MATHEMATICS OF COMPUTATION
Volume 69, Number 230, Pages 653–666
S 0025-5718(99)01150-3
Article electronically published on April 28, 1999

SEMI-DISCRETIZATION OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS ON $\mathbb{R}^1$ BY A FINITE-DIFFERENCE METHOD

HYEK YOO

Abstract. The paper concerns finite-difference scheme for the approximation of partial differential equations in $\mathbb{R}^1$, with additional stochastic noise. By replacing the space derivatives in the original stochastic partial differential equation (SPDE, for short) with difference quotients, we obtain a system of stochastic ordinary differential equations. We study the difference between the solution of the original SPDE and the solution to the corresponding equation obtained by discretizing the space variable. The need to approximate the solution in $\mathbb{R}^1$ with functions of compact support requires us to introduce a scale of weighted Sobolev spaces. Employing the weighted L_p-theory of SPDE, a sup-norm error estimate is derived and the rate of convergence is given.

1. Introduction

The mathematical modeling of stochastic systems can be realized in such a fashion that the time and space behavior of the dependent variable is defined by the superposition of a deterministic evolution and an additional weighted noise. In this case, the weighted noise simulates the existence of the external field and the interaction between the actual system and the outer ambient. Quite often, the deterministic evolution is described by a partial differential equation of parabolic type. Thus, we are led to the following stochastic partial differential equation (we allow the coefficients a, b, c and f in the "deterministic" part to be random):

$$\begin{cases} du = (Lu + f)\,dt + \sum_{k=1}^{d'} (\Lambda^k u + g^k)\,dw_t^k, \\ u(0,\cdot) = u_0, \end{cases} \tag{1.1}$$

where $Lu = au'' + bu' + cu$, $\Lambda^k u = \sigma^k u' + \nu^k u$ and a, b, c, σ^k, ν^k, f, g^k are real valued functions defined on $\Omega \times [0,T] \times \mathbb{R}^1$. The w_t^k's are one-dimensional independent Wiener processes.

Stochastic PDEs of the form (1.1) have been extensively studied. Here we just mention Krylov [11] and Rozovskii [17], in which the reader can find further information.

Our aim is to study an approximate solution of (1.1) given by a finite-difference method, and to analyze the sup-norm error and the rate of convergence. The finite-difference scheme is one of the most frequently used methods for a finite dimensional

Received by the editor March 3, 1998 and, in revised form, July 10, 1998.
1991 *Mathematics Subject Classification.* Primary 35R60, 60H15, 65M06, 65M15.
Key words and phrases. Stochastic partial differential equations, finite-difference method, weighted spaces of Bessel potentials, embedding theorems, rate of convergence.

approximation of (deterministic) elliptic and parabolic PDEs. See [13], [18]. There are several recent papers on the finite-difference approximation of stochastic PDEs, Gyöngy [8], Davie-Gaines [5], Gaines [6]. These authors consider equations of the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f + \sigma \frac{\partial^2 W}{\partial t \partial x}, \tag{1.2}$$

where $\partial^2 W/\partial t\partial x$ is the space-time white noise. Since the coefficient of $\partial^2 u/\partial x^2$ is a constant, one can write down the solution of (1.2) and the solution of the finite dimensional approximation of (1.2) explicitly using the (explicit) Green's functions, and obtain the error bound and the rate of convergence using estimates of the Green's functions.

The study of numerical solutions of stochastic PDEs is a very active ongoing research area. There is an extensive literature on numerical methods for the Zakai equation of the filtering problem. We only mention [3] (Galerkin approximation), [7] (finite-element method), [4] (splitting-up method), [15] (Wiener chaos decomposition). We remark that in the above works L_2-space is used to measure the difference between the exact solution and the approximate solution.

For applications to continuum physics, see [1], [2], where the so-called stochastic interpolation method is developed and applied to many models of stochastic systems in continuum physics. In [2], Bellomo and Flandoli studied one-dimensional SPDE (in a bounded interval) of the form

$$du = [a(x)u'' + b(x)u' + f(t,x,u)]\,dt + \sum_{k=1}^{d'} \phi(t,x,u)\,dw_t^k.$$

They obtained some estimates of the error bound under suitable regularity assumptions on a, b, f, ϕ. Their approach is based on semigroup theory and stochastic interpolation methods. In their analysis, it was crucial that $a(x)$ and $b(x)$ are functions of x only.

Now we briefly describe the organization of the paper. In section 2, we obtain the existence of the solution of (1.1) in a weighted Sobolev space. The weight is introduced to deal with the fact that the solution in $\mathbb{R}^1$ has to be approximated by a function defined at a finite number of points. Using Sobolev-type embedding theorems, we prove that the solution is classical. The advantage of the L_p-theory (over L_2-theory) that we are using in this paper is that one can get the classical solution under much less restrictive conditions on the smoothness of the coefficients and the nonhomogeneous terms by taking sufficiently large p. This fact can be seen, for instance, from the embedding $W^{n,p}(\mathbb{R}^d) \subset C^{n-d/p}(\mathbb{R}^d)$, if $pn > d$. In section 3, we present a finite-difference scheme for (1.1). As a consequence of discretization, we obtain an Itô stochastic differential equation. We also show that the (average) error satisfies a "discrete parabolic equation". In section 4, we analyze the error bound. By the discrete maximum principle and results from sections 2 and 3, error estimates and the rate of convergence are obtained.

In this paper, we only present the semi-discretization in space of (1.1). One can obtain a fully discrete problem by discretizing time also. The reader is referred to [9], [16] and references therein. We also remark that we can consider a d-dimensional equation as well, without any additional difficulty. We consider the 1-dimensional case only for simplicity of notation.

We employ the summation convention throughout; the letter $N(\cdots)$ denotes various constants depending only on the quantities inside the parenthesis.

2. Weighted L_p-theory of SPDE

Let $\mathbb{R}^1$ be 1-dimensional Euclidean space, T a fixed positive number, $(\Omega, \mathcal{F}, P)$ a complete probability space, $(\{\mathcal{F}_t\}, t \geqslant 0)$ an increasing filtration of σ-fields $\mathcal{F}_t \subset \mathcal{F}$ containing all P-null subsets of Ω, and $\mathcal{P}$ the predictable σ-field generated by $\{\mathcal{F}_t\}$. Let $\{w_t^k; k = 1, 2, \cdots, d'\}$ be independent one-dimensional $\mathcal{F}_t$-adapted Wiener processes defined on $(\Omega, \mathcal{F}, P)$. For the above standard terminologies, the reader is referred to [12].

The argument $\omega \in \Omega$ is usually omitted. In places where there is no danger of confusion, other arguments may also be omitted.

