CONVERGENCE STUDY OF THE CHORIN-MARSDEN FORMULA

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ABSTRACT. Using the fundamental solution of the heat equation, we give an expression of the solutions to two-dimensional initial-boundary value problems of the Navier-Stokes equations, where the vorticity is expressed in terms of a Poisson integral, a Newtonian potential, and a single layer potential. The density of the single layer potential is the solution to an integral equation of Volterra type along the boundary. We prove there is a unique solution to the integral equation. One fractional time step approximation is given, based on this expression. Error estimates are obtained for linear and nonlinear problems. The order of convergence is $\frac{1}{4}$ for the Navier-Stokes equations. The result is in the direction of justifying the Chorin-Marsden formula for vortex methods. It is shown that the density of the vortex sheet is twice the tangential velocity for the half plane, while in general the density differs from it by one additional term.

1. INTRODUCTION

The convergence and stability problems for the vortex method have been studied by many authors. Usually a fractional steps approach is applied in the vortex method for viscous flows; therefore viscous splitting is a basic problem in the mathematical theory. The convergence problem of viscous splitting was first studied in [2], and since then there have been quite a number of works on this subject (see [10] and references therein).

Vorticity is the main variable in the computation of the vortex method. However, a velocity boundary condition is applied on a solid boundary and the boundary condition for vorticity is indirect. To approximate the velocity boundary condition, it was proposed that the velocity field be extended oddly after each convection step; then one vortex sheet was created, which resulted in an approximate zero velocity on the boundary (see [4]). This scheme was formulated as

$$(1) \quad \omega_i = (H_k \circ \phi \circ E_k)^i \omega_0,$$

where $k > 0$ is the length of temporal steps, ω_i is the vorticity at $t_i = ik$, ω_0 is the initial data, E_k is the solver of the Euler equation, H_k is the solver of the heat equation on the entire space, and ϕ is the “vorticity creation operator”. Stability and convergence of (1) for linear equations was proved in [5].

Convergence of the Chorin-Marsden formula for a half plane domain was proved in [3], with a slight modification, where the tangential component of velocity was

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extended oddly and the normal component was extended evenly. An error estimate with upper bound $O(k^{\frac{1}{4}})$ was obtained. Partial analysis for convex domains was given in [9], also with some modification, where the extended velocity field was multiplied by a Jacobian determinant. It was observed (see [9]) that both odd extension and even extension of the normal component of velocity yielded convergent results for the half plane domain. We notice that the way to create a vortex sheet is not unique, but all of them should be close enough so that the differences between them approach zero as $k \rightarrow 0$.

The aim of this paper is to study the problem for general bounded domains. The idea is to derive an expression of the density of vortex sheets for exact solutions, then make a time discretization to obtain an approximate one. In our scheme the derivation of vortex sheet is indirect, since it satisfies an integral equation of Volterra type along the boundary. A direct one may be possible. However, since we have the exact expression, the approximate one is required to be consistent with it. In [5], [3], [9] the density of the vortex sheet equals twice the tangential velocity after each convection step. We will make a remark in the final section, indicating that the density of the vortex sheet of our scheme is the same if the domain is a half plane. However, in the general case they are different by one additional term, which does not vanish in general. We hope this additional term approaches zero as $k \rightarrow 0$ for some cases. This will be the subject of our further study.

In order to state our scheme and convergence result, let us write down the equations first. Let Ω be a bounded domain in $\mathbb{R}^2$, and $\partial\Omega$ its boundary. The initial-boundary value problems of the Navier-Stokes equations for viscous incompressible flow are

$$(2) \quad \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \Delta u,$$

$$(3) \quad \nabla \cdot u = 0,$$

$$(4) \quad u|_{x \in \partial\Omega} = 0, \quad u|_{t=0} = u_0,$$

where u is the velocity and p is the pressure. Without loss of generality we set the density and viscosity equal to one. Let $\nabla \wedge = \left(\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1} \right)$, and let the vorticity $\omega = -\nabla \wedge u$; then the vorticity formulation of the Navier-Stokes equation is

$$(5) \quad \frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \Delta \omega,$$

$$(6) \quad -\Delta \psi = \omega, \quad u = \nabla \wedge \psi,$$

$$(7) \quad \psi|_{x \in \partial\Omega} = 0,$$

$$(8) \quad u|_{x \in \partial\Omega} = 0,$$

$$(9) \quad \omega|_{t=0} = \omega_0 = -\nabla \wedge u_0.$$

Let $\rho(\xi, t), \xi \in \partial\Omega, t \in [0, T]$ be the density of the vortex sheet of an exact solution; then the contribution of ρ to the solution is

$$(10) \quad \int_0^t \int_{\partial\Omega} K(x - \xi, t - \tau) \rho(\xi, \tau) ds_\xi d\tau,$$

where K is the fundamental solution of the heat equation:

$$K(x, t) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}}.$$

Let $F = u \cdot \nabla \omega$, and we extend ω_0 smoothly to $\mathbb{R}^2$ so that it is compactly supported; then we will prove that the solution to (5)-(9) can be expressed in terms of the sum of a Poisson integral, a Newtonian potential, and (10):

$$(11) \quad \begin{aligned} \omega = & \int_{\mathbb{R}^2} K(x - \xi, t) \omega_0(\xi) d\xi - \int_0^t \int_{\Omega} K(x - \xi, t - \tau) F(\xi, \tau) d\xi d\tau \\ & + \int_0^t \int_{\partial\Omega} K(x - \xi, t - \tau) \rho(\xi, \tau) ds_{\xi} d\tau. \end{aligned}$$

For exact solutions the velocity vanishes on the boundary, so ρ cannot be expressed in terms of velocity. Instead we will prove that ρ satisfies

$$(12) \quad \int_0^t \int_{\partial\Omega} L(\eta, \xi, t - \tau) \rho(\xi, \tau) ds_{\xi} d\tau + f(\eta, t) = 0,$$

where

$$(13) \quad \begin{aligned} f(\eta, t) = & \int_{\Omega} M(x, \eta) \left\{ \int_{\mathbb{R}^2} K(x - \xi, t) \omega_0(\xi) d\xi \right. \\ & \left. - \int_0^t \int_{\Omega} K(x - \xi, t - \tau) F(\xi, \tau) d\xi d\tau \right\} dx, \end{aligned}$$

and the kernels L, M will be given later.

The following scheme is essentially the same as (1), except the method of vorticity creation is different. By induction we assume that the approximate solution $\omega_k(t)$ is already known for $t < t_i$, $t_i = ik, i = 0, 1, \dots$; then $\omega_k(t)$ is obtained in $[t_i, t_{i+1})$ by the following fractional time step scheme.

Convection step. Solve the inviscid problem

$$(14) \quad \frac{\partial \tilde{\omega}_k}{\partial t} + \tilde{u}_k \cdot \nabla \tilde{\omega}_k = 0, \quad x \in \Omega, \quad t \in [t_i, t_{i+1}),$$

$$(15) \quad -\Delta \tilde{\psi}_k = \tilde{\omega}_k, \quad \tilde{u}_k = \nabla \wedge \tilde{\psi}_k, \quad x \in \Omega, \quad t \in [t_i, t_{i+1}),$$

$$(16) \quad \tilde{\psi}_k|_{x \in \partial\Omega} = 0,$$

$$(17) \quad \tilde{\omega}_k(t_i) = \omega_k(t_i - 0),$$

where $\omega_k(-0) = \omega_0$.

Vorticity creation step. Solve

$$(18) \quad \int_0^t \int_{\partial\Omega} L(\eta, \xi, t - \tau) \rho_k(\xi, \tau) ds_{\xi} d\tau + f_k(\eta, t) = 0,$$

where $\rho_k(\xi, t)$ is obtained for $t \in [0, t_i)$ by the previous steps and f_k is a linear interpolation

$$(19) \quad f_k(\eta, \tau) = \frac{1}{k}((t_{i+1} - \tau)f_{k,i} + (\tau - t_i)f_{k,i+1}), \quad \tau \in [t_i, t_{i+1}],$$

and $f_{ki}, f_{k,i+1}$ is given by

$$(20) \quad f_{ki}(\eta) = \int_{\Omega} M(x, \eta) \left\{ \int_{\mathbb{R}^2} K(x - \xi, t_i) \omega_0(\xi) d\xi - \sum_{j=0}^{i-1} \int_{\Omega} K(x - \xi, t_i - t_j) \int_{t_j}^{t_{j+1}} F(\xi, \tau) d\xi d\tau \right\} dx,$$

where $F = \tilde{u}_k \cdot \nabla \tilde{\omega}_k$.

Diffusion step. Solve the heat equation

$$(21) \quad \frac{\partial \omega_k}{\partial t} = \Delta \omega_k, \quad x \in \mathbb{R}^2, \quad t \in [t_i, t_{i+1}),$$

with initial data

$$(22) \quad \omega_k(t_i) = \omega_k(t_i - 0) + \omega'_1 + \omega'_2,$$

where

$$\omega'_1(x) = \begin{cases} \tilde{\omega}_k(t_{i+1} - 0) - \omega_k(t_i - 0), & x \in \Omega, \\ 0, & x \notin \Omega, \end{cases}$$

and

$$\omega'_2(\xi + s_n n_\xi) = \int_{t_i}^{t_{i+1}} \rho_k(\xi, t) dt \delta(s_n), \quad \forall \xi \in \partial\Omega, -\infty < s_n < +\infty,$$

where n_ξ is the outward unit normal vector at ξ , and δ is the Dirac δ -function. More precisely,

$$\langle \omega'_2, \varphi \rangle = \int_{\partial\Omega} \left(\int_{t_i}^{t_{i+1}} \rho_k(\xi, t) dt \right) \varphi(\xi) ds_\xi, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^2).$$

We are now ready to state our convergence and error estimate results. For simplicity we assume that the domain Ω is simply connected, $\partial\Omega$ is sufficiently smooth, and the solution to (2)-(4) is sufficiently smooth. We will use the usual notations of Sobolev spaces, $H^s(\Omega)$, and norms, $\|\cdot\|_s$, where s is a real number.

Theorem 1.1. *If $1 < \sigma < \frac{3}{2}$, then*

$$(23) \quad \|(\omega - \omega_k)(t_i - 0)\|_{\sigma-1} \leq C^* k^{\frac{1}{4}}$$

for $0 \leq i \leq [\frac{T}{k}]$, provided $k \leq k_0$, where $k_0 > 0$ is a fixed constant and C^* is a constant independent of k .

We remark that the rate of $\frac{1}{4}$ is the same as that in [3], but the norm is different.

We will derive the expression (11) and the equation (12) for vortex sheets in the next section. Using those expressions, we will study the approximate scheme (14)-(22) in Section 3. We will estimate the upper bound and error for linear problems in Section 4 and Section 5, then study nonlinear problems in Section 6. Finally we will make a remark on the vortex sheet in Section 7. Throughout this paper we will always denote by C a generic positive constant, and $C_0, C_1, \dots, M_0, M_1, \dots$, will be different positive constants.

