{\partial}{\partial x_j}\chi(x_0) = \int_{\tau_h^i} \frac{\partial \chi}{\partial x_j} \tilde{\delta}_{1,j}$$

and (A.2) holds.

To verify A.5, say, in the second form, for a typical finite element space, it suffices to consider a reference element $\hat{\tau}$ with $h = 1$; the general case then follows by mapping and scaling. Let ω be a fixed nonnegative C^1-function with compact support in $\hat{\tau}$ and $\int \omega = 1$, and let $\langle v, w\rangle = \int_{\hat{\tau}} v w \omega \, dx$ be the corresponding weighted inner product. Let $\pi_1, \ldots, \pi_D$ be an orthonormal basis for the finite-dimensional space $\frac{\partial \chi}{\partial x_j}\big|_{\hat{\tau}}$ with respect to the weighted inner product above. Then

$$\tilde{\delta}_{1,j}(x) := \sum_{i=1}^{D} \pi_i(x_0) \pi_i(x) \omega(x)$$

is the desired function.

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