We need some notations and definitions to determine in what sense a solution of the problem (1.1) should be understood and to formulate results on its solvability. The scales of function spaces defined below are straightforward generalizations of the (stochastic) Banach spaces introduced by Krylov [10], [11] and by Krylov and Rozovskii [14].

Let $\mathcal{D}$ be the set of real-valued Schwartz distributions defined on $C_0^\infty(\mathbb{R}^1)$. For given $p \geqslant 2$, $r \geqslant 0$ and a nonnegative real number n, define the space $H_{p,r}^n = H_{p,r}^n(\mathbb{R}^1)$ (called the weighted space of Bessel potentials or the weighted Sobolev space with fractional derivatives) as the space of all generalized functions u such that $(1-\Delta)^{n/2}(1+x^2)^{r/2}u \in L_p = L_p(\mathbb{R}^1)$. For $u \in H_{p,r}^n$ and $\phi \in C_0^\infty$, by definition

$$(2.1)\qquad \begin{aligned}(u, \phi) &= ((1-\Delta)^{n/2}(1+x^2)^{r/2}u, (1-\Delta)^{-n/2}\phi) \\ &= \int_{\mathbb{R}^1} [(1-\Delta)^{n/2}(1+x^2)^{r/2}u](x)(1-\Delta)^{-n/2}\phi(x)\, dx.\end{aligned}$$

For $u \in H_{p,r}^n$ one introduces the norm

$$\| u \|_{n,p,r} := \| (1-\Delta)^{n/2}(1+x^2)^{r/2}u \|_p,$$

where $\| \cdot \|_p$ is the norm in L_p. One can easily check that $H_{p,r}^n$ is a Banach space with the norm $\| \cdot \|_{n,p,r}$ and the set C_0^∞ is dense in $H_{p,r}^n$.

Note that for integers $n \geqslant 0$ the space $H_{p,r}^n$ coincides with the weighted Sobolev space $W_{p,r}^n = W_{p,r}^n(\mathbb{R}^1)$. Observe also that $\| u \|_{n,p,r} = \| (1+x^2)^{r/2}u \|_{n,p}$.

We now define

$$\mathbb{H}_{p,r}^n(T) := L_p(\Omega \times [0,T], \mathcal{P}; H_{p,r}^n).$$

If $n = 0$, we use $\mathbb{L}$ instead of $\mathbb{H}^0$. The norms in these spaces are defined in an obvious way.

Definition 2.1. For a $\mathcal{D}$-valued function $u \in \mathbb{H}_{p,r}^n(T)$, we write $u \in \mathcal{H}_{p,r}^n(T)$ if there exists $(f, g) \in \mathcal{F}_{p,r}^{n-2}(T) := \mathbb{H}_{p,r}^{n-2}(T) \times (\mathbb{H}_{p,r}^{n-1}(T))^{d'}$ such that for any $\phi \in C_0^\infty$, with probability 1 the equality

$$(2.2)\qquad (u(t,\cdot), \phi) = (u(0,\cdot), \phi) + \int_0^t (f(s,\cdot), \phi)\, ds + \sum_{k=1}^{d'} \int_0^t (g^k(s,\cdot), \phi)\, dw_s^k$$

holds for all $t \leqslant T$ and $u(0,\cdot) \in L_p(\Omega, \mathcal{F}_0; H_{p,r}^{n-2/p})$. We also define

$$\mathcal{H}_{p,r,0}^n(T) = \mathcal{H}_{p,r}^n(T) \cap \{u : u(0,\cdot) = 0\},$$

$$
\begin{aligned}
(2.3) \qquad \| u \|_{\mathcal{H}^n_{p,r}(T)} = \| u \|_{\mathbb{H}^n_{p,r}(T)} + \| f \|_{\mathbb{H}^{n-2}_{p,r}(T)} + \sum_{k=1}^{d'} \| g^k \|_{\mathbb{H}^{n-1}_{p,r}(T)} \\
+ (E \| u(0,\cdot) \|^p_{H_p^{n-2/p}})^{1/p}.
\end{aligned}
$$

Remarks. 1. If $r = 0$, we write $\mathcal{H}^n_p(T)$ instead of $\mathcal{H}^n_{p,0}(T)$. The spaces $\mathcal{H}^n_p(T)$ were first introduced by Krylov in [10], [11].

2. For $u \in \mathcal{H}^n_{p,r}(T)$, if (2.2) holds, we write $f = \mathbb{D}\, u$ and $g^k = \mathbb{S}^k\, u$.

3. Observe that $\| u \|_{\mathcal{H}^n_{p,r}(T)} = \| (1+x^2)^{r/2} u \|_{\mathcal{H}^n_p(T)}$.

Definition 2.2. An *r-generalized solution* of the problem (1.1) is a function $u \in \mathcal{H}^n_{p,r}(T)$ such that $u(0,\cdot) = u_0$, $\mathbb{D}\, u = au'' + bu' + cu + f$ and $\mathbb{S}^k\, u = \sigma^k u' + \nu^k u + g^k$.

If an r-generalized solution u belongs to $C^{0,2}([0,T] \times \mathbb{R}^1)$, it is a solution of the corresponding problem in the classical sense; that is, it satisfies (1.1) for all t, x and ω from a set of full probability. In this case, we say that u is a *classical solution.*

We make the following assumptions.

Assumption 2.1 (uniform ellipticity)**.** For any $\omega \in \Omega$, $t \geqslant 0$, $x \in \mathbb{R}^1$, we have

$$\lambda \leqslant (a - \frac{1}{2}\sigma^k \sigma^k)(\omega, t, x) \leqslant \Lambda,$$

where λ and Λ are fixed positive constants.

Assumption 2.2 (uniform continuity)**.** For any $\varepsilon > 0$, there exists a $\kappa_\varepsilon > 0$ such that

$$|a(t,x) - a(t,y)| < \varepsilon \text{ and } |\sigma^k(t,x) - \sigma^k(t,y)| < \varepsilon$$

whenever $|x - y| < \kappa_\varepsilon$, $\omega \in \Omega$, $t \geqslant 0$.

Assumption 2.3. a, b, c, σ^k, ν^k are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^1)$-measurable functions, $c \leqslant 0$, and for any $\omega \in \Omega$, $t \geqslant 0$, we have $a(t,\cdot), b(t,\cdot), c(t,\cdot), \sigma^k(t,\cdot), \nu^k(t,\cdot) \in C^n(\mathbb{R}^1)$.

$f(t,x)$, $g^k(t,x)$ are predictable as functions taking values in $H^n_{p,r}$ and $H^{n+1}_{p,r}$, respectively.