2. AN EXPRESSION OF THE EXACT SOLUTIONS

To derive an equivalent formulation of (5)-(9) we prove

Lemma 2.1. *Let $\omega \in L^2(\Omega)$ and $\psi \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfy*

$$-\Delta \psi = \omega, \quad x \in \Omega.$$

Then $\nabla \psi|_{\partial\Omega} = 0$ if and only if $\int_{\Omega} \omega h \, dx = 0$ for all harmonic functions $h \in H^1(\Omega)$.

Proof. We denote by n the outward unit normal vector on $\partial\Omega$, and by ν the anti-clockwise unit tangential vector on $\partial\Omega$. We have

$$(24) \quad \begin{aligned} \int_{\Omega} \omega h \, dx &= - \int_{\Omega} \Delta \psi h \, dx = \int_{\Omega} \nabla \psi \cdot \nabla h \, dx - \int_{\partial\Omega} \frac{\partial \psi}{\partial n} h \, ds \\ &= - \int_{\partial\Omega} \frac{\partial \psi}{\partial n} h \, ds. \end{aligned}$$

Since $h|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$ is arbitrary, $\frac{\partial \psi}{\partial n}|_{\partial\Omega} = 0$ is equivalent to $\int_{\Omega} \omega h \, dx = 0$, and $\frac{\partial \psi}{\partial \nu}|_{\partial\Omega} = 0$ is obvious. $\square$

We set $H(\Omega) = \{h \in H^1(\Omega); \Delta h = 0\}$; then (5)-(9) is equivalent to (5)-(7), (9) and

$$(25) \quad \int_{\Omega} \omega h \, dx = 0, \quad \forall h \in H(\Omega).$$

Let us express a harmonic function in terms of a double layer potential. We define

$$M(x, \eta) = \frac{x - \eta}{2\pi|x - \eta|^2} \cdot n_{\eta}, \quad x \in \Omega, \quad \eta \in \partial\Omega.$$

Then $h \in H(\Omega)$ can be expressed as

$$(26) \quad h(x) = \int_{\partial\Omega} M(x, \eta) \varphi(\eta) \, ds_{\eta},$$

where φ is the density.

We express the solution as (11); then (25)-(26) implies

$$\begin{aligned} \int_{\Omega} \int_{\partial\Omega} M(x, \eta) \left\{ \int_{\mathbb{R}^2} K(x - \xi, t) \omega_0(\xi) \, d\xi - \int_0^t \int_{\Omega} K(x - \xi, t - \tau) F(\xi, \tau) \, d\xi d\tau \right. \\ \left. + \int_0^t \int_{\partial\Omega} K(x - \xi, t - \tau) \rho(\xi, \tau) \, ds_{\xi} d\tau \right\} \varphi(\eta) \, ds_{\eta} \, dx = 0, \end{aligned}$$

Let

$$L(\eta, \xi, t) = \int_{\Omega} M(x, \eta) K(x - \xi, t) \, dx.$$

Then

$$\int_{\partial\Omega} \int_0^t \int_{\partial\Omega} L(\eta, \xi, t - \tau) \rho(\xi, \tau) \varphi(\eta) ds_\xi d\tau ds_\eta + \int_{\partial\Omega} f(\eta, t) \varphi(\eta) ds_\eta = 0,$$

where f is given by (13). Since φ is arbitrary, we obtain (12), an integral equation for ρ .

There is a variational form of (12). We use the conventional notations of Sobolev spaces $H^{s, \frac{s}{2}}(\Omega \times (0, T))$ or $H^{s, \frac{s}{2}}(\partial\Omega \times (0, T))$ related to parabolic equations,

$$H^{s, \frac{s}{2}}(\Omega \times (0, T)) = L^2(0, T; H^s(\Omega)) \cap H^{\frac{s}{2}}(0, T; L^2(\Omega)),$$

equipped with the norm

$$\|f\|_{s, \frac{s}{2}} = \left(\|f\|_{L^2(0, T; H^s(\Omega))}^2 + \|f\|_{H^{\frac{s}{2}}(0, T; L^2(\Omega))}^2 \right)^{\frac{1}{2}}.$$

As usual, $H_0^{s, \frac{s}{2}}(\Omega \times (0, T))$ is the closure of $C_0^\infty(\Omega \times (0, T])$ with respect to the norm $\|\cdot\|_{s, \frac{s}{2}}$, and $H^{-s, -\frac{s}{2}}(\Omega \times (0, T))$ is defined as the dual space of $H_0^{s, \frac{s}{2}}(\Omega \times (0, T))$. Let us define

$$a(\rho, \varphi) = \int_0^T \int_{\partial\Omega} \int_0^t \int_{\partial\Omega} L(\eta, \xi, t - \tau) \rho(\xi, \tau) \varphi(\eta, t) ds_\xi d\tau ds_\eta dt.$$

Then (12) can be written as: find $\rho \in H_0^{s-2, \frac{s}{2}-1}$, satisfying

$$(27) \quad a(\rho, \varphi) + \langle f, \varphi \rangle = 0, \quad \forall \varphi \in H^{-s, -\frac{s}{2}}.$$

Since (5)-(9) is well-posed, and (27) is equivalent to it, (27) is also well-posed. However we will use (27) to define approximate solutions, the existence and uniqueness of which is unknown, so we will prove the existence and uniqueness of a solution to (27) in the next section.

3. APPROXIMATE SOLUTIONS

We will use the generalized Lax-Milgram theorem (see [1]) to study (27). For the reader's convenience we recall it here.

Proposition 3.1. *If H_1 and H_2 are two real Hilbert spaces, and $a(u, v)$ is a bilinear form on $H_1 \times H_2$ such that*

$$|a(u, v)| \leq C \|u\|_{H_1} \|v\|_{H_2},$$

$$(28) \quad \inf_{u \in H_1, u \neq 0} \sup_{v \in H_2, v \neq 0} \frac{|a(u, v)|}{\|u\|_{H_1} \|v\|_{H_2}} \geq C_0,$$

$$\sup_{u \in H_1} |a(u, v)| > 0, \quad v \neq 0,$$

and if $f \in H_2'$, then there exists a unique solution to the equation

$$a(u, v) = f(v), \quad \forall v \in H_2,$$

such that

$$\|u\|_{H_1} \leq \frac{1}{C_0} \|f\|_{H_2'}.$$

We prove the following fact.

Lemma 3.1. *If $s > \frac{3}{2}$ and $f \in H_0^{s, \frac{s}{2}}$, then the problem (27) admits a unique solution $\rho \in H_0^{s-2, \frac{s}{2}-1}$, and*

$$(29) \quad \|\rho\|_{s-2, \frac{s}{2}-1} \leq C \|f\|_{s, \frac{s}{2}}.$$

Proof. Let us verify (28). For an arbitrary $\rho \in H_0^{s-2, \frac{s}{2}-1}$ we construct a single layer potential

$$(30) \quad \omega = \int_0^t \int_{\partial\Omega} K(x - \xi, t - \tau) \rho(\xi, \tau) ds_\xi d\tau.$$

Then it is known that

$$(31) \quad \|\omega\|_{s-\frac{1}{2}, \frac{s}{2}-\frac{1}{4}} \leq C \|\rho\|_{s-2, \frac{s}{2}-1}.$$

Let a function u be solved from (6), (7). Then $\nabla \cdot u = 0$ and $\omega = -\nabla \wedge u$. From the equation

$$\frac{\partial \omega}{\partial t} = \Delta \omega$$

we get

$$\nabla \wedge \left(\frac{\partial u}{\partial t} - \Delta u \right) = 0,$$

so there exists a function p such that

$$(32) \quad \frac{\partial u}{\partial t} - \Delta u = \nabla p,$$

and hence u is the solution to the Stokes equation.

Let $x \rightarrow \partial\Omega$ in (26); then we get a Fredholm integral equation for φ :

$$h(x) = -\frac{1}{2}\varphi(x) + \int_{\partial\Omega} M(x, \eta) \varphi(\eta) ds_\eta, \quad x \in \partial\Omega.$$

Owing to potential theory the mapping from h to φ is an isomorphism in H^s , where s is a real number. By (24) we have

$$\int_{\Omega} \omega h dx = \int_{\partial\Omega} u \cdot \nu h ds.$$

We take $h \in H^{-s, -\frac{s}{2}}$ such that

$$(33) \quad \begin{aligned} \int_0^T \int_{\Omega} \omega h dx dt &\geq \frac{1}{2} \|u \cdot \nu\|_{s, \frac{s}{2}} \|h\|_{-s, -\frac{s}{2}} \\ &\geq C \|u \cdot \nu\|_{s, \frac{s}{2}} \|\varphi\|_{-s, -\frac{s}{2}}. \end{aligned}$$

The estimate for the solutions to the Stokes equation yields

$$(34) \quad \begin{aligned} \|\omega\|_{s-\frac{1}{2}, \frac{s}{2}-\frac{1}{4}} &\leq C \|u\|_{s, \frac{s}{2}, \partial\Omega \times (0, T)} \\ &= C \|u \cdot \nu\|_{s, \frac{s}{2}, \partial\Omega \times (0, T)}. \end{aligned}$$

There is a one-to-one correspondence between ρ and $\frac{\partial \omega}{\partial n}|_{\partial \Omega}$ in $H^{s-2, \frac{s}{2}-1}$, so

$$(35) \quad \|\rho\|_{s-2, \frac{s}{2}-1} \leq C \|\omega\|_{s-\frac{1}{2}, \frac{s}{2}-\frac{1}{4}}$$

for $s > 2$. (35) is also true for $s \in (\frac{3}{2}, 2]$. Let us prove this assertion. Let $\varphi \in C_0^\infty(\partial \Omega \times (0, T))$; then we extend it and define v so that $v|_{\partial \Omega \times (0, T)} = \varphi$, $v \in C_0^\infty(\bar{\Omega} \times (0, T))$ and

$$\|v\|_{-s+\frac{5}{2}, -\frac{s}{2}+\frac{5}{4}} \leq C \|\varphi\|_{-s+2, -\frac{s}{2}+1}.$$

We have

$$\begin{aligned} 0 &= \int_0^T \int_{\Omega} \left(\frac{\partial \omega}{\partial t} - \Delta \omega \right) v \, dx \, dt \\ &= \int_0^T \int_{\Omega} \left(\frac{\partial \omega}{\partial t} v + \nabla \omega \cdot \nabla v \right) \, dx \, dt - \int_0^T \int_{\partial \Omega} \frac{\partial \omega}{\partial n} \varphi \, ds_x \, dt, \end{aligned}$$

and

$$\left| \int_0^T \int_{\Omega} \nabla \omega \cdot \nabla v \, dx \, dt \right| \leq C \|v\|_{-s+\frac{5}{2}, -\frac{s}{2}+\frac{5}{4}} \|\omega\|_{s-\frac{1}{2}, \frac{s}{2}-\frac{1}{4}},$$

$$\begin{aligned} \left| \int_0^T \int_{\Omega} \frac{\partial \omega}{\partial t} v \, dx \, dt \right| &\leq C \|v\|_{H^{-\frac{s}{2}+\frac{5}{4}}(0, T; L^2(\Omega))} \left\| \frac{\partial \omega}{\partial t} \right\|_{H^{\frac{s}{2}-\frac{5}{4}}(0, T; L^2(\Omega))} \\ &\leq C \|v\|_{-s+\frac{5}{2}, -\frac{s}{2}+\frac{5}{4}} \|\omega\|_{s-\frac{1}{2}, \frac{s}{2}-\frac{1}{4}}. \end{aligned}$$

Hence

$$\left\| \frac{\partial \omega}{\partial n} \right\|_{s-2, \frac{s}{2}-1, \partial \Omega \times (0, T)} \leq C \|\omega\|_{s-\frac{1}{2}, \frac{s}{2}-\frac{1}{4}},$$

which yields (35).

