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A MIXED FINITE ELEMENT METHOD FOR A STRONGLY NONLINEAR SECOND-ORDER ELLIPTIC PROBLEM

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ABSTRACT. The approximation of the solution of the first boundary value problem for a strongly nonlinear second-order elliptic problem in divergence form by the mixed finite element method is considered. Existence and uniqueness of the approximation are proved and optimal error estimates in L^2 are established for both the scalar and vector functions approximated by the method. Error estimates are also derived in L^q , $2 \leq q \leq +\infty$.

1. INTRODUCTION

A large number of physical phenomena are modeled by partial differential equations or systems of parabolic type—in evolution, or elliptic—at steady state. Most of the models have quite strong nonlinearities which, typically, are either weakened or removed (by linearization) before the problem is treated analytically or numerically.

It is frequently the case that it is at least as important (if not more so) to obtain a good approximation of some function of the gradient of the solution of the differential equation (which may represent, for example, a velocity field or electric field) as an approximation of the solution itself (which may represent, respectively, a pressure or an electric potential). The mixed finite element method computes both approximations simultaneously and with the same order of accuracy, be it directly or through postprocessing and, for some problems, it seems to yield better results than standard finite element methods. For second-order elliptic problems, the mixed method was described and analyzed by many authors [3, 5, 7, 11] in the case of linear equations in divergence form, as well as in [4, 8, 9] for quasilinear problems in divergence form.

In this paper we shall start to study the applicability of the mixed method to more strongly nonlinear problems. Specifically, we consider the following boundary value problem:

$$\begin{cases} -\operatorname{div}(\underset{\sim}{a}(\underset{\sim}{\nabla} p)) = f & \text{in } \Omega, \\ p = -g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^2$ is a bounded, convex domain with C^2-boundary $\partial\Omega$; a tilde under a symbol is used to indicate a vector, $\underset{\sim}{a} : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}^2$ is twice continuously differentiable with bounded derivatives through second order, $f \in H^{1/2+\varepsilon_0}(\Omega)$ and $g \in H^{2+\varepsilon_0}(\partial\Omega)$, $\varepsilon_0 > 0$. Note that this implies $p \in H^{5/2+\varepsilon_0}(\Omega)$ [6]. We shall not indicate explicitly the dependence of any function on the spatial variables $\underset{\sim}{x}$. Furthermore, we shall assume that $\underset{\sim}{a}$ has a bounded positive definite Jacobian with respect to the second argument, that is, with $\underset{\sim}{a}(\underset{\sim}{z}) = (a_1(\underset{\sim}{z}), a_2(\underset{\sim}{z}))$, $\underset{\sim}{z} \in \mathbb{R}^2$,

$$0 < \lambda(\underset{\sim}{z}) \le \frac{\partial(a_1, a_2)}{\partial(z_1, z_2)} \le \Lambda(\underset{\sim}{z}).$$

These conditions are satisfied, for example, for $\underset{\sim}{a}(\underset{\sim}{\nabla} p) = \kappa(|\underset{\sim}{\nabla} p|)\underset{\sim}{\nabla} p$, with κ a nonnegative function. Such is the problem, for example, in the model for minimal surfaces, where $f = 0$ and

$$\underset{\sim}{a}(\underset{\sim}{\nabla} p) = \frac{\underset{\sim}{\nabla} p}{\sqrt{1 + |\underset{\sim}{\nabla} p|^2}}.$$

Our assumption on the Jacobian $\frac{\partial(a_1, a_2)}{\partial(z_1, z_2)}$ implies (using the implicit function theorem) that $\underset{\sim}{\nabla} p$ can be locally represented as a function of the "flux"

$$\underset{\sim}{u} = -\underset{\sim}{a}(\underset{\sim}{\nabla} p), \tag{1.2}$$

say

$$\underset{\sim}{\nabla} p = -\underset{\sim}{b}(\underset{\sim}{u}). \tag{1.3}$$

We shall assume that this representation is global, and that $\underset{\sim}{u} \in H^{3/2+\varepsilon_0}(\Omega)^2 \cap C^{0,1}(\overline{\Omega})^2$.

Remark 1.1. Note that the domain of $\underset{\sim}{b}$ is not known and, though it may be small in specific cases, it always contains a ball centered at $\underset{\sim}{u}$ in $L^\infty(\Omega)$. Let us denote such a ball by $\mathscr{B}_0$.

Combining (1.1)–(1.3), we arrive at the following coupled system of first-order equations for $\underset{\sim}{u}$ and p:

$$\begin{cases} \underset{\sim}{b}(\underset{\sim}{u}) + \underset{\sim}{\nabla} p = 0 & \text{in } \Omega, \\ \operatorname{div} \underset{\sim}{u} = f & \text{in } \Omega, \\ p = -g & \text{on } \partial\Omega. \end{cases}$$

We now let $\underset{\sim}{V} = \underset{\sim}{H}(\operatorname{div}; \Omega) = \{\underset{\sim}{v} \in L^2(\Omega)^2 : \operatorname{div} \underset{\sim}{v} \in L^2(\Omega)\}$, $W = L^2(\Omega)$, and arrive at the mixed weak form of (1.1) we shall use: $(\underset{\sim}{u}, p) \in \underset{\sim}{V} \times W$ is the solution of the system

$$\begin{cases} (\underset{\sim}{b}(\underset{\sim}{u}), \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, p) = \langle g, \underset{\sim}{v} \cdot \underset{\sim}{n} \rangle, & \underset{\sim}{v} \in \underset{\sim}{V}, \\ (\operatorname{div} \underset{\sim}{u}, w) = (f, w), & w \in W, \end{cases} \tag{1.4}$$

where $\underset{\sim}{n}$ is the unit exterior normal vector on $\partial\Omega$, $(\cdot\,,\cdot)$ and $\langle\cdot\,,\cdot\rangle$ denote, respectively, the $L^2(\Omega)$-inner product and the $L^2(\partial\Omega)$-inner product. We consider a family of subspaces $\underset{\sim}{V}_h \times W_h$ of $\underset{\sim}{V} \times W$ (they may be Raviart-Thomas [11] or Brezzi-Douglas-Marini [1] families, for instance) associated with a quasi-uniform family of polygonal decompositions of Ω by triangles or quadrilaterals, with boundary elements allowed to have one curved side. The mixed finite element method we shall analyze is the discrete form of (1.4) and is given by: *Find* $(\underset{\sim}{u}_h, p_h) \in \underset{\sim}{V}_h \times W_h$ such that

$$
(1.5)\qquad \begin{cases} (\underset{\sim}{b}(\underset{\sim}{u}_h), \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, p_h) = \langle g, \underset{\sim}{v}\cdot\underset{\sim}{n}\rangle, & \underset{\sim}{v} \in \underset{\sim}{V}_h, \\ (\operatorname{div} \underset{\sim}{u}_h, w) = (f, w), & w \in W_h. \end{cases}
$$

This is a nonlinear algebraic system in the components of $(\underset{\sim}{u}_h, p_h)$, which we must prove is uniquely solvable. The plan of the paper is as follows: in the next section we prove the unique solvability of (1.5); in §3 we derive L^2-error estimates for $\underset{\sim}{u}_h$ and p_h, and in §4 we derive L^∞-error estimates for $\underset{\sim}{u}_h$. Finally, in §5 we discuss how these results apply to the model for minimal surfaces.