Assumption 2.4. For any $t \geqslant 0, \omega \in \Omega$,

$$
\begin{aligned}
&\| a(t,\cdot) \|_{C^n} + \| b(t,\cdot) \|_{C^n} + \| c(t,\cdot) \|_{C^n} \\
&+ \| \sigma^k(t,\cdot) \|_{C^n} + \| \nu^k(t,\cdot) \|_{C^n} \leqslant K, \text{ and } (f(\cdot,\cdot), g(\cdot,\cdot)) \in \mathcal{F}^n_{p,r}(T).
\end{aligned}
$$

Theorem 2.3. *Let Assumptions 2.1-2.4 be satisfied and let*

$$u_0 \in L_p(\Omega, \mathcal{F}_0; H^{n+2-2/p}_{p,r}).$$

Then the Cauchy problem for equation (1.1) *on* $[0,T]$ *with the initial condition* $u(0,\cdot) = u_0$ *has a unique r-generalized solution* $u \in \mathcal{H}^{n+2}_{p,r}(T)$*. For this solution, we have*

$$
\begin{aligned}
\| u \|_{\mathcal{H}^{n+2}_{p,r}(T)} \leqslant N\{\| f \|_{\mathbb{H}^n_{p,r}(T)} + \sum_{k=1}^{d'} \| g^k \|_{\mathbb{H}^{n+1}_{p,r}(T)} \\
+ (E \| u_0 \|^p_{n+2-2/p,p,r})^{1/p}\},
\end{aligned}
$$

where the constant N *depends only on* $n, p, \lambda, \Lambda, K, T, r$ *and the function* κ_ε.

Proof. Let $\bar{f} := (1+x^2)^{r/2} f$ and $\bar{g^k} := (1+x^2)^{r/2} g^k$. Then $(\bar{f}, \bar{g^k}) \in \mathcal{F}_p^n(T)$. We also define $\bar{a} := a$, $\bar{b} := b - 2ra\frac{x}{1+x^2}$, $\bar{c} := c - \frac{ra}{1+x^2} + r(r+2)a\frac{x^2}{(1+x^2)^2} - rb\frac{x}{1+x^2}$, $\bar{\sigma^k} := \sigma^k$, $\bar{\nu^k} := \nu^k - r\sigma^k\frac{x}{1+x^2}$ and $\bar{u_0} := (1+x^2)^{r/2} u_0$.

Clearly $\bar{a}, \bar{b}, \bar{c}$ satisfy Assumptions 2.1–2.4. Thus, by Theorem 3.2 (4.1) of [10] ([11]), there exists a unique solution $\bar{u} \in \mathcal{H}_p^{n+2}(T)$ of

$$\begin{cases} d\bar{u} = (\bar{a}\bar{u}'' + \bar{b}\bar{u}' + \bar{c}\bar{u} + \bar{f})\, dt + (\bar{\sigma^k} u' + \bar{\nu^k} u + \bar{g}^k)\, dw_t^k, \\ \bar{u}(0,\cdot) = \bar{u_0}. \end{cases}$$

One can easily check that $u := (1+x^2)^{-r/2}\,\bar{u} \in \mathcal{H}_{p,r}^{n+2}(T)$ satisfies (1.1). Moreover, since

$$\| \bar{u} \|_{\mathcal{H}_p^{n+2}(T)} \leqslant N\{\| \bar{f} \|_{\mathbb{H}_p^n(T)} + \sum_{k=1}^{d'} \| \bar{g}^k \|_{\mathbb{H}_p^{n+1}(T)} + (E \| \bar{u_0} \|_{n+2-2/p,p}^p)^{1/p}\},$$

we get the desired estimate for u by the definition of the weighted spaces. □

Theorem 2.4 (Embedding theorem)**.** *If $p > 2$, $1/2 > \beta > \alpha > 1/p$, then for any function $u \in \mathcal{H}_p^{n+2}(T)$, we have $u \in C^{\alpha-1/p}([0,T], H_p^{n+2-2\beta})$ (a.s.) and for any $t, s \leqslant T$,*

$$E \| u(t,\cdot) - u(s,\cdot) \|_{n+2-2\beta,p}^p \leqslant N(\beta,p,T)|t-s|^{\beta p-1} \| u \|_{\mathcal{H}_p^{n+2}(T)}^p,$$

$$E \| u(t,\cdot) \|_{C^{\alpha-1/p}([0,T],H_p^{n+2-2\beta})}^p \leqslant N(\beta,\alpha,p,T) \| u \|_{\mathcal{H}_p^{n+2}(T)}^p .$$

Proof. See Theorem 3.1 (iii) of [10] or Theorem 6.2 of [11]. □

Corollary 2.5 (Existence of a classical solution)**.** *Suppose that $1/2 > \beta > \alpha > 1/p$ and $n + 2 - 2\beta - 1/p \geqslant 2$. Then the r-generalized solution $u \in \mathcal{H}_{p,r}^{n+2}(T)$ of* (1.1) *is the classical solution.*

Proof. This corollary follows from Theorem 2.4 and the Sobolev embedding theorem, $H_p^{n+2-2\beta} \subset C^{n+2-2\beta-1/p}$. □

Corollary 2.6. *If $p > 2$, $1/2 > \beta > \alpha > 1/p$ and $n + 2 - 2\beta - 1/p > 0$, then for any function $u \in \mathcal{H}_{p,r}^{n+2}(T)$, we have*

$$E \sup_{0\leqslant t\leqslant T} \sup_{|x|\geqslant R} |u(t,x)|^p \leqslant \frac{N(p,T,n,\alpha,\beta)}{(1+R^2)^{rp/2}} \| u \|_{\mathcal{H}_{p,r}^{n+2}(T)}^p .$$

Proof. Let $v(t,x) := (1+x^2)^{r/2}u(t,x)$. Then,

$$\sup_{|x|\geqslant R} |u(t,x)| \leqslant \sup_{|x|\geqslant R} \frac{(1+x^2)^{r/2}}{(1+R^2)^{r/2}}|u(t,x)| = \frac{1}{(1+R^2)^{r/2}} \sup_{|x|\geqslant R} |v(t,x)|.$$

Thus,

$$E \sup_{0\leqslant t\leqslant T} \sup_{|x|\geqslant R} |u(t,x)|^p \leqslant \frac{1}{(1+R^2)^{rp/2}} E \sup_{0\leqslant t\leqslant T} \sup_{|x|\geqslant R} |v(t,x)|^p.$$