It follows from (33)-(35) that

$$\int_0^T \int_{\Omega} \omega h \, dx \, dt \geq C \|\rho\|_{s-2, \frac{s}{2}-1} \|\varphi\|_{-s, -\frac{s}{2}}.$$

It is easy to see that

$$a(\rho, \varphi) = \int_0^T \int_{\Omega} \omega h \, dx \, dt,$$

so (28) is verified. To verify the other conditions of Proposition 3.1 is straightforward. $\square$

Using the fundamental solution K , the solution ω_k of the scheme (14)-(22) can be expressed as

$$(36) \quad \begin{aligned} \omega_k(x, t_i - 0) = & \int_{\mathbb{R}^2} K(x - \xi, t_i) \omega_0(\xi) d\xi \\ & - \sum_{j=0}^{i-1} \int_{\Omega} K(x - \xi, t_i - t_j) \int_{t_j}^{t_{j+1}} F(\xi, \tau) d\xi d\tau \\ & + \sum_{j=0}^{i-1} \int_{\partial\Omega} K(x - \xi, t_i - t_j) \int_{t_j}^{t_{j+1}} \rho_k(\xi, \tau) d\tau ds_\xi, \end{aligned}$$

where $F = \tilde{u}_k \cdot \nabla \tilde{\omega}_k$ and ρ_k is the solution to (18).

4. UPPER BOUND ESTIMATE FOR LINEAR PROBLEMS

To analyse the above scheme for the Navier-Stokes equations, we study the same scheme for linear problems first. We replace (5) by

$$(37) \quad \frac{\partial \omega}{\partial t} + F(x, t) = \Delta \omega,$$

where F is a known function. This makes (37) (6)-(9) a linear problem. We use the same scheme to solve it, and estimate the approximate solution in this and the next section. Likewise, we assume that the exact solution is sufficiently smooth.

To estimate the approximate solution, let us first define some operators. We define $A = -\Delta + I$, where I is the identity. The semigroup of operators generated from A in $L^2(\mathbb{R}^2)$ is denoted by e^{-At} . We define $B : H^s(\Omega) \rightarrow H^{s+\frac{1}{2}}(\partial\Omega)$ by $B : \omega \rightarrow g$,

$$g(\eta) = \int_{\Omega} M(x, \eta) \omega(x) dx,$$

and the restriction operator from $H^s(\mathbb{R}^2)$ to $H^s(\Omega)$ is denoted by R .

Let us estimate f_k first. The first term in (20) is sufficiently regular, so we need to estimate the second term only. Let $i_0 = [\frac{T}{k}]$, and

$$h_i = \sum_{j=0}^{i-1} B R e^{-(A-I)(t_i-t_j)} \int_{t_j}^{t_{j+1}} F(\tau) d\tau,$$

where we understand $F(\tau)$ to be the zero extension of F to the space $\mathbb{R}^2$.

Lemma 4.1. *If $2 \leq s < 3$, then*

$$\sum_{i=1}^{i_0} \|h_i\|_{s, \partial\Omega}^2 \leq C \|F\|_{L^2(0, T; H^\beta(\Omega))}^2,$$

where $\frac{1}{2} > \beta > s - \frac{5}{2}$.

Proof. By the properties of analytic semigroups of operators (see [6]) we have

$$\begin{aligned}
\sum_{i=1}^{i_0} \|h_i\|_{s, \partial\Omega}^2 k &\leq C \sum_{i=1}^{i_0} \left\| \sum_{j=0}^{i-1} e^{-(A-I)(t_i-t_j)} \int_{t_j}^{t_{j+1}} F(\tau) d\tau \right\|_{s-\frac{1}{2}}^2 k \\
&\leq C \sum_{i=1}^{i_0} \left\| \sum_{j=0}^{i-1} A^{\frac{s}{2}-\frac{1}{4}} e^{-(A-I)(t_i-t_j)} \int_{t_j}^{t_{j+1}} F(\tau) d\tau \right\|_0^2 k \\
&\leq C \sum_{i=1}^{i_0} \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{s-\beta}{2}+\frac{1}{4}} \|A^{\frac{\beta}{2}} F(\tau)\|_0 d\tau \right)^2 k \\
&\leq C \sum_{i=1}^{i_0} \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{s-\beta}{2}+\frac{1}{4}} \|F(\tau)\|_{\beta, \Omega} d\tau \right)^2 k \\
&\leq C \sum_{i=1}^{i_0} \left\{ \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{s-\beta}{2}+\frac{1}{4}} d\tau \right\} \\
&\quad \cdot \left\{ \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{s-\beta}{2}+\frac{1}{4}} \|F(\tau)\|_{\beta, \Omega}^2 d\tau \right\} k \\
&\leq C \sum_{i=1}^{i_0} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{s-\beta}{2}+\frac{1}{4}} \|F(\tau)\|_{\beta, \Omega}^2 d\tau k \\
&= C \sum_{j=0}^{i_0-1} \int_{t_j}^{t_{j+1}} \|F(\tau)\|_{\beta, \Omega}^2 d\tau \sum_{i=j+1}^{i_0} (t_i - t_j)^{-\frac{s-\beta}{2}+\frac{1}{4}} k \\
&\leq C \|F\|_{L^2(0, T; H^\beta(\Omega))}^2.
\end{aligned}$$

□

To estimate $\|fk\|_{H^{\frac{s}{2}}(0, T; L^2(\partial\Omega))}$ we prove one auxiliary lemma first.

Lemma 4.2. *If $\beta \in (0, \frac{1}{2})$ and $g : [0, t_{i_0}] \rightarrow \mathbb{R}$ is a piecewise constant function, $g(t) = g_i$, $t \in (t_i, t_{i+1})$, then*

$$\int_0^{t_{i_0}} \int_0^{t_{i_0}} \frac{(g(t) - g(\tau))^2}{|t - \tau|^{1+2\beta}} dt d\tau \leq C \sum_{i \neq j} \frac{(g_i - g_j)^2}{|t_i - t_j|^{1+2\beta}} k^2.$$

Proof. The left hand side can be divided into two parts:

$$\begin{aligned}
\sum_{i=1}^{i_0-1} \int_{t_{i-1}}^{t_i} \int_{t_i}^{t_{i+1}} \frac{(g_{i-1} - g_i)^2}{|t - \tau|^{1+2\beta}} dt d\tau &\leq \frac{k^{1-2\beta}}{2\beta(1-2\beta)} \sum_i (g_{i+1} - g_i)^2, \\
\sum_{|i-j| \geq 2} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} \frac{(g_i - g_j)^2}{|t - \tau|^{1+2\beta}} dt d\tau &\leq C \sum_{|i-j| \geq 2} \frac{(g_i - g_j)^2}{|t_i - t_j|^{1+2\beta}} k^2.
\end{aligned}$$

Then the conclusion follows. □

By (19), $\frac{\partial f_k}{\partial \tau}$ is a piecewise constant function,

$$\frac{\partial f_k}{\partial \tau} = \frac{1}{k}(f_{k\ i+1} - f_{k\ i}), \quad \tau \in (t_i, t_{i+1}).$$

We define

$$(38) \quad g_i = \frac{1}{k}(h_{i+1} - h_i).$$

Lemma 4.3. *If $\frac{5}{2} < s < 3$, then*

$$\sum_{i \neq j} \frac{\|g_i - g_j\|_0^2}{|t_i - t_j|^{s-1}} k^2 \leq C \|F\|_{H^{\frac{s}{2}-1}(0,T;H^{-\frac{1}{2}+\varepsilon}(\Omega))}^2,$$

where $\varepsilon > 0$.

Proof. The left hand side is equal to

$$\begin{aligned} & 2 \sum_{j < i} \frac{\|g_i - g_j\|_0^2}{|t_i - t_j|^{s-1}} k^2 \\ &= 2 \sum_{j < i} \frac{1}{|t_i - t_j|^{s-1}} \left\| \left(\sum_{l=0}^i BRe^{-(A-I)(t_{i+1}-t_l)} \right. \right. \\ & \quad \left. \left. - \sum_{l=0}^{i-1} BRe^{-(A-I)(t_i-t_l)} \right) \int_{t_l}^{t_{i+1}} F(\tau) d\tau \right. \\ & \quad \left. - \left(\sum_{l=0}^j BRe^{-(A-I)(t_{j+1}-t_l)} - \sum_{l=0}^{j-1} BRe^{-(A-I)(t_j-t_l)} \right) \int_{t_l}^{t_{j+1}} F(\tau) d\tau \right\|_0^2. \end{aligned}$$

We extend F by zero for $\tau < 0$; then the right hand side is equal to

$$\begin{aligned} & 2 \sum_{j < i} \frac{1}{|t_i - t_j|^{s-1}} \left\| \sum_{l=j-i}^j BRe^{-(A-I)(t_{j+1}-t_l)} \int_{t_l}^{t_{i+1}} (F(\tau + t_i - t_j) - F(\tau)) d\tau \right. \\ & \quad \left. - \sum_{l=j-i}^{j-1} BRe^{-(A-I)(t_j-t_l)} \int_{t_l}^{t_{i+1}} (F(\tau + t_i - t_j) - F(\tau)) d\tau \right\|_0^2 \\ & \leq 4 \sum_{j < i} \frac{1}{|t_i - t_j|^{s-1}} \left\| \sum_{l=j-i}^{j-1} BR(e^{-(A-I)(t_{j+1}-t_l)} \right. \\ & \quad \left. - e^{-(A-I)(t_j-t_l)}) \int_{t_l}^{t_{i+1}} (F(\tau + t_i - t_j) - F(\tau)) d\tau \right\|_0^2 \\ & \quad + 4 \sum_{j < i} \frac{1}{|t_i - t_j|^{s-1}} \left\| BRe^{-(A-I)k} \int_{t_j}^{t_{j+1}} (F(\tau + t_i - t_j) - F(\tau)) d\tau \right\|_0^2 \\ & \equiv a_1 + a_2. \end{aligned}$$