2. Existence and uniqueness

We shall follow some of the ideas of [9] to use a fixed point argument for the proof of existence. First, we derive from (1.4) and (1.5) the following error equations:

$$
(2.1)\qquad \begin{cases} (\underset{\sim}{b}(\underset{\sim}{u}) - \underset{\sim}{b}(\underset{\sim}{u}_h), \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, p - p_h) = 0, & \underset{\sim}{v} \in \underset{\sim}{V}_h, \\ (\operatorname{div}(\underset{\sim}{u} - \underset{\sim}{u}_h), w) = 0, & w \in W_h. \end{cases}
$$

We shall need the following relations, which are integral forms of Taylor's formula: for $\underset{\sim}{\mu} \in \mathscr{B}_0$,

$$
(2.2)\qquad \begin{aligned} \underset{\sim}{b}(\underset{\sim}{\mu}) - \underset{\sim}{b}(\underset{\sim}{u}) &= -B(\underset{\sim}{u})(\underset{\sim}{u} - \underset{\sim}{\mu}) + (\underset{\sim}{u} - \underset{\sim}{\mu})^T[\widetilde{H}_1(\underset{\sim}{\mu}), \widetilde{H}_2(\underset{\sim}{\mu})](\underset{\sim}{u} - \underset{\sim}{\mu}) \\ &= -\widetilde{B}(\underset{\sim}{\mu})(\underset{\sim}{u} - \underset{\sim}{\mu}), \end{aligned}
$$

where

$$
B(\underset{\sim}{u}) = \frac{\partial \underset{\sim}{b}}{\partial \underset{\sim}{u}} = \frac{\partial(b_1, b_2)}{\partial(u_1, u_2)} \text{ is the Jacobian of } \underset{\sim}{b}, \text{ a positive definite matrix},
$$

$$
(2.3)\qquad H_j = \frac{\partial^2 b_j}{\partial \underset{\sim}{u}^2} \quad (j = 1, 2) \text{ is the Hessian of } b_j,
$$

$$
\underset{\sim}{\zeta}^T[H_1, H_2]\underset{\sim}{\zeta} = (\underset{\sim}{\zeta}^T H_1 \underset{\sim}{\zeta}, \underset{\sim}{\zeta}^T H_2 \underset{\sim}{\zeta}) \in \mathbb{R}^2,
$$

for $j = 1, 2$,

(2.4)

$$\widetilde{H}_j(\underset{\sim}{\mu}) = \begin{bmatrix} \int_0^1 (1-t)\frac{\partial^2 b_j}{\partial u_1^2}(\underset{\sim}{u} + t[\underset{\sim}{\mu} - \underset{\sim}{u}])\,dt & \int_0^1 (1-t)\frac{\partial^2 b_j}{\partial u_1 \partial u_2}(\underset{\sim}{u} + t[\underset{\sim}{\mu} - \underset{\sim}{u}])\,dt \\ \int_0^1 (1-t)\frac{\partial^2 b_j}{\partial u_1 \partial u_2}(\underset{\sim}{u} + t[\underset{\sim}{\mu} - \underset{\sim}{u}])\,dt & \int_0^1 (1-t)\frac{\partial^2 b_j}{\partial u_2^2}(\underset{\sim}{u} + t[\underset{\sim}{\mu} - \underset{\sim}{u}])\,dt \end{bmatrix},$$

$$(2.5) \qquad \widetilde{B}(\underset{\sim}{\mu}) = \begin{bmatrix} \int_0^1 \frac{\partial b_1}{\partial u_1}(\underset{\sim}{\mu} + t[\underset{\sim}{u} + t[\underset{\sim}{u} - \underset{\sim}{\mu}])\,dt & \int_0^1 \frac{\partial b_1}{\partial u_2}(\underset{\sim}{\mu} + t[\underset{\sim}{u} - \underset{\sim}{\mu}])\,dt \\ \int_0^1 \frac{\partial b_2}{\partial u_1}(\underset{\sim}{\mu} + t[\underset{\sim}{u} - \underset{\sim}{\mu}])\,dt & \int_0^1 \frac{\partial b_2}{\partial u_2}(\underset{\sim}{\mu} + t[\underset{\sim}{u} - \underset{\sim}{\mu}])\,dt \end{bmatrix}.$$

Note that $\widetilde{B}(\underset{\sim}{\mu})$ and $\widetilde{H}_j(\underset{\sim}{\mu})$, $j = 1, 2$, are well defined since they involve evaluations of first and second derivatives of $\underset{\sim}{b}$ along the segment joining the center $\underset{\sim}{u}$ of $\mathscr{B}_0$ with the point $\underset{\sim}{\mu}$ interior to the ball. Furthermore, they are bounded (matrix) functions since $\underset{\sim}{a}$ has two continuous and bounded derivatives, and its (continuous) Jacobian is bounded away from 0. Combining (2.1)–(2.5), we arrive at a more useful set of error equations:

$$(2.6) \qquad \begin{cases} (B(\underset{\sim}{u})(\underset{\sim}{u} - \underset{\sim}{u}_h), \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, p - p_h) \\ \qquad = ((\underset{\sim}{u} - \underset{\sim}{u}_h)^T[\widetilde{H}_1(\underset{\sim}{u}_h), \widetilde{H}_2(\underset{\sim}{u}_h)](\underset{\sim}{u} - \underset{\sim}{u}_h), \underset{\sim}{v}), & \underset{\sim}{v} \in \underset{\sim}{V}_h, \\ (\operatorname{div}(\underset{\sim}{u} - \underset{\sim}{u}_h), w) = 0, & w \in W_h. \end{cases}$$

Let us choose now $\underset{\sim}{V}_h \times W_h$ as the Raviart-Thomas space of index $k > 0$ and introduce the L^2-projection

$$P_h \colon W \to W_h,$$

and the Raviart-Thomas projection [11]

$$\pi_h \colon H^1(\Omega)^2 \to \underset{\sim}{V}_h,$$

which have the following useful commuting property:

$$(2.7) \qquad \operatorname{div} \circ \pi_h = P_h \circ \operatorname{div} \colon H^1(\Omega)^2 \to W_h.$$

These projections have the following approximation properties [4, 9, 11]:

$$(2.8) \quad \|w - P_h w\|_{0,r} \le C h^\alpha \|w\|_{\alpha,r}, \qquad 0 \le \alpha \le k+1, \qquad 1 \le r \le +\infty,$$

$$(2.9) \quad \|\underset{\sim}{v} - \pi_h \underset{\sim}{v}\|_{0,r} \le C h^\alpha \|\underset{\sim}{v}\|_{\alpha,r}, \qquad \frac{1}{r} < \alpha \le k+1, \qquad 1 \le r \le +\infty,$$

where $\|\cdot\|_{\alpha,r}$ denotes the standard norm in the Sobolev space $W^{\alpha,r}(\Omega)$ ($r = 2$ being omitted in this notation).

Using (2.7), we rewrite (2.6) as

(2.10)

$$\begin{cases} (B(\underset{\sim}{u})[\pi_h \underset{\sim}{u} - \underset{\sim}{u}_h], \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, P_h p - p_h) \\ = ((\underset{\sim}{u} - \underset{\sim}{u}_h)^T[\widetilde{H}_1(\underset{\sim}{u}_h), \widetilde{H}_2(\underset{\sim}{u}_h)](\underset{\sim}{u} - \underset{\sim}{u}_h) + B(\underset{\sim}{u}) + B(\underset{\sim}{u})[\pi_h \underset{\sim}{u} - \underset{\sim}{u}], \underset{\sim}{v}), & \underset{\sim}{v} \in \underset{\sim}{V}_h, \\ (\operatorname{div}[\pi_h \underset{\sim}{u} - \underset{\sim}{u}_h], w) = 0, & w \in W_h. \end{cases}$$

Consider now the following (selfadjoint) operator $M = M^*: H^2(\Omega) \to L^2(\Omega)$,

$$Mw = -\operatorname{div}(B^{-1}(\underset{\sim}{u}) \underset{\sim}{\nabla} w), \tag{2.11}$$

whose restriction to $H^2(\Omega) \cap H_0^1(\Omega)$ has a bounded inverse. That is, for any $\psi \in L^2(\Omega)$, there is a unique $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $M\phi = \psi$ and $\|\phi\|_2 \le Q\|\psi\|_0$. This is guaranteed by the assumption that $B^{-1}(\underset{\sim}{u}) = A(\underset{\sim}{u}) = \frac{\partial(a_1, a_2)}{\partial(z_1, z_2)}(\underset{\sim}{u})$ is positive definite, since $\underset{\sim}{u}$ is uniformly Lipschitz [6].