But since $v \in \mathcal{H}_p^{n+2}(T)$, by Theorem 2.4,

$$\begin{aligned} E \sup_{0\leqslant t\leqslant T} \sup_{|x|\geqslant R} |v(t,x)|^p &\leqslant E|v|_{C^{\alpha-1/p}([0,T],C^{n+2-2\beta-1/p})}^p \\ &\leqslant N \| v \|_{\mathcal{H}_p^{n+2}(T)}^p = N \| u \|_{\mathcal{H}_{p,r}^{n+2}(T)}^p . \end{aligned}$$

The corollary is proved. □

3. Discretization

We begin our discussion of a finite-difference scheme for (1.1) by defining a grid of points. Take a number $h \in (0,1]$. Define a uniform h-grid on $[-hM, hM]$ by

$$\mathbb{Z}_h^M := \{x_i = hi,\ i = 0, \pm 1, \pm 2, \cdots, \pm M\}.$$

Let $R := hM$. For a random function v defined on $\Omega \times [0,T] \times \mathbb{Z}_h^M$, we write $v_i(\omega, t)$ for the value of v at (ω, t, x_i). For a function v defined on $\Omega \times [0,T] \times \mathbb{R}^1$, we define finite-difference operators L_h and Λ_h^k by

$$\begin{aligned} L_h v(t,x) &= a(t,x)\frac{1}{h^2}[v(t,x+h) - 2v(t,x) + v(t,x-h)] \\ &\qquad + |b(t,x)|\frac{1}{h}[v(t,x+h \text{ sign } b) - v(t,x)] + c(t,x)v(t,x), \end{aligned}$$

$$\Lambda_h^k v(t,x) = \sigma^k(t,x)\frac{1}{h}[v(t,x+h) - v(t,x)] + \nu^k(t,x)v(t,x).$$

Note that L_h and Λ_h^k are obtained by replacing the space derivatives in the operator L and Λ^k in (1.1) by the corresponding difference quotients. Note also that $L_h v(t,x_i)$ and $\Lambda_h^k v(t,x_i)$ make sense for a function v defined on $\Omega \times [0,T] \times \mathbb{Z}_h^M$ if $i \neq \pm M$.

Lemma 3.1. *Let δ be an arbitrary number in $(0,1)$. For any fixed ω, t, and $u(t,\cdot) \in C^{2+\delta}$, we have*

$$|Lu(t,\cdot) - L_h u(t,\cdot)|_{C^0} \leqslant Nh^\delta |u(t,\cdot)|_{C^{2+\delta}},$$

where $N = N(|a(t,\cdot)|_{C^0}, |b(t,\cdot)|_{C^0})$.

Proof.

$$\begin{aligned} &|a(t,x)u''(t,x) - a(t,x)\frac{1}{h^2}[u(t,x+h) - 2u(t,x) + u(t,x-h)]| \\ &\leqslant |a(t,\cdot)|_{C^0}|u'' - \frac{1}{h^2}[u(t,x+h) - 2u(t,x) + u(t,x-h)]| \\ &\leqslant |a(t,\cdot)|_{C^0}\frac{h^\delta}{3}|u''(t,\cdot)|_{C^\delta}. \end{aligned}$$

Also,

$$\begin{aligned} &|b(t,x)u'(t,x) - |b(t,x)|\frac{1}{h}[u(t,x+h \text{ sign } b) - u(t,x)]| \\ &\leqslant \begin{cases} |b(t,x)|\,|u'(t,x) - \frac{1}{h}[u(t,x+h) - u(t,x)]|, & \text{if } b(t,x) \geqslant 0 \\ |b(t,x)|\,|u'(t,x) + \frac{1}{h}[u(t,x-h) - u(t,x)]|, & \text{if } b(t,x) < 0 \end{cases} \\ &\leqslant |b(t,\cdot)|_{C^0}\, h|u''(t,\cdot)|_{C^0}. \end{aligned}$$

Now notice that $h \leqslant h^\delta$. $\square$

From now on, we assume that Assumptions 2.1–2.4 are satisfied with n and p such that the conditions in Corollary 2.5 are satisfied.

Let u_h be a function in $\Omega \times [0,T] \times \mathbb{Z}_h^M$. We consider the following Itô stochastic differential equation in $[0,T]$:

$$\begin{cases} du_h(t,x) = [L_h u_h(t,x) + f(t,x)]\,dt + \sum_{k=1}^{d'} [\Lambda_h^k u_h(t,x) + g^k(t,x)]\,dw_t^k, \\ \text{ for } x \neq x_{\pm M}, \\ u_h(t, x_{\pm M}) = 0, \text{ for all } t \in [0,T], \\ u_h(0,x) = u_0(x), \text{ for all } x \in \mathbb{Z}_h^{M-1}. \end{cases} \tag{3.1}$$

Note that under our assumption, $u_0(x)$, $f(\cdot,x)$ and $g^k(\cdot,x)$ make sense pointwise by the Sobolev embedding theorem.

Lemma 3.2. *There exists a unique solution*

$$(u_h(\cdot,\cdot,x_{-M}), u_h(\cdot,\cdot,x_{-M+1}), \cdots, u_h(\cdot,\cdot,x_{M-1}), u_h(\cdot,\cdot,x_M)) \in \mathbb{R}^{2M+1}$$

of (3.1), *and*

$$E\int_0^T \sum_{j=-M}^{M} (u_h(t,x_j))^2\,dt < \infty. \tag{3.2}$$

Proof. First notice that

$$\begin{aligned} &|b(t,x)|\frac{1}{h}[u_h(t,x+h \text{ sign } b) - u_h(t,x)] \\ &\quad = b_+(t,x)\frac{1}{h}[u_h(t,x+h) - u_h(t,x)] + b_-(t,x)\frac{1}{h}[u_h(t,x-h) - u_h(t,x)], \end{aligned}$$

where $b_+ = \frac{|b|+b}{2}$ and $b_- = \frac{|b|-b}{2}$. Thus, (3.1) is a linear system of stochastic differential equations in $\mathbb{R}^{2M+1}$. Note also that the coefficients a, b, c, σ^k, ν^k are bounded (by a constant K), and by the Sobolev embedding theorem

$$\begin{aligned} E\int_0^T \sum_{j=-M}^{M} f^2(t,x_j)\,dt &\leqslant N(T,M,p)\,(E\int_0^T |f(t,\cdot)|_{C^0}^p\,dt)^{2/p} \\ &\leqslant N(T,M,p)\ \| f \|_{\mathbb{H}^n_{p,r}(T)}^2, \end{aligned} \tag{3.3}$$

$$\begin{aligned} E\int_0^T \sum_{j=-M}^{M}\sum_{k=1}^{d'} (g^k)^2(t,x_j)\,dt &\leqslant N(E\int_0^T \sum_{k=1}^{d'} |g^k(t,\cdot)|_{C^0}^p\,dt)^{2/p} \\ &\leqslant N\sum_{k=1}^{d'} \| g \|_{\mathbb{H}^{n+1}_{p,r}(T)}^2 . \end{aligned} \tag{3.4}$$