Due to the properties of analytic semigroups we have

$$\begin{aligned} & \left\| BR(e^{-(A-I)(t_{j+1}-t_l)} - e^{-(A-I)(t_j-t_l)})(F(\tau+t_i-t_j) - F(\tau)) \right\|_{0,\partial\Omega} \\ & \leq Ck \left\| A^{\frac{3}{4}} e^{-(A-I)(t_j-t_l)}(F(\tau+t_i-t_j) - F(\tau)) \right\|_0 \\ & \leq Ck(t_j-t_l)^{-1+\frac{\varepsilon}{2}} \|F(\tau+t_i-t_j) - F(\tau)\|_{-\frac{1}{2}+\varepsilon}; \end{aligned}$$

then, following the same lines of the proof of Lemma 4.1, we get

$$a_1 \leq C \sum_{j < i} \frac{k^2}{|t_i - t_j|^{s-1}} \sum_{l=j-i}^{j-1} \int_{t_l}^{t_{l+1}} (t_j - t_l)^{-1+\frac{\varepsilon}{2}} \|F(\tau+t_i-t_j) - F(\tau)\|_{-\frac{1}{2}+\varepsilon}^2 d\tau.$$

Let $i' = i - j$, then the right hand side is equal to

$$\begin{aligned} & C \sum_{i'=1}^{i_0-1} \sum_{l=-i'}^{i_0-1-i'} \sum_{j>l} \frac{k^2}{(i'k)^{s-1}} \int_{t_l}^{t_{l+1}} (t_j - t_l)^{-1+\frac{\varepsilon}{2}} \|F(\tau+i'k) - F(\tau)\|_{-\frac{1}{2}+\varepsilon}^2 d\tau \\ & \leq C \sum_{i'=1}^{i_0-1} k \sum_{l=-i'}^{i_0-1-i'} \int_{t_l}^{t_{l+1}} \frac{\|F(\tau+i'k) - F(\tau)\|_{-\frac{1}{2}+\varepsilon}^2}{(i'k)^{s-1}} d\tau \\ & \leq C \|F\|_{H^{\frac{s}{2}-1}(0,T;H^{-\frac{1}{2}+\varepsilon}(\Omega))}^2. \end{aligned}$$

For the second term,

$$\begin{aligned} a_2 & \leq C \sum_{j < i} \frac{1}{|t_i - t_j|^{s-1}} \left(\int_{t_j}^{t_{j+1}} \|F(\tau+t_i-t_j) - F(\tau)\|_{-\frac{1}{2}+\varepsilon} d\tau \right)^2 \\ & \leq C \sum_{i=1}^{i_0} \sum_{j < i} \frac{k \int_{t_j}^{t_{j+1}} \|F(\tau+t_i-t_j) - F(\tau)\|_{-\frac{1}{2}+\varepsilon}^2 d\tau}{|t_i - t_j|^{s-1}} \\ & = C \sum_{i'=1}^{i_0-1} \sum_{j=0}^{i_0-1-i'} \frac{k \int_{t_j}^{t_{j+1}} \|F(\tau+i'k) - F(\tau)\|_{-\frac{1}{2}+\varepsilon}^2 d\tau}{(i'k)^{s-1}} \\ & \leq C \|F\|_{H^{\frac{s}{2}-1}(0,T;H^{-\frac{1}{2}+\varepsilon}(\Omega))}^2. \end{aligned}$$

Thus the proof is complete. $\square$

Combining Lemma 4.1, Lemma 4.2, and Lemma 4.3, we get for $\frac{5}{2} < s < 3$ that

$$(39) \quad \|f_k\|_{s, \frac{s}{2}} \leq C \left\{ \|F\|_{L^2(0,T;H^\beta(\Omega))} + \|F\|_{H^{\frac{s}{2}-1}(0,T;H^{-\frac{1}{2}+\varepsilon}(\Omega))} \right\} + C(\omega_0),$$

where $C(\omega_0)$ is a constant depending on ω_0 only. Lemma 3.1 implies

$$(40) \quad \|\rho_k\|_{s-2, \frac{s}{2}-1} \leq C \left\{ \|F\|_{L^2(0,T;H^\beta(\Omega))} + \|F\|_{H^{\frac{s}{2}-1}(0,T;H^{-\frac{1}{2}+\varepsilon}(\Omega))} \right\} + C(\omega_0).$$

We are going to estimate the solutions ω_k and $\tilde{\omega}_k$. Equation (14) is reduced to

$$\frac{\partial \tilde{\omega}_k}{\partial t} + F = 0;$$

hence

$$\tilde{\omega}_k(\tau) = - \int_{t_i}^{\tau} F d\tau + \omega_k(t_i - 0),$$

so $\tilde{\omega}_k$ can be estimated in terms of F and ω_k . It suffices to estimate ω_k . We define a vector function χ_k satisfying

$$\begin{aligned}\frac{\partial \chi_k}{\partial t} &= \Delta \chi_k, \\ \chi_k|_{x \in \partial \Omega} &= \rho_k n, \\ \chi_k|_{t=0} &= 0.\end{aligned}$$

Lemma 4.4. *We have*

$$(41) \quad \|\chi_k\|_{s-\frac{3}{2}, \frac{s}{2}-\frac{3}{4}, \Omega \times (0, T)} \leq C \|\rho_k\|_{s-2, \frac{s}{2}-1, \partial \Omega \times (0, T)}$$

for all $s \in \mathbb{R}$, provided $\rho_k \in H_0^{s-2, \frac{s}{2}-1}$ and χ_k is a classical solution.

Proof. By classical potential theory, (41) is valid for $s > 2$. If $s < -2$, we take $\varphi \in C_0^\infty(\Omega \times (0, T))$ and let v satisfy

$$\begin{aligned}\frac{\partial v}{\partial t} + \Delta v &= \varphi, \\ v|_{x \in \partial \Omega} &= 0, \\ v|_{t=T} &= 0.\end{aligned}$$

We have

$$\begin{aligned}\int_0^T \int_\Omega \chi_k \varphi \, dx \, dt &= \int_0^T \int_\Omega \chi_k \left(\frac{\partial v}{\partial t} + \Delta v \right) \, dx \, dt \\ &= \int_0^T \int_\Omega \left(-v \frac{\partial \chi_k}{\partial t} + v \Delta \chi_k \right) \, dx \, dt + \int_0^T \int_{\partial \Omega} \chi_k \frac{\partial v}{\partial n} \, ds_x \, dt \\ &= \int_0^T \int_{\partial \Omega} \rho_k n \frac{\partial v}{\partial n} \, ds_x \, dt.\end{aligned}$$

Then

$$\begin{aligned}\left| \int_0^T \int_\Omega \chi_k \varphi \, dx \, dt \right| &\leq C \|\rho_k\|_{s-2, \frac{s}{2}-1} \|v\|_{3-s, \frac{3}{2}-\frac{s}{2}} \\ &\leq C \|\rho_k\|_{s-2, \frac{s}{2}-1} \|\varphi\|_{\frac{3}{2}-s, \frac{3}{4}-\frac{s}{2}},\end{aligned}$$

which yields (41). For $s \in [-2, 2]$, (41) can be obtained by interpolation. $\square$

Theorem 4.1. *If $1 < \sigma < \frac{3}{2}$, then*

$$\|\omega_k(t_i - 0)\|_{\sigma, \Omega} \leq C \|F\|_{L^2(0, T; H^{s-\frac{5}{2}}(\Omega))} + C \|\rho_k\|_{s-2, \frac{s}{2}-1} + C(\omega_0),$$

for all i , where $\sigma + \frac{3}{2} < s < 3$.

Proof. We apply (36). Let us consider the third term first. By integration by parts, we have

$$\begin{aligned}(42) \quad &\int_{\partial \Omega} K(x - \xi, t_i - t_j) \rho_k(\xi, \tau) \, ds_\xi \\ &= \int_\Omega \nabla_\xi K(x - \xi, t_i - t_j) \cdot \chi_k \, d\xi + \int_\Omega K(x - \xi, t_i - t_j) \nabla_\xi \cdot \chi_k \, d\xi.\end{aligned}$$

Let χ_k be extended by zero to $\mathbb{R}^2$. In view of Lemma 4.4 we get

$$\begin{aligned} & \left\| \sum_{j=0}^{i-1} R A^{\frac{1}{2}} e^{-(A-I)(t_i-t_j)} \int_{t_j}^{t_{j+1}} \chi_k(\tau) d\tau \right\|_{\sigma} \\ & \leq C \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \|A^{\frac{1+\sigma}{2}} e^{-(A-I)(t_i-t_j)} \chi_k(\tau)\|_0 d\tau \equiv a_1. \end{aligned}$$

We take $r = s - \frac{5}{2} \in (\sigma - 1, \frac{1}{2})$; then

$$\begin{aligned} a_1 & \leq C \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{1+\sigma-r}{2}} \|\chi_k(\tau)\|_r d\tau \\ & \leq C \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{1+\sigma-r}{2} p} d\tau \right)^{\frac{1}{p}} \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \|\chi_k(\tau)\|_r^q d\tau \right)^{\frac{1}{q}}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. We take $p = \frac{4}{2s-1-2r}$; then $\frac{1+\sigma-r}{2}p < 1$, and due to the interpolation of Sobolev spaces we get

$$a_1 \leq C \|\chi_k\|_{L^q(0,T;H^r(\Omega))} \leq C \|\chi_k\|_{s-\frac{3}{2}, \frac{s}{2}-\frac{3}{4}}.$$

For the second term of (42) we extend $\nabla \cdot \chi_k$ by zero to $\mathbb{R}^2$ and note that $\|\nabla \cdot \chi_k\|_{r,\mathbb{R}^2} \leq C \|\chi_k\|_{1+r,\Omega}$, then take $p = q = 2$ to obtain

$$\begin{aligned} & \left\| \sum_{j=0}^{i-1} R e^{-(A-I)(t_i-t_j)} \int_{t_j}^{t_{j+1}} \nabla \cdot \chi_k(\tau) d\tau \right\|_{\sigma} \\ & \leq C \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{\sigma-r}{2}} \|\chi_k(\tau)\|_{1+r} d\tau \\ & \leq C \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\sigma+r} d\tau \right)^{\frac{1}{2}} \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \|\chi_k(\tau)\|_{1+r}^2 d\tau \right)^{\frac{1}{2}} \\ & \leq C \|\chi_k\|_{L^2(0,T;H^{s-\frac{3}{2}})}. \end{aligned}$$

The second term of (36) can be estimated in the same way. $\square$

5. ERROR ESTIMATE FOR LINEAR PROBLEMS

Lemma 5.1. *If $0 < \beta < \frac{1}{2}$ and $\varepsilon > 0$, then*

$$(43) \quad \|f - f_k\|_{\frac{3}{2}+\varepsilon, \frac{3}{4}+\frac{\varepsilon}{2}} \leq C k^{\frac{7}{16}+\frac{3\beta}{8}-\frac{\varepsilon}{2}}.$$