We shall show that (2.1) is uniquely solvable by using a fixed point argument. Let h be small enough that $\pi_h \underset{\sim}{u} \in \underset{\sim}{V}_h \cap \mathscr{B}_0$, and choose a ball $\mathscr{B}_1$ centered at $\pi_h \underset{\sim}{u}$ such that $\mathscr{B}_1 \subset \mathscr{B}_0$ with respect to the L^∞-norm. Let now

$$\Phi: (\mathscr{B}_1 \cap \underset{\sim}{V}_h) \times W_h \to \underset{\sim}{V}_h \times W_h$$

be given by

$$\Phi((\underset{\sim}{\mu}, \rho)) = (\underset{\sim}{y}, q),$$

where $(\underset{\sim}{y}, q)$ is the solution of the system

(2.12)
$$\begin{cases} (B(\underset{\sim}{u})[\pi_h\underset{\sim}{u} - \underset{\sim}{y}], \underset{\sim}{v}) - (\operatorname{div}\underset{\sim}{v}, P_h p - q) \\ \qquad = (B(\underset{\sim}{u})[\pi_h\underset{\sim}{u} - \underset{\sim}{u}] + (\underset{\sim}{u} - \underset{\sim}{\mu})^T[\widetilde{H}_1(\underset{\sim}{\mu}), \widetilde{H}_2(\underset{\sim}{\mu})](\underset{\sim}{u} - \underset{\sim}{\mu}), \underset{\sim}{v}), & \underset{\sim}{v} \in \underset{\sim}{V}_h, \\ (\operatorname{div}[\pi_h\underset{\sim}{u} - \underset{\sim}{y}], w) = 0, & w \in W_h. \end{cases}$$

Note that, since the left-hand side of (2.12) corresponds to the mixed finite element method for the operator M given by (2.11) with $B(\underset{\sim}{u})$ smooth and $\widetilde{H}_1(\cdot)$ and $\widetilde{H}_2(\cdot)$ uniformly bounded on $\mathscr{B}_1$, the operator Φ is well defined [3]. Clearly, in order to establish the solvability of (1.5), it suffices to prove the following theorem (compare (2.10) with (2.12)):

Theorem 2.1. *For h sufficiently small, Φ has a fixed point.*

We shall need the following technical result, the proof of which follows in an analogous way to the one used in [9].

Lemma 2.2. *Let $\underset{\sim}{\omega} \in \underset{\sim}{V}$, $\underset{\sim}{l} \in L^2(\Omega)^2$, and $m \in L^2(\Omega)$. If $\tau \in W_h$ satisfies the relations*

$$\begin{cases} (B(\underset{\sim}{u})\underset{\sim}{\omega}, \underset{\sim}{v}) - (\operatorname{div}\underset{\sim}{v}, \tau) = (\underset{\sim}{l}, \underset{\sim}{v}), & \underset{\sim}{v} \in \underset{\sim}{V}_h, \\ (\operatorname{div}\underset{\sim}{\omega}, w) = (m, w), & w \in W_h, \end{cases} \tag{2.13}$$

then there exists a constant $C > 0$, independent of h, but dependent on $\| \underset{\sim}{u} \|_{C^{0,1}(\overline{\Omega})^2}$, such that

$$\|\tau\|_0 \le C[h\| \underset{\sim}{\omega} \|_0 + h^2\| \operatorname{div} \underset{\sim}{\omega} \|_0 + \| \underset{\sim}{l} \|_0 + \|m\|_0].$$

Let now $\underset{\sim}{\mathscr{V}}_h = \underset{\sim}{V}_h$ endowed with the strong norm

$$\| \underset{\sim}{v} \|_{\underset{\sim}{\mathscr{V}}_h} = \| \underset{\sim}{v} \|_{0,4+\varepsilon} + \| \operatorname{div} \underset{\sim}{v} \|_0.$$

Theorem 2.1 will be true if we prove the following result.

Theorem 2.3. *For $\delta > 0$ sufficiently small (dependent on h via the inverse inequality* (2.18)*, and smaller than the radius of $\mathscr{B}_1$ so that Φ is well defined on $\mathscr{B}_1$), Φ maps the ball of radius δ of $\underset{\sim}{\mathscr{V}}_h \times W_h$, centered at $(\pi_h \underset{\sim}{u}, P_h p)$, into itself.*

Proof. Let $\|\pi_h \underset{\sim}{u} - \underset{\sim}{\mu}\|_{\underset{\sim}{\mathscr{V}}_h} \le \delta$ and $\|P_h p - \rho\|_{W_h} \le \delta$. We use in the sequel $\theta = 4 + \varepsilon$ $(0 < \varepsilon \ll 1)$ and $s = 1 + \varepsilon + \frac{\varepsilon}{4+\varepsilon}$. Then, $s - \frac{2}{\theta} = \frac{1}{2} + \varepsilon_0$, where $0 < \varepsilon_0 = \varepsilon + \frac{3\varepsilon}{8+2\varepsilon} \ll 1$. Therefore, the Sobolev embedding theorem implies that

$$H^{3/2+\varepsilon_0}(\Omega) \subset W^{s,\theta}(\Omega) \quad \text{and} \quad \|\chi\|_{s,\theta} \le Q_\varepsilon \|\chi\|_{3/2+\varepsilon_0}.$$

Note that the second equation in (2.12) implies that

$$\operatorname{div}(\pi_h \underset{\sim}{u} - \underset{\sim}{y}) \equiv 0. \tag{2.14}$$

We now apply Lemma 2.2 to (2.12), using $\underset{\sim}{\omega} = \pi_h \underset{\sim}{u} - \underset{\sim}{y}$, $\tau = P_h p - q$, $m = 0$, and $\underset{\sim}{l} = B(\underset{\sim}{u})[\pi_h \underset{\sim}{u} - \underset{\sim}{u}] + (\underset{\sim}{u} - \underset{\sim}{\mu})^T[\widetilde{H}_1(\underset{\sim}{\mu}), \widetilde{H}_2(\underset{\sim}{\mu})](\underset{\sim}{u} - \underset{\sim}{\mu})$ in (2.13). It follows from (2.8) and (2.9) that

$$\begin{aligned} \|P_h p - q\|_0 &\le C[h\|\pi_h \underset{\sim}{u} - \underset{\sim}{y}\|_0 + h^s\|\underset{\sim}{u}\|_s + \|\underset{\sim}{u} - \underset{\sim}{\mu}\|_{0,4}^2] \\ &\le C[h\|\pi_h \underset{\sim}{u} - \underset{\sim}{y}\|_0 + h^s\|\underset{\sim}{u}\|_s + \|\underset{\sim}{u} - \underset{\sim}{\mu}\|_{0,\theta}^2] \\ &\le C[h\|\pi_h \underset{\sim}{u} - \underset{\sim}{y}\|_0 + (h^s + \delta^2)(1 + \|\underset{\sim}{u}\|_{s,\theta})^2], \end{aligned} \tag{2.15}$$

where C depends on $\|\underset{\sim}{u}\|_{C^{0,1}(\overline{\Omega})^2}$, as we have used Lemma 2.2. Note that, since $s > 1$ is being required in the approximation by projection, we may not use the Raviart-Thomas space of index $k = 0$, which is why we have assumed $k > 0$. Next, it is easy to show (see [9]) that

$$\begin{aligned} \|\pi_h \underset{\sim}{u} - \underset{\sim}{y}\|_0 &\le C\|\underset{\sim}{l}\|_0 \\ &\le C(h^s + \delta^2)(1 + \|\underset{\sim}{u}\|_{3/2+\varepsilon_0})^2, \end{aligned} \tag{2.16}$$

where C depends on $\|\underset{\sim}{u}\|_{C^{0,1}(\overline{\Omega})^2}$. Substituting (2.16) in (2.15) yields the relation

$$\|P_h p - q\|_0 \le K_1[h^s + \delta^2], \tag{2.17}$$

where K_1 depends on $\|\underset{\sim}{u}\|_{3/2+\varepsilon_0}$ and $\|\underset{\sim}{u}\|_{C^{0,1}(\overline{\Omega})^2}$. We use now the following "inverse-type" estimate [2]: for $2 \le \nu \le \theta$,

$$\|\pi_h \underset{\sim}{u} - \underset{\sim}{y}\|_{0,\theta} \le C h^{2/\theta - 2/\nu} \|\pi_h \underset{\sim}{u} - \underset{\sim}{y}\|_{0,\nu}. \tag{2.18}$$