Let $\mathbf{u}_h(t) := (u_h(t,x_{-M}), u_h(t,x_{-M+1}), \cdots, u_h(t,x_{M-1}), u_h(t,x_M)) \in \mathbb{R}^{2M+1}$, and let $u_{h,i}(t)$ be the i-th component of $\mathbf{u}_h(t)$. We rewrite (3.1):

$$d\mathbf{u}_h(t) = A_h(t,\mathbf{u}_h(t))\,dt + \sum_{k=1}^{d'} B_h^k(t,\mathbf{u}_h(t))\,dw_t^k, \tag{3.5}$$

where A_h and B_h^k are the drift and diffusion terms in (3.1), respectively. Note that A_h and B_h^k are of the following forms:

$$\begin{cases} A_h^i(t,\mathbf{u}) = \sum_{j=-M}^{M} \alpha_{ij}(t) u_j + f_i(t), \\ B_h^{k,i}(t,\mathbf{u}) = \sum_{j=-M}^{M} \beta_{ij}^k(t) u_j + g_i^k(t), \end{cases}$$

for $\mathbf{u} = (u_i), \mathbf{f}(t) = (f_i(t)), \mathbf{g}^k(t) = (g_i^k(t)) \in \mathbb{R}^{2M+1}$. To show the unique solvability of (3.5), we apply Theorem 5.1.1 of [12]. To employ this theorem, we need to check

(i). (monotonicity condition) There exists a function $K(t) \geqslant 0$ such that $E\int_0^T K(t)\,dt < \infty$ and for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{2M+1}$, $t \in [0,T]$, $\omega \in \Omega$,

$$2(\mathbf{u}-\mathbf{v}, A_h(t,\mathbf{u}) - A_h(t,\mathbf{v})) + \| B_h(t,\mathbf{u}) - B_h(t,\mathbf{v}) \| \leqslant K(t)|\mathbf{u}-\mathbf{v}|^2,$$

where $|\cdot|$, $\|\cdot\|$ are norms in $\mathbb{R}^d$ and $\mathbb{R}^{(2M+1)\times d'}$, respectively.

(ii). (growth condition) For all $\mathbf{u}$, $t \in [0,T]$, $\omega \in \Omega$,

$$2(\mathbf{u}, A_h(t,\mathbf{u})) + \| B_h(t,\mathbf{u}) \|^2 \leqslant K(t)(1+|u|^2).$$

Let $K(t) := N(K)(1 + |f(t,\cdot)|^2_{C^0} + \sum_{k=1}^{d'} |g^k(t,\cdot)|^2_{C^0})$. By the previous observation, $E\int_0^T K(t)\,dt < \infty$. (i) and (ii) follow from the linearity of A_h, B_h^k, the boundedness of $\alpha_{ij}, \beta_{ij}^k$ and the definition of $K(t)$ with an obvious choice of $N(K)$. The unique solvability is proved.

Now we show (3.2). Recall that $\mathbf{u}_h(t)$ is a $(2M+1)$-dimensional continuous stochastic process and satisfies

$$\mathbf{u}_h(t) = \mathbf{u}_0 + \int_0^t A_h(s,\mathbf{u}_h(s))\,ds + \int_0^t B_h^k(t,\mathbf{u}_h(s))\,dw_s^k \tag{3.6}$$

almost surely. Since $\mathbf{u}_h(t)$ is also a locally square integrable local martingale, there exists a sequence of Markov times τ_n such that $\tau_n \uparrow T$ a.s. and

$$E\int_0^{\tau_n} |\mathbf{u}_h(t)|^2\,dt < \infty.$$

We square both sides of (3.6) and then take expectations. Then for all $t \leqslant T$ and n we have

$$\begin{aligned}
&E|\mathbf{u}_h(t\wedge\tau_n)|^2 \\
&\leqslant 3E|\mathbf{u}_0|^2 + 3E(\int_0^{t\wedge\tau_n} A_h(s,\mathbf{u}_h(s))\,ds)^2 + 3E(\int_0^{t\wedge\tau_n} B_h^k(t,\mathbf{u}_h(s))\,dw_s^k)^2 \\
&\leqslant 3E|\mathbf{u}_0|^2 + N(T)E\int_0^{t\wedge\tau_n} (A_h)^2(s,\mathbf{u}_h(s))\,ds + 3E\int_0^{t\wedge\tau_n} (B_h^k)^2(t,\mathbf{u}_h(s))\,ds.
\end{aligned} \tag{3.7}$$

In the last inequality we used the Cauchy-Schwarz inequality and the L^2-isometry property of the stochastic integral. Since α_{ij} and β_{ij}^k are bounded, we obtain from (3.7)

$$\begin{aligned}
&E|\mathbf{u}_h(t\wedge\tau_n)|^2 \leqslant 3E|\mathbf{u}_0|^2 + N(K,h,T,M)\,E\int_0^{t\wedge\tau_n} |\mathbf{u}_h(s)|^2\,ds \\
&\quad + N(T)\,E\int_0^{t\wedge\tau_n} |\mathbf{f}(s)|^2 + |\mathbf{g}^k(s)|^2\,ds.
\end{aligned} \tag{3.8}$$

Now by the Gronwall inequality, we obtain from (3.8)

$$E|\mathbf{u}_h(t\wedge\tau_n)|^2 \leqslant N(K,h,T,M)\,(E|\mathbf{u}_0|^2 + E\int_0^T |\mathbf{f}(s)|^2 + |\mathbf{g}^k(s)|^2\,ds).$$

Thus, we get

$$E\int_0^{t\wedge\tau_n} |\mathbf{u}_h(s)|^2\,ds \leqslant N(K,h,T,M)\,(E|\mathbf{u}_0|^2 + E\int_0^T |\mathbf{f}(s)|^2 + |\mathbf{g}^k(s)|^2\,ds). \tag{3.9}$$

Note that by (3.3) and (3.4), the right hand side of (3.9) is bounded by a constant independent of n. We let $n \to \infty$ in (3.9) and apply Fatou's lemma. Finally, (3.2) is proved. $\square$

Now let u be the classical solution of (1.1), which exists by the assumption on n, p and Corollary 2.5. Then for all $t \in [0,T]$, $x \in \mathbb{Z}_h^{M-1}$,

$$\begin{aligned} u(t,x) &= u_0(x) + \int_0^t Lu(s,x) + f(s,x)\,ds \\ &\quad + \sum_{k=1}^{d'} \int_0^t \Lambda^k u(s,x) + g^k(s,x)\,dw_s^k, \text{ a.s.} \end{aligned} \tag{3.10}$$