Proof. We get from (13) and (20) that

$$f(t_i) - f_{k,i} = \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} B R (e^{-(A-I)(t_i-\tau)} - e^{-(A-I)(t_i-t_j)}) F(\tau) d\tau.$$

Let $G(\tau) = (I - e^{-(A-I)(\tau-t_j)})F(\tau)$ and

$$a_1 = \sum_{j=0}^{i-2} \int_{t_j}^{t_{j+1}} BRe^{-(A-I)(t_i-\tau)} G(\tau) d\tau,$$

$$a_2 = \int_{t_{i-1}}^{t_i} BRe^{-(A-I)(t_i-\tau)} G(\tau) d\tau;$$

then $f(t_i) - f_{k,i} = a_1 + a_2$. We have

$$\begin{aligned} \|a_1\|_{\frac{3}{2}+\varepsilon, \partial\Omega} &\leq C\|a_1\|_{2+\varepsilon, \Omega} \leq C \sum_{j=0}^{i-2} \int_{t_j}^{t_{j+1}} \|e^{-(A-I)(t_i-\tau)} G(\tau)\|_{1+\varepsilon} d\tau \\ &\leq C \sum_{j=0}^{i-2} \int_{t_j}^{t_{j+1}} (t_i - \tau)^{-1+\frac{\varepsilon}{2}} \|G(\tau)\|_{-1+2\varepsilon} d\tau \\ &\leq C \sum_{j=0}^{i-2} \int_{t_j}^{t_{j+1}} (t_i - \tau)^{-1+\frac{\varepsilon}{2}} k^{\frac{1+\beta}{2}-\varepsilon} \|F(\tau)\|_{\beta} d\tau, \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{i=1}^{i_0} \|a_1\|_{\frac{3}{2}+\varepsilon, \partial\Omega}^2 k &\leq C \sum_{i=1}^{i_0} \left(\sum_{j=0}^{i-2} \int_{t_j}^{t_{j+1}} (t_i - \tau)^{-1+\frac{\varepsilon}{2}} k^{\frac{1+\beta}{2}-\varepsilon} \|F(\tau)\|_{\beta} d\tau \right)^2 k \\ &\leq C \sum_{i=1}^{i_0} \left(\sum_{j=0}^{i-2} \int_{t_j}^{t_{j+1}} (t_i - \tau)^{-1+\frac{\varepsilon}{2}} d\tau \right) \\ &\quad \cdot \left(\sum_{j=0}^{i-2} \int_{t_j}^{t_{j+1}} (t_i - \tau)^{-1+\frac{\varepsilon}{2}} k^{1+\beta-2\varepsilon} \|F(\tau)\|_{\beta}^2 d\tau \right) k \\ &\leq C \sum_{j=0}^{i_0-2} \int_{t_j}^{t_{j+1}} \|F(\tau)\|_{\beta}^2 k^{1+\beta-2\varepsilon} \sum_{i=j+2}^{i_0} (t_i - \tau)^{-1+\varepsilon} k d\tau \\ &\leq C k^{1+\beta-2\varepsilon} \|F\|_{L^2(0,T;H^{\beta}(\Omega))}^2. \end{aligned}$$

Besides, we have

$$\begin{aligned} \sum_{i=1}^{i_0} \|a_2\|_{\frac{3}{2}+\varepsilon, \partial\Omega}^2 k &\leq \sum_{i=1}^{i_0} \left(\int_{t_{i-1}}^{t_i} \|e^{-(A-I)(t_i-\tau)} (I - e^{-(A-I)(\tau-t_{i-1})}) F(\tau)\|_{1+\varepsilon} d\tau \right)^2 k \\ &\leq C \sum_{i=1}^{i_0} \int_{t_{i-1}}^{t_i} \|F(\tau)\|_{\beta}^2 d\tau \int_{t_{i-1}}^{t_i} (t_i - \tau)^{-1-\varepsilon+\beta} k d\tau \\ &\leq C k^{1+\beta-\varepsilon} \|F\|_{L^2(0,T;H^{\beta}(\Omega))}^2. \end{aligned}$$

Consequently,

$$(44) \quad \sum_{i=1}^{i_0} \|f(t_i) - f_{k,i}\|_{\frac{3}{2}+\varepsilon, \partial\Omega}^2 k \leq Ck^{1+\beta-2\varepsilon} \|F\|_{L^2(0,T;H^\beta(\Omega))}^2.$$

In the same way,

$$(45) \quad \sum_{i=1}^{i_0} \|f(t_i) - f_{k,i}\|_{0,\partial\Omega}^2 k \leq Ck^2 \|F\|_{L^2(0,T;H^\beta(\Omega))}^2.$$

Let

$$\begin{aligned} h_i &= \frac{1}{k}(f(t_i) - f_{k,i} - f(t_{i-1}) + f_{k,i-1}) \\ &= \frac{1}{k} \sum_{j=0}^{i-2} \int_{t_j}^{t_{j+1}} BR(e^{-(A-I)(t_i-\tau)} - e^{-(A-I)(t_{i-1}-\tau)})G(\tau) d\tau + \frac{a_2}{k}. \end{aligned}$$

Following the same lines, we can prove that

$$(46) \quad \sum_{i=1}^{i_0} \|h_i\|_0^2 k \leq Ck^{\frac{1}{2}+\beta} \|F\|_{L^2(0,T;H^\beta(\Omega))}^2.$$

The function f_k is linear in each interval $[t_{i-1}, t_i]$. The function f is not linear, but f is sufficiently smooth. It is easy to estimate the difference between f and its linear interpolation; then by (44)-(46) we get

$$\|f - f_k\|_{L^2(0,T;H^{\frac{3}{2}+\varepsilon}(\partial\Omega))} \leq Ck^{\frac{1+\beta}{2}-\varepsilon},$$

$$\|f - f_k\|_{L^2(0,T;L^2(\partial\Omega))} \leq Ck,$$

$$\|f - f_k\|_{H^1(0,T;L^2(\partial\Omega))} \leq Ck^{\frac{1}{4}+\frac{\beta}{2}}.$$

By interpolation we get

$$\|f - f_k\|_{H^{\frac{3}{4}+\frac{\varepsilon}{2}}(0,T;L^2(\partial\Omega))} \leq Ck^{\frac{7}{16}+\frac{3\beta}{8}-\frac{\varepsilon}{2}}.$$

Then (43) follows. □

Lemma 5.1 and Lemma 3.1 imply

$$(47) \quad \|\rho - \rho_k\|_{-\frac{1}{2}+\varepsilon, -\frac{1}{4}+\frac{\varepsilon}{2}} \leq Ck^{\frac{7}{16}+\frac{3\beta}{8}-\frac{\varepsilon}{2}}.$$

We turn now to estimating the error $\omega - \omega_k$.

Theorem 5.1. *If $1 < \sigma < \frac{3}{2}$, then*

$$(48) \quad \|\omega(t_i) - \omega_k(t_i - 0)\|_{\sigma-2, \Omega} \leq Ck^{\frac{7}{16} + \frac{3\beta}{8} - \frac{\varepsilon}{2}},$$

for all i , where $0 < \beta < \frac{1}{2}, \varepsilon > 0$.

Proof. We get from (11) and (36) that

$$\begin{aligned} & \omega(x, t_i) - \omega_k(x, t_i - 0) \\ &= - \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \int_{\Omega} (K(x - \xi, t_i - \tau) - K(x - \xi, t_i - t_j)) F(\xi, \tau) d\xi d\tau \\ & \quad + \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \int_{\partial\Omega} (K(x - \xi, t_i - \tau) \rho(\xi, \tau) - K(x - \xi, t_i - t_j) \rho_k(\xi, \tau)) ds_{\xi} d\tau \\ &= - \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \int_{\Omega} (K(x - \xi, t_i - \tau) - K(x - \xi, t_i - t_j)) F(\xi, \tau) d\xi d\tau \\ & \quad + \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \int_{\partial\Omega} (K(x - \xi, t_i - \tau) - K(x - \xi, t_i - t_j)) \rho_k(\xi, \tau) ds_{\xi} d\tau \\ & \quad + \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \int_{\partial\Omega} K(x - \xi, t_i - \tau) (\rho(\xi, \tau) - \rho_k(\xi, \tau)) ds_{\xi} d\tau \\ &\equiv a_1 + a_2 + a_3. \end{aligned}$$

The sum in a_1, a_2 can be divided into two parts: $j = 0, \dots, i-2$, and $j = i-1$. For $j = i-1$ we have

$$\begin{aligned} & \left\| \int_{t_{i-1}}^{t_i} \left(e^{-(A-I)(t_i-\tau)} - e^{-(A-I)t_i} \right) F(\tau) d\tau \right\|_{\sigma-2} \\ & \leq C \int_{t_{i-1}}^{t_i} \|F(\tau)\|_{\beta} d\tau \leq Ck, \end{aligned}$$

and for $j = 0, \dots, i-2$ we have

$$\begin{aligned} & \left\| \sum_{j=0}^{i-2} \int_{t_j}^{t_{j+1}} \left(e^{-(A-I)(t_i-\tau)} - e^{-(A-I)(t_i-t_j)} \right) F(\tau) d\tau \right\|_{\sigma-2} \\ & \leq C \sum_{j=0}^{i-2} \int_{t_j}^{t_{j+1}} \left\| \int_{t_i-t_j}^{t_i-\tau} e^{-(A-I)\zeta} d\zeta F(\tau) \right\|_{\sigma} d\tau \\ & \leq Ck \sum_{j=0}^{i-2} \int_{t_j}^{t_{j+1}} (t_i - \tau)^{-\frac{\sigma-\beta}{2}} \|F(\tau)\|_{\beta} d\tau \leq Ck. \end{aligned}$$

We get for the first part of a_2 that

$$\begin{aligned}
\|a_{21}\|_{\sigma-2} &= \left\| \int_{t_{i-1}}^{t_i} RA^{\frac{1}{2}} \left(e^{-(A-I)(t_i-\tau)} - e^{-(A-I)k} \right) \chi_k(\tau) d\tau \right. \\
&\quad \left. + \int_{t_{i-1}}^{t_i} R \left(e^{-(A-I)(t_i-\tau)} - e^{-(A-I)k} \right) \nabla \cdot \chi_k(\tau) d\tau \right\|_{\sigma-2} \\
&\leq C \int_{t_{i-1}}^{t_i} \left\| A^{\frac{\sigma}{2}-\frac{1}{2}} \left(e^{-(A-I)(t_i-\tau)} - e^{-(A-I)k} \right) \chi_k(\tau) \right\|_0 d\tau \\
&\quad + C \int_{t_{i-1}}^{t_i} \left\| A^{\frac{\sigma}{2}-1} \left(e^{-(A-I)(t_i-\tau)} - e^{-(A-I)k} \right) \nabla \cdot \chi_k(\tau) \right\|_0 d\tau \\
&\leq C \int_{t_{i-1}}^{t_i} (t_i - \tau)^{\frac{s-\sigma}{2}-\frac{3}{4}-\frac{\varepsilon}{2}} \|\chi_k(\tau)\|_{s-\frac{5}{2}-\varepsilon} d\tau \\
&\quad + C \int_{t_{i-1}}^{t_i} (t_i - \tau)^{\frac{s-\sigma}{2}-\frac{1}{4}} \|\chi_k(\tau)\|_{s-\frac{3}{2}} d\tau \\
&\leq C \int_{t_{i-1}}^{t_i} (t_i - \tau)^{\frac{s-\sigma}{2}-\frac{3}{4}-\frac{\varepsilon}{2}} d\tau \sup_{\tau} \|\chi_k(\tau)\|_{s-\frac{5}{2}-\varepsilon} \\
&\quad + C \left(\int_{t_{i-1}}^{t_i} (t_i - \tau)^{s-\sigma-\frac{1}{2}} d\tau \right)^{\frac{1}{2}} \left(\int_{t_{i-1}}^{t_i} \|\chi_k(\tau)\|_{s-\frac{3}{2}}^2 d\tau \right)^{\frac{1}{2}}.
\end{aligned}$$

We take $s = \sigma + \frac{3}{2}$ and notice (41), then get

$$\|a_{21}\|_{\sigma-2} \leq Ck^{1-\frac{\varepsilon}{2}}.$$

For the second part of a_2 we can repeat the proof of Theorem 4.1 and get an upper bound of Ck . The details are omitted here.