Combining (2.14) and (2.18), we see that (2.17) yields

$$\begin{aligned} \|\pi_h \underset{\sim}{u} - \underset{\sim}{y}\|_{\underset{\sim}{\mathscr{V}}_h} &= \|\pi_h \underset{\sim}{u} - \underset{\sim}{y}\|_{0,4+\varepsilon} \\ &\le K_\varepsilon K_1 h^{\frac{2}{4+\varepsilon} - 1}[h^s + \delta^2] \\ &= K_2[h^{s - \frac{2+\varepsilon}{4+\varepsilon}} + h^{-\frac{2+\varepsilon}{4+\varepsilon}}\delta^2] \\ &= K_2[h^{\frac{1}{2} + \varepsilon + \frac{\varepsilon}{8+2\varepsilon}} + h^{-\frac{2+\varepsilon}{4+\varepsilon}}\delta^2]. \end{aligned} \tag{2.19}$$

Since we want that $K_2 h^{\frac{1}{2}+\varepsilon+\frac{\varepsilon}{8+2\varepsilon}} \le \frac{\delta}{2}$ and $K_2 h^{-\frac{2+\varepsilon}{4+\varepsilon}}\delta^2 \le \frac{\delta}{2}$ in (2.19), we need

$$2K_2 h^{\frac{1}{2}+\varepsilon+\frac{\varepsilon}{8+2\varepsilon}} \le \delta \le \frac{1}{2K_2} h^{\frac{1}{2}+\frac{\varepsilon}{8+2\varepsilon}}. \tag{2.20}$$

Let $h \le (2K_2)^{-2/\varepsilon}$ and $\delta = 2K_2 h^{\frac{1}{2}+\varepsilon+\frac{\varepsilon}{8+2\varepsilon}}$. It follows that (2.20) holds, and then (2.19) yields

$$\|\pi_h \underset{\sim}{u} - \underset{\sim}{v}\|_{\underset{\sim}{\mathscr{V}}_h} \le \delta.$$

Similarly (by choosing $K_2 \le K_1$), (2.17) gives the bound

$$\|P_h p - q\|_{W_h} \le \delta,$$

which concludes the proof. □

Remark 2.4. Note that Theorem 2.3 not only proves that (1.5) is solvable, but also that the solution is close to $(\underset{\sim}{u}, p)$. Specifically, for small h,

$$\|\pi_h \underset{\sim}{u} - \underset{\sim}{u}_h\|_{\underset{\sim}{\mathscr{V}}_h} + \|P_h p - p_h\|_0 \le K h^{\frac{1}{2}+\varepsilon+\frac{\varepsilon}{8+2\varepsilon}}.$$

By the inverse inequality (2.18), this implies that

$$\begin{aligned} \|\pi_h \underset{\sim}{u} - \underset{\sim}{u}_h\|_{0,\infty} &\le C h^{-\frac{2}{4+\varepsilon}} \|\pi_h \underset{\sim}{u} - \underset{\sim}{u}_h\|_{0,4+\varepsilon} \\ &\le \overline{C} h^{\varepsilon\frac{5+\varepsilon}{4+\varepsilon}}, \end{aligned} \tag{2.21}$$

where $\overline{C}$ depends on $\|\underset{\sim}{u}\|_{3/2+\varepsilon_0}$ and $\|\underset{\sim}{u}\|_{C^{0,1}(\overline{\Omega})^2}$. We shall need this estimate in the next section.

We can also show that the solution of (1.5) is unique (near $(\underset{\sim}{u}, p)$).

Theorem 2.5. *Let* $(\underset{\sim}{u}_h, p_h)$ *and* $(\underset{\sim}{v}_h, q_h)$ *be solutions of* (1.5). *Then,* $\underset{\sim}{u}_h = \underset{\sim}{v}_h$ *and* $p_h = q_h$.

Proof. Let $\underset{\sim}{U} = \underset{\sim}{u}_h - \underset{\sim}{v}_h$ and $P = p_h - q_h$. Then, (1.5) implies that $(\underset{\sim}{U}, P) \in \underset{\sim}{V}_h \times W_h$ satisfies the relations

$$\begin{cases} (\underset{\sim}{b}(\underset{\sim}{u}_h) - \underset{\sim}{b}(\underset{\sim}{v}_h), \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, P) = 0, & \underset{\sim}{v} \in \underset{\sim}{V}_h, \\ (\operatorname{div} \underset{\sim}{U}, w) = 0, & w \in W_h, \end{cases}$$

and, using the mean value theorem, (2.2), we obtain

$$\begin{cases} (\overline{B}(\underset{\sim}{u}_h)\underset{\sim}{U}, \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, P) = 0, & \underset{\sim}{v} \in \underset{\sim}{V}_h, \\ (\operatorname{div} \underset{\sim}{U}, w) = 0, & w \in W_h, \end{cases} \tag{2.22}$$

where $\overline{B}(\underset{\sim}{u}_h)$ is given by (2.5) with $\underset{\sim}{\mu} = \underset{\sim}{u}_h$ and $\underset{\sim}{u}$ replaced by $\underset{\sim}{v}_h$.

The crucial observation is now that, for h sufficiently small, the positive definiteness of $B(\underset{\sim}{u})$ together with (2.21) imply the positive definiteness of $\overline{B}(\underset{\sim}{u}_h)$. That is,

$$\tilde{\lambda}\|\underset{\sim}{v}\|_0^2 \le (\overline{B}(\underset{\sim}{u}_h), \underset{\sim}{v}, \underset{\sim}{v}), \qquad \underset{\sim}{v} \in \underset{\sim}{V}, \tag{2.23}$$

where $\tilde{\lambda} > 0$ is independent of h and $\underset{\sim}{v}$. Take now $\underset{\sim}{v} = \underset{\sim}{U}$ and $w = P$ in (2.22). Using (2.23), we see that $\underset{\sim}{U} = \underset{\sim}{0}$ and hence $(\operatorname{div} \underset{\sim}{v}, P) = 0$, $\underset{\sim}{v} \in \underset{\sim}{V}_h$. Choose $\underset{\sim}{v} \in \underset{\sim}{V}_h$ such that $\operatorname{div} \underset{\sim}{v} = P \in W_h$. Then $\|P\|_0 = 0$ gives $P = 0$, as needed. □

3. L^2-ERROR ESTIMATES

Let $\underset{\sim}{\zeta} = \underset{\sim}{u} - \underset{\sim}{u}_h$, $\underset{\sim}{\sigma} = \pi_h \underset{\sim}{u} - \underset{\sim}{u}_h$, $\xi = p - p_h$, and $\tau = P_h p - p_h$, and rewrite (2.1) as

$$
(3.1) \qquad \begin{cases} (\widetilde{B}(\underset{\sim}{u}_h)\underset{\sim}{\zeta}, \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, \xi) = 0, & \underset{\sim}{v} \in \underset{\sim}{V}_h, \\ (\operatorname{div} \underset{\sim}{\zeta}, w) = 0, & w \in W_h \end{cases}
$$

or

$$
(3.2) \qquad \begin{cases} (\widetilde{B}(\underset{\sim}{u}_h)\underset{\sim}{\sigma}, \underset{\sim}{v}) - (\operatorname{div} v, \tau) = (\widetilde{B}(\underset{\sim}{u}_h)[\pi_h \underset{\sim}{u} - \underset{\sim}{u}], \underset{\sim}{v}), & \underset{\sim}{v} \in \underset{\sim}{V}_h, \\ (\operatorname{div} \underset{\sim}{\sigma}, w) = 0, & w \in W_h. \end{cases}
$$

With the choices $\underset{\sim}{v} = \underset{\sim}{\sigma}$ and $w = \tau$ in (3.2), we see that

$$
(\widetilde{B}(\underset{\sim}{u}_h)\underset{\sim}{\sigma}, \underset{\sim}{\sigma}) = (\widetilde{B}(\underset{\sim}{u}_h)[\pi_h \underset{\sim}{u} - \underset{\sim}{u}], \underset{\sim}{\sigma}).
$$