And by Lemma 3.2,

$$\begin{aligned} u_h(t,x) &= u_0(x) + \int_0^t L_h u_h(s,x) + f(s,x)\,ds \\ &\quad + \sum_{k=1}^{d'} \int_0^t \Lambda_h^k u_h(s,x) + g^k(s,x)\,dw_s^k, \text{ a.s.} \end{aligned} \tag{3.11}$$

Theorem 3.3. *Let $e(t,x) := u(t,x) - u_h(t,x)$. Assume $x \neq x_{\pm M}$.*

(i). $Ee(t,x)$ satisfies

$$\begin{cases} \frac{d}{dt}Ee(t,x) = L_h Ee(t,x) + E(Lu(t,x) - L_h u(t,x)), \\ Ee(0,x) = 0. \end{cases}$$

(ii). If $\sigma^k = \nu^k = 0$, $e(t,x)$ almost surely satisfies

$$\begin{cases} \frac{de}{dt}(t,x) = L_h e(t,x) + Lu(t,x) - L_h u(t,x), \\ e(0,x) = 0. \end{cases}$$

Proof. (i). First we claim that

$$E\int_0^t \Lambda^k u(s,x) - \Lambda_h^k u_h(s,x)\,dw_s^k = 0.$$

Since a stochastic integral is a local martingale, it suffices to show that

$$E\int_0^T \{\Lambda^k u(s,x) - \Lambda_h^k u_h(s,x)\}^2\,ds < \infty.$$

We calculate

$$\begin{aligned} E\int_0^T |\Lambda^k u(s,x)|^2\,ds &= E\int_0^T |\sigma^k(s,x)u'(s,x) + \nu^k(s,x)u(s,x)|^2\,ds \\ &\leqslant 2K^2 E\int_0^T |u'(s,x)|^2 + |u(s,x)|^2\,ds \\ &\leqslant N(K,T,p)\,(E\int_0^T |u'(s,x)|^p + |u(s,x)|^p\,ds)^{2/p} \\ &\leqslant N(K,T,p)\ \| u \|^2_{\mathcal{H}^{n+2}_{p,r}(T)} < \infty, \end{aligned} \tag{3.12}$$

$$
\begin{aligned}
&E\int_0^T |\Lambda_h^k u_h(s,x)|^2\,ds\\
&= E\int_0^T |\sigma^k(s,x)\frac{1}{h}\{u_h(s,x+h)-u_h(s,x)\}+\nu^k(s,x)u_h(s,x)|^2\,ds \qquad (3.13)\\
&\leqslant N(K,h)E\int_0^T |u_h(s,x+h)|^2+|u_h(s,x)|^2\,ds<\infty,
\end{aligned}
$$

by Lemma 3.2. The claim is proved. If we subtract (3.11) from (3.10), we get

$$
\begin{aligned}
e(t,x) &= \int_0^t Lu(s,x)-L_hu_h(s,x)\,ds+\sum_{k=1}^{d'}\int_0^t \Lambda^k u(s,x)-\Lambda_h^k u_h(s,x)\,dw_s^k\\
&= \int_0^t L_he(s,x)\,ds+\int_0^t Lu(s,x)-L_hu(s,x)\,ds\\
&+\sum_{k=1}^{d'}\int_0^t \Lambda^k u(s,x)-\Lambda_h^k u_h(s,x)\,dw_s^k.
\end{aligned}
$$

By the claim, if we take the expectation in the above equality, we get

$$
Ee(t,x)=\int_0^t EL_he(s,x)\,ds+\int_0^t E(Lu(s,x)-L_hu(s,x))\,ds.
$$

Thus, we see that $Ee(t,x)$ is differentiable in t and

$$
\frac{d}{dt}Ee(t,x)=L_hEe(t,x)+E(Lu(t,x)-L_hu(t,x))
$$

and $Ee(0,x)=0$.

(ii). We subtract (3.11) from (3.10). Then we get

$$
\begin{aligned}
e(t,x) &= \int_0^t Lu(s,x)-L_hu_h(s,x)\,ds\\
&= \int_0^t L_he(s,x)\,ds+\int_0^t Lu(s,x)-L_hu(s,x)\,ds,
\end{aligned}
\qquad (3.14)
$$

for all $t\in[0,T]$, $x\in\mathbb{Z}_h^{M-1}$. Now if we differentiate (3.14), we get the desired equation. □

4. Error estimates

We begin with a lemma which is a standard tool in the study of (deterministic) partial differential equations.

Lemma 4.1 (Discrete maximum principle)**.** *Fix ω. Suppose that a function v defined on $\Omega\times[0,T]\times\mathbb{Z}_h^M$ satisfies*

$$
\begin{cases}
\frac{dv}{dt}(t,x)\leqslant L_hv(t,x) \text{ for all } t\in[0,T] \text{ and } x\in\mathbb{Z}_h^{M-1},\\
v(0,x)\leqslant 0 \text{ for all } x\in\mathbb{Z}_h^M, \text{ and } v(t,x_{\pm M})\leqslant 0, \text{ for all } t\in[0,T].
\end{cases}
$$

Then $v(t,x)\leqslant 0$ for all $(t,x)\in[0,T]\times\mathbb{Z}_h^M$.

Proof. Take a constant $\gamma > 0$ and define $\bar{v} = v - \frac{\gamma}{T-t}$. Let (t^*, x_l) be a point at which $\bar{v}$ takes its maximum value. Observe that $t^* < T$. Actually if $\bar{v}(t^*, x_l) \geqslant 0$, then $t^* = 0$ or $l = M$ or $l = -M$. Indeed, if $t^* > 0$ and $l \neq \pm M$, then

$$a(t^*, x_l)\frac{1}{h^2}[\bar{v}(t^*, x_{l+1}) - 2\bar{v}(t^*, x_l) + \bar{v}(t^*, x_{l-1})] \leqslant 0,$$

$$|b(t^*, x_l)|\frac{1}{h}[\bar{v}(t^*, x_{l+\text{sign } b(t^*,x_l)}) - \bar{v}(t^*, x_l)] \leqslant 0,$$

$$c(t^*, x_l)\bar{v}(t^*, x_l) \leqslant 0 \quad \text{and} \quad \frac{d\bar{v}}{dt}(t^*, x_l) = 0.$$