Finally, by potential theory and (47) we have

$$\|a_3\|_{1+\varepsilon, \frac{1}{2}+\frac{\varepsilon}{2}} \leq Ck^{\frac{7}{16}+\frac{3\beta}{8}-\frac{\varepsilon}{2}}.$$

Then by the embedding theorem we get

$$\|a_3\|_{0,\Omega} \leq Ck^{\frac{7}{16}+\frac{3\beta}{8}-\frac{\varepsilon}{2}}.$$

Combining the above results, we obtain (48). $\square$

6. ESTIMATES FOR THE NAVIER-STOKES EQUATION

For the Navier-Stokes equation we denote $F_0 = u \cdot \nabla \omega$, where u and ω are exact solutions; then we regard F_0 as a known function. We denote by f^*, ρ^* and ω^* the approximate solutions to

$$\frac{\partial \omega}{\partial t} + F_0 = \Delta \omega$$

and (6)-(9). Then we define

$$(49) \quad \tilde{\omega}^*(\tau) = - \int_{t_i}^{\tau} F_0 d\tau + \omega^*(t_i - 0), \quad \tau \in [t_i, t_{i+1}).$$

The estimates in Section 4 and Section 5 are valid for ω^* and $\tilde{\omega}^*$, namely, we have

$$(50) \quad \|\omega^*(t_i - 0)\|_\sigma, \|\tilde{\omega}^*(t)\|_\sigma \leq C_1,$$

$$(51) \quad \|\omega(t_i) - \omega^*(t_i - 0)\|_{\sigma-2, \Omega} \leq C_2 k^{\frac{7}{16} + \frac{3\beta}{8} - \frac{\epsilon}{2}},$$

for $0 \leq i \leq i_0$ and $0 \leq t \leq T$. Certainly the constants C_1, C_2 depend on the exact solution. Using this auxiliary solution we estimate $\omega^* - \omega_k$; then we obtain the estimates for $\omega - \omega_k$.

We need one fact about Sobolev spaces. For the reader's convenience we recall it here:

Lemma 6.1. *If $h, g \in H^r(\Omega)$ with $1 < r < 2$, then*

$$(52) \quad \|hg\|_r \leq C\|h\|_r\|g\|_r,$$

and if $h \in H^{1+\varepsilon}(\Omega)$, $g \in H^r(\Omega)$ with $\varepsilon > 0$, $0 \leq r \leq 1$, then

$$(53) \quad \|hg\|_r \leq C\|h\|_{1+\varepsilon}\|g\|_r.$$

To estimate the upper bound and error for non-linear problems, let us prove the following lemmas first.

Lemma 6.2. *Let g_i be defined by (38). Then*

$$(54) \quad \sum_{i=1}^{i_0} \|g_i\|_{0, \partial\Omega}^2 k \leq C\|F\|_{L^2(0, T; H^{-\frac{1}{2}+\varepsilon}(\Omega))}^2,$$

where $\varepsilon > 0$.

Proof. We notice that

$$\begin{aligned} g_i &= \frac{1}{k} \sum_{j=0}^{i-1} BR \left(e^{-(A-I)(t_{i+1}-t_j)} - e^{-(A-I)(t_i-t_j)} \right) \int_{t_j}^{t_{j+1}} F(\tau) d\tau \\ &\quad + \frac{1}{k} BR e^{-(A-I)k} \int_{t_i}^{t_{i+1}} F(\tau) d\tau \\ &= - \sum_{j=0}^{i-1} BR \frac{1}{k} \int_{t_i-t_j}^{t_{i+1}-t_j} (A-I) e^{-(A-I)\zeta} d\zeta \int_{t_j}^{t_{j+1}} F(\tau) d\tau \\ &\quad + BR e^{-(A-I)k} \frac{1}{k} \int_{t_i}^{t_{i+1}} F(\tau) d\tau. \end{aligned}$$

Following the same lines as the proof of Lemma 4.1, we get (54). $\square$

For the Navier-Stokes equation, $F = \tilde{u}_k \cdot \nabla \tilde{\omega}_k$. We apply Lemma 4.1 and Lemma 6.2 to obtain

$$(55) \quad \|f_k\|_{2,1} \leq C\|F\|_{L^2(0, T; H^{-\frac{1}{2}+\varepsilon}(\Omega))} + C(\omega_0);$$

then Lemma 3.1 gives

$$(56) \quad \|\rho_k\|_{0,0} \leq C\|F\|_{L^2(0, T; H^{-\frac{1}{2}+\varepsilon}(\Omega))} + C(\omega_0).$$

Lemma 6.3.

$$\|\tilde{u}_k \cdot \nabla \tilde{\omega}_k\|_{-\frac{1}{2}+\varepsilon} \leq C\|\tilde{\omega}_k\|_{\frac{1}{2}+\varepsilon}^2.$$

Proof. We have

$$\tilde{u}_k \cdot \nabla \tilde{\omega}_k = -\nabla \wedge ((\tilde{u}_k \cdot \nabla) \tilde{u}_k) = -\nabla \wedge (\nabla(\tilde{u}_k \otimes \tilde{u}_k)).$$

Lemma 6.1 implies

$$\|\tilde{u}_k \cdot \nabla \tilde{\omega}_k\|_{-\frac{1}{2}+\varepsilon} = \|\tilde{u}_k \otimes \tilde{u}_k\|_{\frac{3}{2}+\varepsilon} \leq C \|\tilde{u}_k\|_{\frac{3}{2}+\varepsilon}^2 \leq C \|\tilde{\omega}_k\|_{\frac{1}{2}+\varepsilon}^2.$$

□

Lemma 6.4. Assume that $1 < \sigma < \frac{3}{2}$, $0 < \varepsilon \leq \frac{\sigma-1}{2}$, and

$$(57) \quad \|\tilde{\omega}_k\|_{L^2(0,T;H^{\sigma-\varepsilon}(\Omega))} + \|\tilde{\omega}_k\|_{\sigma-1-\varepsilon} \leq M_0.$$

Then

$$(58) \quad \left(\sum_{l=0}^i \|\omega_k(t_l - 0)\|_{\sigma}^2 k \right)^{\frac{1}{2}} + \|\omega_k(t_i - 0)\|_{\sigma-1} + k \|\omega_k(t_i - 0)\|_{\sigma+1} \leq M_1,$$

where M_1 depends on M_0 .

Proof. We apply (36) and (42) to estimate $\omega_k(t_i - 0)$. We take $\beta \in (\sigma - 1, \frac{1}{2})$, then get

$$\left\| \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} A^{\frac{1}{2}} e^{-(A-I)(t_i-t_j)} \chi_k(\tau) d\tau \right\|_{\sigma} \leq C \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{1+\sigma-\beta}{2}} \|\chi_k(\tau)\|_{\beta} d\tau.$$

Then

$$\begin{aligned} & \sum_i \left\| \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} A^{\frac{1}{2}} e^{-(A-I)(t_i-t_j)} \chi_k(\tau) d\tau \right\|_{\sigma}^2 k \\ & \leq C \int_0^T \|\chi_k(\tau)\|_{\beta}^2 d\tau = C \|\chi_k\|_{L^2(0,T;H^{\beta}(\Omega))}^2. \end{aligned}$$

The other terms in (36) and (42) can be estimated in the same way. By Lemma 4.4,

$$(59) \quad \|\chi_k\|_{L^2(0,T;H^{\beta}(\Omega))} \leq C \|\rho_k\|_{\beta-\frac{1}{2}, \frac{\beta}{2}-\frac{1}{4}}.$$

Now (56) can be applied. We obtain

$$\sum_l \|\omega_k(t_l - 0)\|_{\sigma}^2 k \leq C \|F\|_{L^2(0,T;H^{-\frac{1}{2}+\varepsilon}(\Omega))}^2 + C(\omega_0).$$

Due to Lemma 6.3,

$$\sum_l \|\omega_k(t_l - 0)\|_{\sigma}^2 k \leq C \|\tilde{\omega}_k\|_{L^4(0,T;H^{\frac{1}{2}+\varepsilon}(\Omega))}^4 + C(\omega_0).$$

We apply the interpolation inequality to obtain

$$\int_0^T \|\tilde{\omega}_k\|_{\frac{1}{2}+\varepsilon}^4 dt \leq \int_0^T \|\tilde{\omega}_k\|_{\sigma-\varepsilon}^{6-4\sigma+8\varepsilon} \|\tilde{\omega}_k\|_{\sigma-1-\varepsilon}^{4\sigma-2-8\varepsilon} dt.$$

We notice that $6 - 4\sigma + 8\varepsilon \leq 2$, so the assumption (57) implies the right hand side is bounded by a constant depending on M_0 . Following the proof of Theorem 4.1, the upper bounds for $\|\tilde{\omega}_k(t_i - 0)\|_{\sigma-1}$ and $k \|\tilde{\omega}_k(t_i - 0)\|_{\sigma+1}$ can also be obtained. □

Lemma 6.5. *If (58) holds for $i \leq i_1$, then there exists a constant $k_0 > 0$, depending on M_1 , such that if $k \leq k_0$, then*

$$(60) \quad \|\tilde{\omega}_k(\tau)\|_r \leq C_3 \|\omega_k(t_i - 0)\|_r, \quad t_i \leq \tau < t_{i+1},$$

for $0 \leq r \leq 2$, where C_3 depends on M_1 .