Then, the positive definiteness of $\widetilde{B}(\underset{\sim}{u}_h)$, the analogue of (2.23), implies that $\|\underset{\sim}{\sigma}\|_0 \le C\|\pi_h \underset{\sim}{u} - \underset{\sim}{u}\|_0$. Thus, by (2.9), if $\underset{\sim}{u} \in H^\alpha(\Omega)^2$, $1/2 < \alpha \le k+1$, we have

$$
(3.3) \qquad \|\underset{\sim}{\zeta}\|_0 \le C\|\pi_h \underset{\sim}{u} - \underset{\sim}{u}\|_0 \le Ch^\alpha \|\underset{\sim}{u}\|_\alpha,
$$

where C depends on $\|\underset{\sim}{u}\|_{C^{0,1}(\overline{\Omega})^2}$. Observing that $\operatorname{div} \underset{\sim}{\sigma} = 0$ from (2.7) and (3.2), if $f \in W^{\gamma,q}(\Omega)$, $0 \le \gamma \le k+1$, $1 \le q \le +\infty$, we obtain from (2.7) and (2.8) the estimate

$$
(3.4) \qquad \begin{aligned} \|\operatorname{div} \underset{\sim}{\zeta}\|_{0,q} &= \|\operatorname{div} \underset{\sim}{u} + \operatorname{div} \underset{\sim}{\sigma} - \operatorname{div} \pi_h \underset{\sim}{u}\|_{0,q} \\ &= \|\operatorname{div} \underset{\sim}{u} - P_h \operatorname{div} \underset{\sim}{u}\|_{0,q} \\ &= \|f - P_h f\|_{0,q} \\ &\le Ch^\gamma \|f\|_{\gamma,q} \\ &\le Ch^\gamma \|\underset{\sim}{u}\|_{\gamma+1,q}. \end{aligned}
$$

Error estimates for the scalar function p_h will be obtained by the following Lemma 3.1. Using (2.2), we can rewrite the error equations (3.1) as

$$
(3.5) \qquad \begin{cases} (B(\underset{\sim}{u})\underset{\sim}{\zeta}, \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, \tau) = (\underset{\sim}{\zeta}^T[\widetilde{H}_1(\underset{\sim}{u}_h), \widetilde{H}_2(\underset{\sim}{u}_h)]\underset{\sim}{\zeta}, \underset{\sim}{v}), & \underset{\sim}{v} \in \underset{\sim}{V}_h, \\ (\operatorname{div} \underset{\sim}{\zeta}, w) = 0, & w \in W_h. \end{cases}
$$

We shall prove now an improved version of Lemma 2.2 .

Lemma 3.1. *Assume that the solution of* (1.1) *is sufficiently smooth. Then, there is a constant* C *independent of* h, *but dependent on* $\| \underset{\sim}{u} \|_{3/2+\varepsilon_0}$ *and* $\| \underset{\sim}{u} \|_{C^{s,1}(\overline{\Omega})^2}$, *such that, for* $0 \le s \le k+1$, $\frac{1}{2} < \alpha$, $\beta \le k+1$, $1 \le \gamma \le k+1$,

$$\|\tau\|_{-s} \le C[h^{\gamma+s+1-(s+1-k)^+} \| \underset{\sim}{u} \|_\gamma + h^{\alpha+\beta-\varepsilon} \| \underset{\sim}{u} \|_\alpha \| \underset{\sim}{u} \|_\beta],$$

where $(s+1-k)^+ = \max\{0, s+1-k\}$.

Proof. Recall that

$$\|\tau\|_{-s} = \sup_{\substack{\psi \in H^s(\Omega) \\ \psi \neq 0}} \frac{(\tau, \psi)}{\|\psi\|_s}.$$

We shall employ duality to bound (τ, ψ) for $\psi \in H^s(\Omega)$. Let $\phi \in H^{s+2}(\Omega) \cap H_0^1(\Omega)$ be the unique solution of $M^*\phi = \psi$ in Ω such that $\|\phi\|_{s+2} \le K\|\psi\|_s$, where M^* is given by (2.11), and K depends on $\| \underset{\sim}{u} \|_{C^{s,1}(\overline{\Omega})^2}$. Then, it follows from (2.7) and (3.5) that

$$\begin{aligned}
(\tau, \psi) &= (\tau, M^*\phi) = (\tau, -\operatorname{div}(B^{-1}(\underset{\sim}{u})\underset{\sim}{\nabla}\phi)) \\
&= (\tau, -\operatorname{div}(\pi_h(B^{-1}(\underset{\sim}{u})\underset{\sim}{\nabla}\phi))) \\
&= (\underset{\sim}{\zeta}^T[\widetilde{H}_1(\underset{\sim}{u}_h), \widetilde{H}_2(\underset{\sim}{u}_h)]\underset{\sim}{\zeta} - B(\underset{\sim}{u})\underset{\sim}{\zeta}, \pi_h(B^{-1}(\underset{\sim}{u})\underset{\sim}{\nabla}\phi)) \\
&= (\underset{\sim}{\zeta}^T[\widetilde{H}_1(\underset{\sim}{u}_h), \widetilde{H}_2(\underset{\sim}{u}_h)]\underset{\sim}{\zeta} - B(\underset{\sim}{u})\underset{\sim}{\zeta}, \pi_h(B^{-1}(\underset{\sim}{u})\underset{\sim}{\nabla}\phi) - B^{-1}(\underset{\sim}{u})\underset{\sim}{\nabla}\phi)) \\
&\quad + (\underset{\sim}{\zeta}^T[\widetilde{H}_1(\underset{\sim}{u}_h), \widetilde{H}_2(\underset{\sim}{u}_h)]\underset{\sim}{\zeta}, B^{-1}(\underset{\sim}{u})\underset{\sim}{\nabla}\phi) + (\operatorname{div}\underset{\sim}{\zeta}, \phi - P_h\phi).
\end{aligned} \tag{3.6}$$

Also, Sobolev's embedding theorem, (2.9), (2.18), and (3.3) lead to the following bound: for $\frac{1}{2} < \alpha$, $\beta \le k+1$,

$$\begin{aligned}
&(\underset{\sim}{\zeta}^T[\widetilde{H}_1(\underset{\sim}{u}_h), \widetilde{H}_2(\underset{\sim}{u}_h)]\underset{\sim}{\zeta}, \pi_h B^{-1}(\underset{\sim}{u})\underset{\sim}{\nabla}\phi - B^{-1}(\underset{\sim}{u})\underset{\sim}{\nabla}\phi) \\
&\quad \le C\|\underset{\sim}{\zeta}\|_0\|\underset{\sim}{\zeta}\|_{0,2+\varepsilon}\|\pi_h A(\underset{\sim}{u})\underset{\sim}{\nabla}\phi - A(\underset{\sim}{u})\underset{\sim}{\nabla}\phi\|_{0,\frac{4+2\varepsilon}{\varepsilon}} \\
&\quad \le C\|\underset{\sim}{\zeta}\|_0[\|\underset{\sim}{u}_h - \pi_h\underset{\sim}{u}\|_{0,2+\varepsilon} + \|\pi_h\underset{\sim}{u} - \underset{\sim}{u}\|_{0,2+\varepsilon}]h^{\frac{\varepsilon}{2+\varepsilon}}\|\underset{\sim}{\nabla}\phi\|_{\frac{\varepsilon}{2+\varepsilon},\frac{4+2\varepsilon}{\varepsilon}} \\
&\quad \le Ch^\alpha\|\underset{\sim}{u}\|_\alpha(\|\underset{\sim}{u}_h - \pi_h\underset{\sim}{u}\|_0 + h^\beta\|\underset{\sim}{u}\|_{\beta-\frac{\varepsilon}{2+\varepsilon},2+\varepsilon})\|\phi\|_2 \\
&\quad \le Ch^\alpha\|\underset{\sim}{u}\|_\alpha(\|\underset{\sim}{\zeta}\|_0 + \|\pi_h\underset{\sim}{u} - \underset{\sim}{u}\|_0 + h^\beta\|\underset{\sim}{u}\|_\beta)\|\phi\|_2 \\
&\quad \le Ch^\alpha\|\underset{\sim}{u}\|_\alpha h^\beta\|\underset{\sim}{u}\|_\beta\|\phi\|_{s+2} \\
&\quad \le Ch^{\alpha+\beta}\|\underset{\sim}{u}\|_\alpha\|\underset{\sim}{u}\|_\beta\|\phi\|_{s+2},
\end{aligned} \tag{3.7}$$

while (2.9) and (3.3) yield, for $\frac{1}{2} < \alpha \le k+1$,

$$
\begin{aligned}
(B(\underset{\sim}{u})\underset{\sim}{\zeta}, B^{-1}(\underset{\sim}{u})\underset{\sim}{\nabla}\phi - \pi_h(B^{-1}(\underset{\sim}{u})\underset{\sim}{\nabla}\phi))& \\
&\le C\|\underset{\sim}{\zeta}\|_0 \|B^{-1}(\underset{\sim}{u})\underset{\sim}{\nabla}\phi - \pi_h(B^{-1}(\underset{\sim}{u})\underset{\sim}{\nabla}\phi)\|_0 \\
&\le C\|\underset{\sim}{\zeta}\|_0 h^{s+1-\delta_{s,k+1}} \|\underset{\sim}{\nabla}\phi\|_{s+1-\delta_{s,k+1}} \\
&\le Ch^{\alpha}\|\underset{\sim}{u}\|_{\alpha} h^{s+1-\delta_{s,k+1}} \|\phi\|_{s+2} \\
&\le Ch^{\alpha+s+1-\delta_{s,k+1}} \|\underset{\sim}{u}\|_{\alpha}\|\phi\|_{s+2}.
\end{aligned}
\tag{3.8}
$$