Since $L_h v = L_h(\bar{v} + \frac{\gamma}{T-t}) = L_h\bar{v} + c(t,x)\frac{\gamma}{T-t} \leqslant L_h\bar{v}$, from the assumption and the above inequalities we get

$$\begin{aligned} 0 &\leqslant L_h v(t^*, x_l) - \frac{dv}{dt}(t^*, x_l) \\ &\leqslant L_h\bar{v}(t^*, x_l) - \frac{d\bar{v}}{dt}(t^*, x_l) - \frac{\gamma}{(T-t)^2} \\ &\leqslant -\frac{\gamma}{(T-t)^2} < 0, \end{aligned}$$

which is impossible. Thus, either $\bar{v}(t^*, x_l) < 0$ or $t^* = 0$ or $l = M$ or $l = -M$. In any case, we see that $\bar{v}(t^*, x_l) \leqslant 0$ and $\bar{v}(t, x) \leqslant 0$ for all $(t, x) \in [0, T] \times \mathbb{Z}_h^M$. Since γ is arbitrary, the lemma is proved. □

Lemma 4.2. *The function e defined in Theorem 3.3 satisfies*

$$\sup_{[0,T]\times\mathbb{Z}_h^M} |Ee(t,x)| \leqslant T\, E \sup_{[0,T]\times\mathbb{Z}_h^{M-1}} |Lu - L_h u| + E \sup_{t\in[0,T],k=\pm M} |u(t, x_k)|.$$

If $\sigma^k = \nu^k = 0$, then

$$\sup_{[0,T]\times\mathbb{Z}_h^M} |e(t,x)| \leqslant T \sup_{[0,T]\times\mathbb{Z}_h^{M-1}} |Lu - L_h u| + \sup_{t\in[0,T],k=\pm M} |u(t, x_k)|.$$

Proof. Let

$$N_1 := E \sup_{t\in[0,T],k=\pm M} |e(t, x_k)| = E \sup_{t\in[0,T],k=\pm M} |u(t, x_k)|$$

and

$$N_0 := E \sup_{[0,T]\times\mathbb{Z}_h^{M-1}} |Lu - L_h u|.$$

The last equality for N_1 follows from (3.1). Then for $\bar{e} = e - N_0 t - N_1$, we have

$$\begin{aligned} \frac{d}{dt}E\bar{e} = \frac{d}{dt}Ee - N_0 &= L_h Ee(t,x) + E(Lu - L_h u)(t,x) - N_0 \\ &\leqslant L_h Ee(t,x) = L_h E\bar{e} + c(t,x)(N_0 t + N_1) \leqslant L_h E\bar{e}(t,x). \end{aligned}$$

Obviously, $E\bar{e}(0,x) \leqslant 0$ and $E\bar{e}(t, x_{\pm M}) \leqslant 0$. Therefore, by Lemma 4.1, $E\bar{e}(t,x) \leqslant 0$ and $Ee(t,x) \leqslant N_0 T + N_1$. Similarly, $-Ee(t,x) \leqslant N_0 T + N_1$. The first desired inequality is proved.

To prove the second inequality, we just copy the above proof without writing "E". □

Now we present the main result of this paper.

Theorem 4.3. *Suppose that n and p satisfy*

$$n+2-2\beta-1/p \geqslant 2+\delta \text{ and } 1/2>\beta>\alpha>1/p,$$

for a fixed $\delta\in(0,1)$. Then the function e satisfies

$$\begin{aligned}\sup_{[0,T]\times\mathbb{Z}_h^M}|Ee(t,x)| &\leqslant N(h^\delta+\frac{1}{(1+R^2)^{r/2}})(E\parallel u_0\parallel^p_{n+2-2/p,p,r})^{1/p}\\ &+N(h^\delta+\frac{1}{(1+R^2)^{r/2}})(\parallel f\parallel_{\mathbb{H}^n_{p,r}(T)}+\sum_{k=1}^{d'}\parallel g^k\parallel_{\mathbb{H}^{n+1}_{p,r}(T)}),\end{aligned}\tag{4.1}$$

where $N=N(n,p,T,K,r,\lambda,\Lambda,\kappa_\varepsilon,\alpha,\beta)$.

If $\sigma^k=\nu^k=0$, then

$$\begin{aligned}E\sup_{[0,T]\times\mathbb{Z}_h^M}|e(t,x)|^p &\leqslant N(h^{\delta p}+\frac{1}{(1+R^2)^{rp/2}})E\parallel u_0\parallel^p_{n+2-2/p,p,r}\\ &+N(h^{\delta p}+\frac{1}{(1+R^2)^{rp/2}})(\parallel f\parallel^p_{\mathbb{H}^n_{p,r}(T)}+\sum_{k=1}^{d'}\parallel g^k\parallel^p_{\mathbb{H}^{n+1}_{p,r}(T)}).\end{aligned}\tag{4.2}$$

Proof. We first prove (4.2). The proof of (4.1) is similar and easier. By Lemma 4.2 and Lemma 3.1,

$$\begin{aligned}\sup_{[0,T]\times\mathbb{Z}_h^M}|e(t,x)| &\leqslant T\sup_{t\in[0,T]}|Lu(t,\cdot)-L_hu(t,\cdot)|_{C^0}+\sup_{t\in[0,T],k=\pm M}|u(t,x_k)|\\ &\leqslant N(T,K)h^\delta\sup_{t\in[0,T]}|u(t,\cdot)|_{C^{2+\delta}}+\sup_{t\in[0,T],k=\pm M}|u(t,x_k)|.\end{aligned}$$

We take pth powers and mathematical expectations:

$$\begin{aligned}E\sup_{[0,T]\times\mathbb{Z}_h^M}|e(t,x)|^p &\leqslant N(T,K,p)h^{\delta p}E\sup_{t\in[0,T]}|u(t,\cdot)|^p_{C^{2+\delta}}\\ &+N(p)E\sup_{t\in[0,T],k=\pm M}|u(t,x_k)|^p.\end{aligned}$$

We apply Theorem 2.4 and Corollary 2.6 to the right hand side of the above inequality, to get

$$\begin{aligned}&E\sup_{[0,T]\times\mathbb{Z}_h^M}|e(t,x)|^p\\ &\leqslant N(T,K,p,\alpha,\beta)h^{\delta p}\parallel u\parallel^p_{\mathcal{H}^{n+2}_{p,r}(T)}+\frac{N(p,T,n,\alpha,\beta)}{(1+R^2)^{rp/2}}\parallel u\parallel^p_{\mathcal{H}^{n+2}_{p,r}(T)}.\end{aligned}$$

Finally, we apply Theorem 2.3, and (4.2) is proved.