Proof. It can be proved (see [10]) that for the Euler equation

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0,$$

$$\nabla \cdot u = 0,$$

there are constants C_4, C_5 , such that if $t \leq \frac{1}{C_4 \|u(0)\|_3}$, then $\|u(t)\|_r \leq C_5 \|u(0)\|_r$ for $0 \leq r \leq 3$. By (58) and interpolation we have $\|\omega_k(t_i - 0)\|_2 \leq M_1 k^{\frac{\sigma-3}{2}}$; hence $\|u_k(t_i - 0)\|_3 \leq C(M_1) k^{\frac{\sigma-3}{2}}$. Thus we take

$$k_0 = \left(\frac{1}{C_4 C(M_1)} \right)^{\frac{2}{\sigma-1}}.$$

If $k \leq k_0$, then (60) holds. □

Lemma 6.6. *Under the assumptions of Lemma 6.5,*

$$(61) \quad \left(\sum_{i=0}^{i_1+1} \|(\omega^* - \omega_k)(t_i - 0)\|_{\sigma}^2 k \right)^{\frac{1}{2}} \leq C_6 k^{\frac{5}{8}(\sigma-1)-2\varepsilon},$$

where $\varepsilon > 0$, $k \leq k_0$, and C_6 depends on M_1 .

Proof. Let $\tilde{F} = u \cdot \nabla \omega - \tilde{u}_k \cdot \nabla \tilde{\omega}_k$. We obtain by Lemma 6.1 and interpolation that

$$\begin{aligned} \|\tilde{F}\|_{-\frac{1}{2}+\varepsilon} &= \|\nabla \wedge (\nabla((u - \tilde{u}_k) \otimes u + u \otimes (u - \tilde{u}_k) - (u - \tilde{u}_k) \otimes (u - \tilde{u}_k)))\|_{-\frac{1}{2}+\varepsilon} \\ &\leq C \|u - \tilde{u}_k\|_{\frac{3}{2}+\varepsilon} + C \|u - \tilde{u}_k\|_{\frac{3}{2}+\varepsilon}^2 \\ &\leq C \|\omega - \tilde{\omega}_k\|_{\frac{1}{2}+\varepsilon} + C \|\omega - \tilde{\omega}_k\|_{\frac{1}{2}+\varepsilon}^2 \\ &\leq C \|\omega - \tilde{\omega}_k\|_{\frac{1}{2}+\varepsilon} + C \|\omega - \tilde{\omega}_k\|_{\sigma-1} \|\omega - \tilde{\omega}_k\|_{2-\sigma+2\varepsilon}. \end{aligned}$$

Then we notice (58) and (60), to get for $\tau < t_{i_1+1}$ that

$$\begin{aligned} \|\tilde{F}\|_{-\frac{1}{2}+\varepsilon} &\leq C \|\omega - \tilde{\omega}_k\|_{2-\sigma+2\varepsilon} \\ &\leq C \|\tilde{\omega}^* - \tilde{\omega}_k\|_{2-\sigma+2\varepsilon} + C \|\omega - \tilde{\omega}^*\|_{2-\sigma+2\varepsilon}. \end{aligned}$$

Since ω is sufficiently smooth, (49) and (51) imply that

$$\|\omega(\tau) - \tilde{\omega}^*(\tau)\|_{\sigma-2} \leq C k^{\frac{7}{16} + \frac{3\beta}{8} - \frac{\varepsilon}{2}}, \quad \tau < t_{i_1+1}.$$

Because $\beta \in (0, \frac{1}{2})$ and $\varepsilon > 0$ are arbitrary, we take β close to $\frac{1}{2}$ and ε small; then the interpolation between it and (50) gives

$$\|\omega(\tau) - \tilde{\omega}^*(\tau)\|_{2-\sigma+2\varepsilon} \leq C k^{\frac{5}{8}(\sigma-1)-2\varepsilon},$$

and consequently

$$(62) \quad \|\tilde{F}\|_{-\frac{1}{2}+\varepsilon} \leq C \|\tilde{\omega}^* - \tilde{\omega}_k\|_{2-\sigma+2\varepsilon} + C k^{\frac{5}{8}(\sigma-1)-2\varepsilon}.$$

It follows from (36) that for $i \leq i_1 + 1$

$$\begin{aligned} (\omega^* - \omega_k)(x, t_i - 0) &= - \sum_{j=0}^{i-1} \int_{\Omega} K(x - \xi, t_i - t_j) \int_{t_j}^{t_{j+1}} \tilde{F}(\xi, \tau) d\xi d\tau \\ &\quad + \sum_{j=0}^{i-1} \int_{\partial\Omega} K(x - \xi, t_i - t_j) \int_{t_j}^{t_{j+1}} (\rho^* - \rho_k)(\xi, \tau) ds_{\xi} d\tau. \end{aligned}$$

Then we get

$$\begin{aligned} \|(\omega^* - \omega_k)(t_i - 0)\|_{\sigma} &\leq C \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{\sigma}{2} - \frac{1}{4} + \frac{\varepsilon}{2}} \|\tilde{F}\|_{-\frac{1}{2} + \varepsilon} d\tau \\ (63) \quad &\quad + C \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{1+\sigma}{2} + \frac{\beta}{2}} \|\chi^* - \chi_k\|_{\beta} d\tau, \end{aligned}$$

where $\sigma - 1 < \beta < \frac{1}{2}$. Analogously to (55), (56) and (59), we have

$$(64) \quad \|\chi^* - \chi_k\|_{\beta, \frac{\beta}{2}, \Omega \times (0, t_i)} \leq C \|\tilde{F}\|_{L^2(0, t_i; H^{-\frac{1}{2} + \varepsilon}(\Omega))},$$

where we notice that the term corresponding to ω_0 is cancelled out.

Let i_2 be an index to be determined. We take the sum

$$\begin{aligned} &\sum_{i=0}^{i_2} \|(\omega^* - \omega_k)(t_i - 0)\|_{\sigma}^2 k \\ &\leq C \sum_{i=0}^{i_2} \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{\sigma}{2} - \frac{1}{4} + \frac{\varepsilon}{2}} \|\tilde{F}\|_{-\frac{1}{2} + \varepsilon} d\tau \right)^2 k \\ &\quad + C \sum_{i=0}^{i_2} \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{1+\sigma}{2} + \frac{\beta}{2}} \|\chi^* - \chi_k\|_{\beta} d\tau \right)^2 k \\ &\leq C \sum_{i=0}^{i_2} \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{\sigma}{2} - \frac{1}{4} + \frac{\varepsilon}{2}} d\tau \right) \\ &\quad \cdot \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{\sigma}{2} - \frac{1}{4} + \frac{\varepsilon}{2}} \|\tilde{F}\|_{-\frac{1}{2} + \varepsilon}^2 d\tau \right) k \\ &\quad + C \sum_{i=0}^{i_2} \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{1+\sigma}{2} + \frac{\beta}{2}} d\tau \right) \\ &\quad \cdot \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{1+\sigma}{2} + \frac{\beta}{2}} \|\chi^* - \chi_k\|_{\beta}^2 d\tau \right) k \\ &\leq C t_{i_2}^{\frac{3}{2} - \sigma + \varepsilon} \|\tilde{F}\|_{L^2(0, t_{i_2}; H^{-\frac{1}{2} + \varepsilon}(\Omega))}^2 + C t_{i_2}^{1 - \sigma + \beta} \|\chi^* - \chi_k\|_{L^2(0, t_{i_2}; H^{\beta}(\Omega))}^2. \end{aligned}$$

Then by (64)

$$(65) \quad \sum_{i=0}^{i_2} \|(\omega^* - \omega_k)(t_i - 0)\|_{\sigma}^2 k \leq C t_{i_2}^{\alpha} \|\tilde{F}\|_{L^2(0, t_{i_2}; H^{-\frac{1}{2}+\varepsilon}(\Omega))}^2,$$

where $\alpha = \min(\frac{3}{2} - \sigma + \varepsilon, 1 - \sigma + \beta) > 0$.

Let us estimate the right hand side of (65) through (62). For $t_i \leq \tau < t_{i+1}$ we have

$$\|(\tilde{\omega}^* - \tilde{\omega}_k)(\tau)\|_{2-\sigma+2\varepsilon} \leq \int_{t_i}^{\tau} \|\tilde{F}\|_{2-\sigma+2\varepsilon} d\tau + \|(\omega^* - \omega_k)(t_i - 0)\|_{2-\sigma+2\varepsilon}.$$

Then applying Lemma 6.1 we obtain

$$\begin{aligned} & \|(\tilde{\omega}^* - \tilde{\omega}_k)(\tau)\|_{2-\sigma+2\varepsilon} \\ & \leq \int_{t_i}^{\tau} (C + C\|\tilde{u}_k\|_{1+\varepsilon}\|\tilde{\omega}_k\|_{3-\sigma+2\varepsilon}) d\tau + \|(\omega^* - \omega_k)(t_i - 0)\|_{2-\sigma+2\varepsilon}. \end{aligned}$$

The inequalities (60) and (58) imply

$$(66) \quad \|(\tilde{\omega}^* - \tilde{\omega}_k)(\tau)\|_{2-\sigma+2\varepsilon} \leq C k^{\sigma-1-2\varepsilon} + \|(\omega^* - \omega_k)(t_i - 0)\|_{2-\sigma+2\varepsilon}.$$

We substitute (66) and (62) into (65), and obtain

$$\begin{aligned} & \sum_{i=0}^{i_2} \|(\omega^* - \omega_k)(t_i - 0)\|_{\sigma}^2 k \\ & \leq C t_{i_2}^{\alpha} \left\{ \sum_{i=0}^{i_2-1} \|(\omega^* - \omega_k)(t_i - 0)\|_{2-\sigma+2\varepsilon}^2 k + k^{\frac{5}{4}(\sigma-1)-4\varepsilon} \right\} \\ & \leq C t_{i_2}^{\alpha} \left\{ \sum_{i=0}^{i_2-1} \|(\omega^* - \omega_k)(t_i - 0)\|_{\sigma}^2 k + k^{\frac{5}{4}(\sigma-1)-4\varepsilon} \right\}. \end{aligned}$$

We take t_{i_2} small enough so that $C t_{i_2}^{\alpha} \leq \frac{1}{2}$, then get

$$\sum_{i=0}^{i_2} \|(\omega^* - \omega_k)(t_i - 0)\|_{\sigma}^2 k \leq C k^{\frac{5}{4}(\sigma-1)-4\varepsilon}.$$

We notice that t_{i_2} is independent of k ; thus we can carry out an induction in a finite number, T/t_{i_2} , of steps. Suppose (61) is true for $i \leq N i_2$; then

$$\begin{aligned} & \sum_{i=N i_2+1}^{(N+1)i_2} \|(\omega^* - \omega_k)(t_i - 0)\|_{\sigma}^2 k \\ & \leq C \sum_{i=N i_2+1}^{(N+1)i_2} \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{\sigma}{2}-\frac{1}{4}+\frac{\varepsilon}{2}} \|\tilde{F}\|_{-\frac{1}{2}+\varepsilon} d\tau \right)^2 k \\ & \quad + C \sum_{i=N i_2+1}^{(N+1)i_2} \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{1+\sigma}{2}+\frac{\beta}{2}} \|\chi^* - \chi_k\|_{\beta} d\tau \right)^2 k. \end{aligned}$$