Next, note that Sobolev's embedding theorem, (2.9), (2.18), and (3.3) lead, in a similar way to how (3.7) was derived from them, to the following bound: for $\frac{1}{2} < \alpha,\ \beta \le k+1$,

$$
\begin{aligned}
(\underset{\sim}{\zeta}^T[\widetilde{H}_1(\underset{\sim}{u}_h), \widetilde{H}_2(\underset{\sim}{u}_h)]\underset{\sim}{\zeta}, B^{-1}(\underset{\sim}{u})\underset{\sim}{\nabla}\phi) &\le C\|\underset{\sim}{\zeta}\|_0 \|\underset{\sim}{\zeta}\|_{0,2+\varepsilon} \|\underset{\sim}{\nabla}\phi\|_{0,\frac{4+2\varepsilon}{\varepsilon}} \\
&\le Ch^{\alpha}\|\underset{\sim}{u}\|_{\alpha} h^{\beta-\frac{\varepsilon}{2+\varepsilon}} \|\underset{\sim}{u}\|_{\beta}\|\phi\|_2 \\
&\le Ch^{\alpha+\beta-\varepsilon}\|\underset{\sim}{u}\|_{\alpha}\|\underset{\sim}{u}\|_{\beta}\|\phi\|_{s+2}.
\end{aligned}
\tag{3.9}
$$

Finally, (2.8) and (3.4) yield, for $0 \le \gamma \le k+1$,

$$
\begin{aligned}
(\operatorname{div}\underset{\sim}{\zeta}, \phi - P_h\phi) &\le \|\operatorname{div}\underset{\sim}{\zeta}\|_0 \|\phi - P_h\phi\|_0 \\
&\le Ch^{\gamma+s+2-(s+1-k)^+} \|\underset{\sim}{u}\|_{\gamma+1} \|\phi\|_{s+2-(s+1-k)^+} \\
&\le Ch^{\gamma+s+2-(s+1-k)^+} \|\underset{\sim}{u}\|_{\gamma+1} \|\phi\|_{s+2},
\end{aligned}
\tag{3.10}
$$

and combining (3.6)–(3.10) we conclude the proof. □

Remark 3.2. Lemma 3.1 shows, in particular, that, if $\underset{\sim}{u} \in H^{\gamma}(\Omega)^2$, $1 \le \gamma \le k+1$, then

$$
\|\tau\|_0 \le Kh^{\gamma+1}\|\underset{\sim}{u}\|_{\gamma},
\tag{3.11}
$$

where K depends on $\|\underset{\sim}{u}\|_{3/2+\varepsilon_0}$ and $\|\underset{\sim}{u}\|_{C^{0,1}(\overline{\Omega})^2}$.

We shall now apply Lemma 3.1 to the derivation of L^q-error estimates for the scalar function p_h, for $2 \le q \le +\infty$.

Theorem 3.3. *Let* $2 \le q \le +\infty$. *There exist positive constants* C_q, *independent of* h, *but dependent on* $\|\underset{\sim}{u}\|_{3/2+\varepsilon_0}$ *and* $\|\underset{\sim}{u}\|_{C^{0,1}(\overline{\Omega})^2}$, *such that, if* $p \in W^{r,q}(\Omega)$ *and* $\underset{\sim}{u} \in H^{r-\frac{2}{q}}(\Omega)^2$, $1+\frac{2}{q} \le r \le k+1$, *then*

$$
\|p - p_h\|_{0,q} \le C_q h^r(\|p\|_{r,q} + \|\underset{\sim}{u}\|_{r-\frac{2}{q}}).
$$

Proof. Use Lemma 3.1 with $s = 0$, (2.8), and the estimate (3.11), and set $\gamma + \frac{2}{q} = r$. Then, by the scalar analogue of (2.18), we obtain, for $1+\frac{2}{q} \le r \le k+1$,

$$\begin{aligned}\|p-p_h\|_{0,q} &\le \|p-P_hp\|_{0,q}+\|\tau\|_{0,q}\\ &\le K_q[h^r\|p\|_{r,q}+h^{\frac{2}{q}-1}\|\tau\|_0]\\ &\le K_q[h^r\|p\|_{r,q}+h^{\frac{2}{q}-1}h^{\gamma+1}\|\underset{\sim}{u}\|_\gamma]\\ &\le C_qh^r(\|p\|_{r,q}+\|\underset{\sim}{u}\|_{r-\frac{2}{q}}). \quad \square\end{aligned}$$

Remark 3.4. Note that Theorem 3.3 shows, in particular, that,

$$\|p-p_h\|_0 \le Ch^r(\|p\|_r+\|\underset{\sim}{u}\|_{r-1}), \qquad 2\le r\le k+1,$$

$$\|p-p_h\|_{0,\infty} \le Ch^r(\|p\|_{r,\infty}+\|\underset{\sim}{u}\|_r), \qquad 1\le r\le k+1.$$

Also note that these estimates are both of optimal rate and, in case that $\|\underset{\sim}{u}\|_{r-1}\le C\|p\|_r$, as in the linear case, and $r>3$ (which requires $k>2$), then the L^2 estimate is also optimal in regularity (that is, $\|p-p_h\|_0\le Ch^r\|p\|_r$). This is true because Sobolev's embedding theorem gives $H^{2+\varepsilon}(\Omega)^2\subset C^{0,1}(\overline{\Omega})^2$.

4. L^∞-ERROR ESTIMATES FOR THE VECTOR FUNCTION $\underset{\sim}{u}_h$

In this section, we shall extend some of the ideas of [8]. First, we shall need an L^q version of the Duality Lemma 2.2.

Lemma 4.1. *Let* $2<\theta<\infty$ *and let* τ *be given by* (3.5). *Then the following relation holds*:

$$\|\tau\|_{0,\theta}\le C_\theta(h\|\underset{\sim}{\zeta}\|_{0,\theta}+h^2\|\operatorname{div}\underset{\sim}{\zeta}\|_{0,\theta}),$$

where C_θ *depends on* $\|\underset{\sim}{u}\|_{C^{0,1}(\overline{\Omega})^2}$.

Proof. We shall employ a duality argument. For $\psi\in L^{\theta'}(\Omega)$, let $\phi\in W^{2,\theta'}(\Omega)$ be the unique solution of $M^*\phi=\psi$ in Ω, $\phi=0$ on $\partial\Omega$ such that $\|\phi\|_{2,\theta'}\le K\|\psi\|_{0,\theta'}$. The result now follows from (3.6) by the same argument as used in Lemma 3.1. $\square$

To prove our L^∞-estimates, we shall use Nitsche's weighted L^2-norms [10]. Let $\rho>0$, $\mu(\underset{\sim}{x})=|\underset{\sim}{x}-\underset{\sim}{x}_0|^2+\rho^2$, $\underset{\sim}{x}_0\in\overline{\Omega}$ fixed, $\underset{\sim}{x}\in\overline{\Omega}$. Then, the weighted L^2-norm with weight μ is defined as

$$\|\underset{\sim}{v}\|_{r,\mu}=\|\mu^{-\frac{r}{2}}\underset{\sim}{v}\|_0, \qquad r\in\mathbb{R},\ \underset{\sim}{v}\in L^2(\Omega)^2.$$

We shall need several well-known properties of these norms, which we list as lemmas.