Now we show (4.1). By Lemma 4.2 and Jensen's inequality, we get

$$\begin{aligned}&\sup_{[0,T]\times\mathbb{Z}_h^M}|Ee(t,x)|\\ &\leqslant T\,(E\sup_{[0,T]\times\mathbb{Z}_h^{M-1}}|Lu-L_hu|^p)^{1/p}+(E\sup_{t\in[0,T],k=\pm M}|u(t,x_k)|^p)^{1/p}.\end{aligned}$$

We apply Lemma 3.1 and Corollary 2.6 to estimate the right hand side. Then we get

$$\begin{aligned}&\sup_{[0,T]\times\mathbb{Z}_h^M}|Ee(t,x)|\\ &\leqslant TNh^\delta(E\sup_{t\in[0,T]}|u(t,\cdot)|^p_{C^{2+\delta}})^{1/p}+\frac{N}{(1+R^2)^{r/2}}\parallel u\parallel_{\mathcal{H}^{n+2}_{p,r}(T)}.\end{aligned}$$

We apply Theorem 2.3 and Theorem 2.4. The theorem is proved. ∎

From the above theorem, we see that if $\sigma^k = \nu^k = 0$, one can select the mesh size h and the diameter R to guarantee that up to any specified final time T, the pth moment of the sup-norm error is less than any given $\varepsilon > 0$. If σ^k and ν^k are not identically zero, one can make the sup-norm of the average error arbitrarily small.

Remarks. 1. As we remarked above, when σ^k and ν^k are not identically zero, we only have (4.1), which is much weaker than (4.2). The main idea to obtain these estimates was to derive certain equations for the error and to apply the discrete maximum principle, Lemma 4.1, to these equations. Since one does not have maximum principle for general stochastic PDEs, we had to take the expectation of the error and obtain an equation for $Ee(t,x)$. A new approach based on the L_2-theory of discrete stochastic evolution equations and embedding theorems has recently been investigated by the author [19]. Estimates (full-discretization) similar to (4.2) were obtained for general stochastic PDEs under rather strong regularity assumptions on the coefficients and the data.

2. Lemma 3.1 can be strengthened for $u(t,\cdot) \in C^4$:

$$|Lu(t,\cdot) - L_h u(t,\cdot)|_{C^0} = O(h^2)$$

if we approximate

$$\frac{\partial u}{\partial x}(t,x) \simeq \frac{1}{2h}[u(t,x+h) - u(t,x-h)].$$

But to obtain a solution in C^4, we need to assume that

$$n + 2 - 2\beta - 1/p \geqslant 4$$

instead of $n + 2 - 2\beta - 1/p \geqslant 2 + \delta$; thus we must require higher regularity for the coefficients and the data. As we explained in the Introduction, our aim was to obtain solutions and estimates for them under less restrictive conditions on the smoothness of the coefficients and the data by employing L_p-theory.

Acknowledgments

The author wishes to express his deep gratitude to Professor N. V. Krylov, Professor I. Gyöngy and Professor S. V. Lototsky for their comments. The author would also like to thank the anonymous referee for helpful suggestions and comments.

References

[1] N. Bellomo, Z. Brzezniak, L.M. de Socio, *Nonlinear stochastic evolution problems in applied sciences*, Kluwer, Dordrecht, 1992. MR **94j:**60121

[2] N. Bellomo, F. Flandoli, *Stochastic partial differential equations in continuum physics—on the foundations of the stochastic interpolation methods for Itô type equations*, Math. Comput. Simulation, 31 (1989), 3-17. MR **90e:**35174

[3] J.F. Bennaton, *Discrete time Galerkin approximations to the nonlinear filtering solution*, J. Math. Anal. and Appl. 110 (1985), 364-383. MR **86k:**93147

[4] A. Bensoussan, R. Glowinski, A. Răşcanu, *Approximation of some stochastic differential equations by the splitting up method*, Appl. Math. Optim. 25 (1992), 81-106. MR **92k:**60139

[5] A.M. Davie, J.G. Gaines, *Convergence of numerical schemes for the solution of parabolic stochastic partial differential equations*, Preprint

[6] J.G. Gaines, *Numerical experiments with S(P)DE's*, in Stochastic Partial Differential Equations, A.M. Etheridge. ed., London Mathematical Society Lecture Note Series 216, Cambridge Univ. Press., 1995, pp. 55-71. MR **96k:**60154

[7] A. Germani, M. Piccioni, *Semi-discretization of stochastic partial differential equations on* $\mathbb{R}^d$ *by a finite- element technique*, Stochastics 23 (1988), 131-148. MR **89f:**60063

[8] I. Gyöngy, *Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise I, II*, Potential Analysis 9 (1998), 1–25 and to appear. CMP 99:01
[9] P.E. Kloeden, E. Platen, *Numerical solutions of stochastic differential equations*, Applications of Mathematics series, Vol. 23, Springer-Verlag, Heidelberg, 1992. MR **94b:**60069
[10] N.V. Krylov, *On L_p-theory of stochastic partial differential equations in the whole space*, SIAM J. Math. Anal. 27 (1996), 313-340. MR **97b:**60107
[11] N.V. Krylov, *An analytic approach to SPDEs*, in Stochastic Partial Differential Equations, Six Perspectives, R. A. Carmona and B. Rozovskii, eds., Mathematical Surveys and Monographs, vol. 64, Amer. Math. Soc., Providence, RI, 1999, pp. 185–242.
[12] N.V. Krylov, *Introduction to the theory of diffusion processes*, Amer. Math. Soc., Providence, RI, 1995. MR **96k:**60196
[13] N.V. Krylov, *Lectures on elliptic and parabolic equations in Hölder spaces*, Graduate Studies in Mathematics, Vol. 12, Amer. Math. Soc., 1996. MR **97i:**35001
[14] N.V. Krylov, B.L. Rozovskii, *Stochastic partial differential equations and diffusion processes*, Russian Math. Surveys, 37 (1982), no. 6, 81-105. MR **84d:**60095
[15] S.V. Lototsky, *Problems in statistics of stochastic differential equations*, Thesis, University of Southern California, 1996.
[16] G. Milstein, *Numerical integration of stochastic differential equations*, Kluwer, Dordrecht, 1995. MR **96e:**65003
[17] B.L.Rozovskii, *Stochastic evolution systems*, Kluwer, Dordrecht, 1990. MR **92k:**60136
[18] J.C. Strikwerda, *Finite difference schemes and partial differential equations*, Wadsworth & Brook/Cole, Pacific Grove, CA, 1989. MR **90g:**65004
[19] H. Yoo, *On L_2-theory of discrete stochastic evolution equations and its application to finite difference approximations for SPDEs*, Preprint.

School of Mathematics, University of Minnesota, Minneapolis, MN 55455
E-mail address: `yoo@math.umn.edu`