By (62),(64) and the estimate for $i \leq Ni_2$ we get

$$\begin{aligned}
& \sum_{i=Ni_2+1}^{(N+1)i_2} \|(\omega^* - \omega_k)(t_i - 0)\|_{\sigma}^2 k \\
& \leq C \sum_{i=Ni_2+1}^{(N+1)i_2} \left(\sum_{j=Ni_2+1}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{\sigma}{2} - \frac{1}{4} + \frac{\varepsilon}{2}} \|\tilde{F}\|_{-\frac{1}{2} + \varepsilon} d\tau \right)^2 k \\
& \quad + C \sum_{i=Ni_2+1}^{(N+1)i_2} \left(\sum_{j=Ni_2+1}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{1+\sigma}{2} + \frac{\beta}{2}} \|\chi^* - \chi_k\|_{\beta} d\tau \right)^2 k \\
& \quad + Ck^{\frac{5}{4}(\sigma-1)-4\varepsilon}.
\end{aligned}$$

Following the same lines, we get the estimate for $i \leq (N+1)i_2$. After a finite number of steps we get the estimate up to $i \leq i_1 + 1$. $\square$

Lemma 6.7. *Under the assumptions of Lemma 6.5,*

$$(67) \quad \|(\omega^* - \omega_k)(t_i - 0)\|_{\sigma-1, \Omega} \leq C_7 k^{\frac{5}{8}(\sigma-1)-2\varepsilon},$$

for $i \leq i_1 + 1$.

Proof. Analogously to (63), we have

$$\begin{aligned}
\|(\omega^* - \omega_k)(t_i - 0)\|_{\sigma-1} & \leq C \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{\frac{1}{4} - \frac{\sigma}{2} + \frac{\varepsilon}{2}} \|\tilde{F}\|_{-\frac{1}{2} + \varepsilon} d\tau \\
& \quad + C \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\frac{\sigma}{2} + \frac{\beta}{2}} \|\chi^* - \chi_k\|_{\beta} d\tau,
\end{aligned}$$

and then

$$\begin{aligned}
& \|(\omega^* - \omega_k)(t_i - 0)\|_{\sigma-1} \\
& \leq C \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{\frac{1}{2} - \sigma + \varepsilon} d\tau \right)^{\frac{1}{2}} \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \|\tilde{F}\|_{-\frac{1}{2} + \varepsilon}^2 d\tau \right)^{\frac{1}{2}} \\
& \quad + C \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - t_j)^{-\sigma + \beta} d\tau \right)^{\frac{1}{2}} \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \|\chi^* - \chi_k\|_{\beta}^2 d\tau \right)^{\frac{1}{2}}.
\end{aligned}$$

Then we use (62),(64) to get the desired estimate. $\square$

Theorem 6.1. *If $1 < \sigma < \frac{3}{2}$, then*

$$(68) \quad \left(\sum_{l=0}^i \|(\omega^* - \omega_k)(t_l - 0)\|_{\sigma}^2 k \right)^{\frac{1}{2}} + \|(\omega^* - \omega_k)(t_i - 0)\|_{\sigma-1} \leq Ck^{\frac{1}{4}}$$

for $0 \leq i \leq [\frac{T}{k}]$, provided $k \leq k_0$, where $k_0 > 0$ is a fixed constant and C is independent of k .

Proof. We prove (58), (67) and

$$(69) \quad \left(\sum_{l=0}^i \|(\omega^* - \omega_k)(t_l - 0)\|_{\sigma}^2 k \right)^{\frac{1}{2}} \leq C_8 k^{\frac{5}{8}(\sigma-1)-2\varepsilon}$$

by induction. They are obviously satisfied for $i = 0$.

Let us fix the constant M_0 in (57) first. By (50) it is easy to see that

$$\|\tilde{\omega}^*\|_{L^2(0,T;H^{\sigma-\varepsilon}(\Omega))} + \|\tilde{\omega}^*\|_{\sigma-1-\varepsilon} \leq C_9.$$

We take $M_0 = 2C_9$.

We assume that (58), (67), and (69) are satisfied for $i \leq i_1$. Analogously to (66), we have

$$\|(\tilde{\omega}^* - \tilde{\omega}_k)(\tau)\|_{\sigma-1} \leq Ck^{\frac{1}{2}} + \|(\omega^* - \omega_k)(t_i - 0)\|_{\sigma-1}$$

for $\tau \in [t_i, t_{i+1})$, $i \leq i_1$. We substitute (67) in it and obtain

$$\|(\tilde{\omega}^* - \tilde{\omega}_k)(\tau)\|_{\sigma-1} \leq Ck^{\frac{5}{8}(\sigma-1)-2\varepsilon}.$$

(50), (60) and (58) imply

$$\|(\tilde{\omega}^* - \tilde{\omega}_k)(\tau)\|_{\sigma} \leq C.$$

Then by interpolation we obtain

$$\|(\tilde{\omega}^* - \tilde{\omega}_k)(\tau)\|_{\sigma-\varepsilon} \leq Ck^{\varepsilon(\frac{5}{8}(\sigma-1)-2\varepsilon)}.$$

We take k_0 small enough so that (57) holds for $k \leq k_0$ and $\tau < t_{i_1+1}$. Then from Lemma 6.4, Lemma 6.6, and Lemma 6.7 we conclude that (58), (67), (69) are satisfied for $i \leq i_1 + 1$. Thus the induction is complete.

We take σ close to $\frac{3}{2}$ and ε small enough. Then the exponent is greater than $\frac{1}{4}$. $\square$

Finally let us prove our main result, Theorem 1.1.

Proof of Theorem 1.1. We take β close to $\frac{1}{2}$ and ε small enough in (48). Then

$$\|\omega(t_i) - \omega^*(t_i - 0)\|_{\sigma-2,\Omega} \leq Ck^{\frac{1}{2}}.$$

Theorem 4.1 implies

$$\|\omega^*(t_i - 0)\|_{\sigma,\Omega} \leq C.$$

Then by interpolation we get

$$\|\omega(t_i) - \omega^*(t_i - 0)\|_{\sigma-1,\Omega} \leq Ck^{\frac{1}{4}},$$

which in conjunction with (68) yields (23). $\square$

7. A REMARK ON THE VORTEX SHEETS

We have shown in (36) that the density of vortex sheets is $\int_{t_j}^{t_{j+1}} \rho_k(\xi, \tau) d\tau$. Let us make a heuristic analysis here to compare our result with the others in the literature.

The vorticity in (36) is divided into three parts: by initial data, by convection, and the vortex sheet, $\omega = \omega_1 + \omega_2 + \omega_3$. Suppose the velocity generated by $\omega_1 + \omega_2$ is $u = u_n n + u_\nu \nu$ on the boundary $\partial\Omega$. Then $u_n = 0$, due to the boundary condition of the Euler equation. The tangential component of velocity generated by the vortex sheets should be $-u_\nu$, to keep the no-slip boundary condition.

On the other hand, potential theory tells us that for $x_0 \in \partial\Omega$,

$$(70) \quad \begin{aligned} & \lim_{\substack{x \rightarrow x_0 \\ x \in \Omega}} \int_0^t \int_{\partial\Omega} \frac{\partial}{\partial n_x} K(x - \xi, t - \tau) \rho_k(\xi, \tau) ds_\xi d\tau \\ &= \frac{1}{2} \rho_k(x_0, t) + \int_0^t \int_{\partial\Omega} \frac{\partial}{\partial n_x} K(x_0 - \xi, t - \tau) \rho_k(\xi, \tau) ds_\xi d\tau. \end{aligned}$$

The left hand side is

$$\frac{\partial \omega_3}{\partial n} \Big|_{\partial\Omega} = \frac{\partial}{\partial n} \left(-\frac{\partial \bar{u}_n}{\partial \nu} + \frac{\partial \bar{u}_\nu}{\partial n} \right),$$

where $\bar{u} = (\bar{u}_n, \bar{u}_\nu)$ is generated by ω_3 . Since $\frac{\partial \bar{u}_n}{\partial n} + \frac{\partial \bar{u}_\nu}{\partial \nu} = 0$, we have

$$\frac{\partial \omega_3}{\partial n} \Big|_{\partial\Omega} = \frac{\partial^2 \bar{u}_\nu}{\partial \nu^2} + \frac{\partial^2 \bar{u}_\nu}{\partial n^2} = \Delta \bar{u}_\nu.$$

ω_3 satisfies the heat equation on $\mathbb{R}^2$. $\bar{u}$ satisfies the Stokes equation, but since $\nabla \cdot \bar{u} = 0$ at $t = 0$, $\bar{u}$ also satisfies the heat equation and the pressure equals to zero. We get

$$\frac{\partial \omega_3}{\partial n} \Big|_{\partial\Omega} = \frac{\partial \bar{u}_\nu}{\partial t} \Big|_{\partial\Omega} = -\frac{\partial u_\nu}{\partial t} \Big|_{\partial\Omega}.$$

Let us omit the second term in (70), to get $\frac{\partial \omega_3}{\partial n} \Big|_{\partial\Omega} = \frac{1}{2} \rho_k$. Then

$$\rho_k = -2 \frac{\partial u_\nu}{\partial t} \Big|_{\partial\Omega}.$$

When we set a vortex sheet at $t = t_{j+1}$, we assume that the no-slip condition is satisfied at $t = t_j$. Therefore

$$\int_{t_j}^{t_{j+1}} \rho_k d\tau = -2 \int_{t_j}^{t_{j+1}} \frac{\partial u_\nu}{\partial t} dt = -2u_\nu(\xi, t_{j+1}).$$

We notice that $u_\nu(\xi, t_{j+1})$ is the tangential component of velocity after the convection step. This is exactly the strength of vortex sheets given in some works.

By the above analysis the jump of velocity across the boundary is

$$\begin{aligned} [u_\nu] &= \int_{t_j}^{t_{j+1}} \left\{ \lim_{x \rightarrow x_0, x \in \Omega^c} \frac{\partial \omega_3}{\partial n} - \lim_{x \rightarrow x_0, x \in \Omega} \frac{\partial \omega_3}{\partial n} \right\} d\tau \\ &= - \int_{t_j}^{t_{j+1}} \rho_k(\xi, t) dt = -2u_\nu, \end{aligned}$$

where Ω^c is the exterior of Ω . Therefore it is assumed that $u_\nu^+|_{\partial\Omega} = -u_\nu^-|_{\partial\Omega}$, where “+” and “−” indicate the exterior and the interior.

If Ω is a half plane, the second term of (70) is exactly zero. This observation coincides with that in [3]. However, in general, this term does not vanish.

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