Lemma 4.2. *Let* $\theta\ge 2$. *If* $\underset{\sim}{\phi}\in L^{\theta'}(\Omega)^2$, $\frac{1}{\theta}+\frac{1}{\theta'}=1$, *then*

$$\|\underset{\sim}{\nabla}\mu^{-1}\cdot\underset{\sim}{\phi}\|_{0,\theta'}\le K\rho^{-1-\frac{2}{\theta}}\|\underset{\sim}{\phi}\|_{1,\mu}.$$

Lemma 4.3. *Given* $\theta\ge 2$, *if* $\underset{\sim}{\phi}\in L^\infty(\Omega)^2$, *then*

$$\|\underset{\sim}{\phi}\|_{0,\theta}\le K(\theta)\|\underset{\sim}{\phi}\|_{1,\mu}^{\frac{2}{\theta}}\|\underset{\sim}{\phi}\|_{0,\infty}^{1-\frac{2}{\theta}}.$$

We shall also need the following relations between L^∞-norms and weighted L^2-norms [10]:

$$\|\underset{\sim}{v}\|_{1,\mu} \le C|\ln\rho|^{\frac{1}{2}}\|\underset{\sim}{v}\|_{0,\infty}, \qquad \underset{\sim}{v} \in L^\infty(\Omega)^2; \tag{4.1}$$

if $w \in W_h$, $\underset{\sim}{x}_0 \in \overline{\Omega}$ is chosen such that $\|w\|_{0,\infty} = |w(x_0)|$, then

$$\|w\|_{0,\infty} \le C_\gamma h^{-1}\rho\|w\|_{1,\mu} \quad \text{for } \rho \le \gamma h, \tag{4.2}$$

$$\|\mu^{-1}\underset{\sim}{\eta} - \pi_h(\mu^{-1}\underset{\sim}{\eta})\|_{-1,\mu} \le Ch\rho^{-1}\|\underset{\sim}{\eta}\|_{1,\mu}, \qquad \underset{\sim}{\eta} \in L^2(\Omega)^2. \tag{4.3}$$

Theorem 4.4. *Assume that* $\underset{\sim}{u} \in W^{r,\infty}(\Omega)$, $\frac{1}{2} < r \le k+1$. *Then, for* h *sufficiently small,*

$$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{0,\infty} \le Ch^{r-\frac{1}{2}}|\ln h|^{\frac{1}{2}}\|\underset{\sim}{u}\|_{r,\infty},$$

where C *depends on* $\|\underset{\sim}{u}\|_{C^{0,1}(\overline{\Omega})^2}$.

Proof. In view of (2.9), it suffices to prove the theorem for $\underset{\sim}{\sigma}$ in place of $\underset{\sim}{\zeta} = \underset{\sim}{u} - \underset{\sim}{u}_h$. Note that it follows from (2.23) and (4.3) that

$$\begin{aligned}
\|\underset{\sim}{\sigma}\|_{1,\mu}^2 &= (\widetilde{B}(\underset{\sim}{u}_h)\underset{\sim}{\sigma}, \widetilde{B}^{-1}(\underset{\sim}{u}_h)\mu^{-1}\underset{\sim}{\sigma}) \\
&\le C(\widetilde{B}(\underset{\sim}{u}_h)\underset{\sim}{\sigma}, \mu^{-1}\underset{\sim}{\sigma}) \\
&= C\{(\widetilde{B}(\underset{\sim}{u}_h)\underset{\sim}{\sigma}, \mu^{-1}\underset{\sim}{\sigma} - \pi_h(\mu^{-1}\underset{\sim}{\sigma})) - (\widetilde{B}(\underset{\sim}{u}_h)[\underset{\sim}{u}_h - \underset{\sim}{u}], \pi_h(\mu^{-1}\underset{\sim}{\sigma})) \\
&\qquad - (\widetilde{B}(\underset{\sim}{u}_h)[\underset{\sim}{u} - \pi_h\underset{\sim}{u}], \pi_h(\mu^{-1}\underset{\sim}{\sigma}))\} \\
&\le C\{Ch\rho^{-1}\|\underset{\sim}{\sigma}\|_{1,\mu}^2 - (\widetilde{B}(\underset{\sim}{u}_h)[\underset{\sim}{u}_h - \underset{\sim}{u}], \pi_h(\mu^{-1}\underset{\sim}{\sigma})) \\
&\qquad + (1 + Ch\rho^{-1})\|\underset{\sim}{u} - \pi_h\underset{\sim}{u}\|_{1,\mu}\|\underset{\sim}{\sigma}\|_{1,\mu}\},
\end{aligned} \tag{4.4}$$

and hence, for $h < \gamma\rho$, γ sufficiently small, we get

$$\|\underset{\sim}{\sigma}\|_{1,\mu}^2 \le C\{\|\underset{\sim}{u} - \pi_h\underset{\sim}{u}\|_{1,\mu}^2 + (\widetilde{B}(\underset{\sim}{u}_h)[\underset{\sim}{u} - \underset{\sim}{u}_h], \pi_h(\mu^{-1}\underset{\sim}{\sigma}))\}. \tag{4.5}$$

Next, from (3.1), (2.7), and Lemma 4.2, we see that for any $\delta > 0$,

$$\begin{aligned}
(\widetilde{B}(\underset{\sim}{u}_h)[\underset{\sim}{u} - \underset{\sim}{u}_h], \pi_h(\mu^{-1}\underset{\sim}{\sigma})) &= (\operatorname{div}(\pi_h(\mu^{-1}\underset{\sim}{\sigma})), p - p_h) \\
&= (\operatorname{div}(\pi_h(\mu^{-1}\underset{\sim}{\sigma})), \tau) \\
&= (\operatorname{div}(\mu^{-1}\underset{\sim}{\sigma}), \tau) \\
&= (\underset{\sim}{\nabla}\mu^{-1} \cdot \underset{\sim}{\sigma}, \tau) + (\mu^{-1}\operatorname{div}\underset{\sim}{\sigma}, \tau) \\
&\le \|\underset{\sim}{\nabla}\mu^{-1} \cdot \underset{\sim}{\sigma}\|_{0,\theta'}\|\tau\|_{0,\theta} \\
&\le K\rho^{-1-\frac{2}{\theta}}\|\underset{\sim}{\sigma}\|_{1,\mu}\|\tau\|_{0,\theta} \\
&\le K(\delta)\{\rho^{-2-\frac{4}{\theta}}\|\tau\|_{0,\theta}^2\} + \delta\|\underset{\sim}{\sigma}\|_{1,\mu}^2.
\end{aligned}$$

Thus, (4.1) and (4.5) yield

$$\begin{aligned}\|\underset{\sim}{\sigma}\|_{1,\mu} &\le C\{\|\underset{\sim}{u}-\pi_h\underset{\sim}{u}\|_{1,\mu}+\rho^{-1-\frac{2}{\theta}}\|\tau\|_{0,\theta}\} \\ &\le C\{|\ln\rho|^{\frac12}\|\underset{\sim}{u}-\pi_h\underset{\sim}{u}\|_{0,\infty}+\rho^{-1-\frac{2}{\theta}}\|\tau\|_{0,\theta}\}.\end{aligned} \tag{4.6}$$

Next note that, by Lemmas 4.1 and 4.3, (3.4), and (4.2), we obtain, for $1\le r\le k+2$,

$$\begin{aligned}\|\tau\|_{0,\theta} &\le C[h\|\underset{\sim}{\zeta}\|_{0,\theta}+h^2\|\operatorname{div}\underset{\sim}{\zeta}\|_{0,\theta}] \\ &\le Ch[\|\underset{\sim}{\sigma}\|_{0,\theta}+h^r\|\underset{\sim}{u}\|_{r,\infty}] \\ &\le Ch[(h^{-1}\rho)^{1-\frac{2}{\theta}}\|\underset{\sim}{\sigma}\|_{1,\mu}+h^r\|\underset{\sim}{u}\|_{r,\infty}].\end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7), we arrive, using (2.9), at the relation

$$\|\underset{\sim}{\sigma}\|_{1,\mu}\le C(\theta)[(|\ln h|^{\frac12}+h\rho^{-1-\frac{2}{\theta}})h^r\|\underset{\sim}{u}\|_{r,\infty}+h^{\frac{2}{\theta}}\rho^{-\frac{4}{\theta}}\|\underset{\sim}{\sigma}\|_{1,\mu}], \qquad 0<r\le k+1. \tag{4.8}$$

Now fix $\theta\in(2,\infty)$ and let

$$h^{\frac{2}{\theta}}\rho^{-\frac{4}{\theta}}=\frac{1}{2C(\theta)},\quad \text{that is, } h=\rho^2(2C(\theta))^{-\frac{\theta}{2}}. \tag{4.9}$$

Then, by (4.8),

$$\|\underset{\sim}{\sigma}\|_{1,\mu}\le C|\ln h|^{\frac12}h^r\|\underset{\sim}{u}\|_{r,\infty},\qquad 0<r\le k+1.$$

Finally, by (4.2) and (4.9),

$$\begin{aligned}\|\underset{\sim}{\sigma}\|_{0,\infty} &\le Ch^{-1}\rho\|\underset{\sim}{\sigma}\|_{1,\mu} \\ &\le C|\ln h|^{\frac12}h^{r-\frac12}\|\underset{\sim}{u}\|_{r,\infty},\qquad \tfrac12<r\le k+1. \quad \square\end{aligned}$$

Remark 4.5. The error estimate of Theorem 4.4 is optimal in regularity (only for $r\ge1$, since the constant C depends on $\|\underset{\sim}{u}\|_{C^{0,1}(\overline{\Omega})^2}$ for any r), but it is one half power of h suboptimal in rate. This matches the estimate obtained in [9] for the quasilinear case, but it is not as sharp as that of [4].

We can now combine Lemma 4.1 and Theorem 4.4 to obtain a better L^∞-estimate for the error in p_h than the one given in Remark 3.4. That estimate requires as many derivatives in $\underset{\sim}{u}$ as in p, while the one which follows requires one half derivative less in $\underset{\sim}{u}$ than in p.

Theorem 4.6. *Assume that $p\in W^{r,\infty}(\Omega)$ and $\underset{\sim}{u}\in W^{r-\frac12,\infty}(\Omega)$ for some given ε, $0<\varepsilon\ll1$, and $1\le r\le k+1$. Then, for h sufficiently small,*

$$\|p-p_h\|_{0,\infty}\le Ch^r(\|p\|_{r,\infty}+\|\underset{\sim}{u}\|_{r-\frac12,\infty}),$$

where C depends on $\|\underset{\sim}{u}\|_{C^{0,1}(\overline{\Omega})^2}$.

Proof. It follows from (2.18), Lemma 4.1, and Theorem 4.4 that, for $\frac{1}{2} < \alpha \le k+1$, $0 \le \beta \le k+1$,

$$\begin{aligned}
\|\tau\|_{0,\infty} &\le Ch^{-\frac{\varepsilon}{2}}\|\tau\|_{0,\frac{4}{\varepsilon}} \\
&\le C(h^{1-\frac{\varepsilon}{2}}\|\underset{\sim}{\zeta}\|_{0,\frac{4}{\varepsilon}} + h^{2-\frac{\varepsilon}{2}}\|\operatorname{div}\underset{\sim}{\zeta}\|_{0,\frac{4}{\varepsilon}}) \\
&\le C[h^{1-\frac{\varepsilon}{2}+\alpha-\frac{1}{2}}|\ln h|^{\frac{1}{2}}\|\underset{\sim}{u}\|_{\alpha,\infty} + h^{2-\frac{\varepsilon}{2}}h^{\beta}\|\underset{\sim}{u}\|_{\beta+1,\infty}] \\
&\le C[h^{r}\|\underset{\sim}{u}\|_{r-\frac{1}{2}+\varepsilon,\infty} + h^{r+\frac{1}{2}+\frac{\varepsilon}{2}}\|\underset{\sim}{u}\|_{r-\frac{1}{2}+\varepsilon,\infty}] \\
&\le Ch^{\gamma}\|\underset{\sim}{u}\|_{\gamma-\frac{1}{2}+\varepsilon,\infty}, \qquad 1 \le \gamma \le k+\tfrac{3}{2}-\varepsilon,
\end{aligned}$$

and the result follows from (2.8). □

Remark 4.7. The error estimate for $\tau = P_h p - p_h$ in Theorem 4.6 is superconvergent by almost one half power of h in L^∞. This is an improvement over the first result for the quasi-linear case [8], but it is not as sharp as that of [4], which gives a superconvergent estimate by one power of h.

5. Application to the minimal surface equation

Let $p(\underset{\sim}{x})$ represent the vertical displacement of a surface (such as a membrane or soap film) at the point $\underset{\sim}{x} \in \Omega$, where the displacements at the boundary are prescribed by the function $-g$. Assume Ω is convex. Then, p is determined (uniquely if g and $\partial\Omega$ are sufficiently smooth) as the solution of the following boundary value problem:

$$(5.1) \qquad \begin{cases} -\operatorname{div}\left(\dfrac{\underset{\sim}{\nabla} p}{\sqrt{1+|\underset{\sim}{\nabla} p|^2}}\right) = 0 & \text{in } \Omega, \\ p = -g & \text{on } \partial\Omega. \end{cases}$$

We see that (5.1) corresponds to the general equation (1.1) with

$$\underset{\sim}{a}(\underset{\sim}{\nabla} p) = \frac{\underset{\sim}{\nabla} p}{\sqrt{1+|\underset{\sim}{\nabla} p|^2}}.$$

We let

$$\underset{\sim}{u} = \frac{-\underset{\sim}{\nabla} p}{\sqrt{1+|\underset{\sim}{\nabla} p|^2}},$$

so that

$$|\underset{\sim}{u}|^2 = \frac{|\underset{\sim}{\nabla} p|^2}{1+|\underset{\sim}{\nabla} p|^2}, \qquad |\underset{\sim}{\nabla} p|^2 = \frac{|\underset{\sim}{u}|^2}{1-|\underset{\sim}{u}|^2}.$$

Thus, we obtain

$$\underset{\sim}{b}(\underset{\sim}{u}) = -\underset{\sim}{\nabla} p = \frac{\underset{\sim}{u}}{\sqrt{1-|\underset{\sim}{u}|^2}}.$$

We compute now the Jacobian matrix B. Very straightforward calculations lead to the following expressions:

$$\begin{cases} \frac{\partial b_1}{\partial u_1} = \frac{1-u_2^2}{(1-|\underset{\sim}{u}|^2)^{\frac{3}{2}}}, \\ \frac{\partial b_1}{\partial u_2} = \frac{u_1 u_2}{(1-|\underset{\sim}{u}|^2)^{\frac{3}{2}}}, \\ \frac{\partial b_2}{\partial u_1} = \frac{u_1 u_2}{(1-|\underset{\sim}{u}|^2)^{\frac{3}{2}}}, \\ \frac{\partial b_2}{\partial u_2} = \frac{1-u_1^2}{(1-|\underset{\sim}{u}|^2)^{\frac{3}{2}}}. \end{cases}$$

Therefore, we see that

$$\frac{\partial \underset{\sim}{b}}{\partial \underset{\sim}{u}} = (1 - |\underset{\sim}{u}|^2)^{-\frac{3}{2}} \begin{bmatrix} 1 - u_2^2 & u_1 u_2 \\ u_1 u_2 & 1 - u_1^2 \end{bmatrix},$$

with eigenvalues

$$\begin{cases} \lambda_1 = \frac{1}{2}(\mathrm{tr}(B) - \sqrt{(\mathrm{tr}\, B)^2 - 4 \det B}) = \frac{1}{(1-|\underset{\sim}{u}|^2)^{\frac{1}{2}}} \equiv \lambda, \\ \lambda_2 = \frac{1}{(1-|\underset{\sim}{u}|^2)^{\frac{3}{2}}} \equiv \Lambda, \end{cases}$$

λ being the smaller eigenvalue and Λ the larger. In this case we have $\lambda \geq 1$, which means that

$$(B(\underset{\sim}{u})\underset{\sim}{v}, \underset{\sim}{v}) \geq \|\underset{\sim}{v}\|_0^2, \qquad \underset{\sim}{v} \in L^2(\Omega)^2.$$

However, in order for B to be bounded, we must be sure that $|\underset{\sim}{u}| < 1$. This is equivalent to $|\underset{\sim}{\nabla} p|$ being uniformly bounded in Ω, which holds under our assumptions. It then follows that $|\underset{\sim}{u}| \leq \rho < 1$ (for some positive constant ρ).

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