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A LOWER BOUND FOR RANK 2 LATTICE RULES

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ABSTRACT. We give a lower bound for a quality measure of rank 2 lattice rules which shows that an existence result of Niederreiter is essentially best possible.

1. INTRODUCTION

For the definition and the general theory of lattice rules for multivariate integration we refer to the monographs of Niederreiter [7] and of Sloan and Joe [9].

A rank 2 lattice rule is a quadrature rule for functions f over the s-dimensional unit cube $[0,1]^s$ of the form

$$Q(f) = \frac{1}{N} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} f(\{k_1\mathbf{z}_1/n_1 + k_2\mathbf{z}_2/n_2\}), \tag{1}$$

which cannot be re-expressed in an analogous form with a single sum. Here n_1, n_2 are positive integers such that $n_2|n_1$, $N = n_1n_2$ and $\mathbf{z}_1, \mathbf{z}_2$ are vectors in $\mathbb{Z}^s$. The integers n_1, n_2 are called the invariants of the lattice rule. (For a vector $\mathbf{x} \in \mathbb{R}^s$ the fractional part $\{\mathbf{x}\}$ is defined componentwise.)

For a given rank 2 lattice rule with invariants n_1 and n_2, $N = n_1n_2$ and with $\mathbf{z}_1 = (z_1, \ldots, z_s)$ and $\mathbf{z}_2 = (\zeta_1, \ldots, \zeta_s)$ for $z_i, \zeta_i \in \mathbb{Z}$, we define the quantity

$$R_N(\mathbf{z}_1, \mathbf{z}_2) := \sideset{}{^*}\sum_{\substack{-N<h_1,\ldots,h_s<N \\ h_1z_1+\ldots+h_sz_s\equiv 0 \pmod{n_1} \\ h_1\zeta_1+\ldots+h_s\zeta_s\equiv 0 \pmod{n_2}}} \frac{1}{r(h_1)\ldots r(h_s)},$$

where $\sum^*$ means summation over $(h_1, \ldots, h_s) \neq (0, \ldots, 0)$, and where $r(h) = \max(1, |h|)$ for $h \in \mathbb{Z}$.

Let $f : [0,1]^s \longrightarrow \mathbb{R}$ be a real-valued periodic function with period 1 in each variable and with Fourier-coefficients $\hat{f}(\mathbf{h})$, $\mathbf{h} = (h_1, \ldots, h_s) \in \mathbb{Z}^s$, satisfying $|\hat{f}(\mathbf{h})| = O(r(\mathbf{h})^{-\alpha})$ for some $\alpha > 1$ where $r(\mathbf{h}) = \prod_{i=1}^s r(h_i)$. Then for the integration error of any rank 2 lattice rule (1) we have the relation

$$\left| \int_{[0,1]^s} f(\mathbf{x})\, d\mathbf{x} - Q(f) \right| = O(R_N(\mathbf{z}_1, \mathbf{z}_2)^\alpha).$$

For a proof of this result see [6] or [7].

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Another reason for the importance of the quantity R_N is its relation to the discrepancy D_N of the finite s-dimensional point set

$$\left\{\frac{k_1}{n_1}\mathbf{z}_1 + \frac{k_2}{n_2}\mathbf{z}_2\right\}, \qquad k_i = 1, \ldots, n_i, \ \ 1 \le i \le 2. \tag{2}$$

(For the definition of the discrepancy D_N see, for example, [3] or [7].) In fact, it was shown by Niederreiter and Sloan [8] that the discrepancy of the point set (2) can be estimated by

$$D_N \le \frac{s}{N} + \frac{1}{2} R_N(\mathbf{z}_1, \mathbf{z}_2).$$

(A proof of this estimate can also be found in [7].)

In [6] Niederreiter proved that for every dimension $s \ge 2$ and for any prescribed invariants n_1 and n_2, $N = n_1 n_2$, there exist integer vectors of the form $\mathbf{z}_1 = (z_1, \ldots, z_s)$, $\mathbf{z}_2 = (0, \zeta_2, \ldots, \zeta_s)$ with $\gcd(z_i, n_1) = 1$, $1 \le i \le s$, and $\gcd(\zeta_i, n_2) = 1$, $2 \le i \le s$, such that

$$R_N(\mathbf{z}_1, \mathbf{z}_2) < c_s' \left(\frac{(\log N)^s}{N} + \frac{\log N}{n_1} \right),$$

where $c_s' > 0$ is a constant only depending on s. Note that the lattice rule in Niederreiter's existence result is projection-regular. (See [7] for the definition of projection-regular lattice rules.)

In this paper we prove a lower bound for the quantity $R_N(\mathbf{z}_1, \mathbf{z}_2)$ which shows that Niederreiter's estimate is essentially best possible.

2. Statement and proof of the result

We have

Theorem 2.1. *For every dimension $s \ge 2$ there is a constant $c_s > 0$, depending only on s, with the following property: for any prescribed invariants n_1 and n_2 with $n_2 | n_1$, $N = n_1 n_2$ and for any integer vectors $\mathbf{z}_1 = (z_1, \ldots, z_s)$ and $\mathbf{z}_2 = (\zeta_1, \ldots, \zeta_s)$ such that there is an index $1 \le i_0 \le s$ with $\gcd(z_{i_0}, n_1) = 1$, we have*

$$R_N(\mathbf{z}_1, \mathbf{z}_2) > c_s \frac{(\log N)^s}{N}.$$

Remark 2.2. Note that by [7, Theorem 5.38] there is also a simple lower bound for $R_N(\mathbf{z}_1, \mathbf{z}_2)$ of the order $(\log n_2)/n_1$, which shows that the second term in Niederreiter's upper bound is essentially best possible.

Remark 2.3. In particular the lower bound for $R_N(\mathbf{z}_1, \mathbf{z}_2)$ from Theorem 2.1 is true for all projection-regular rank 2 lattice rules (see [7]), since by a result of Sloan and Lyness [10] a rank 2 lattice rule is projection-regular if and only if the vectors $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{Z}^s$ can be chosen in such a way that $z_1 = 1$, $\zeta_1 = 0$ and $\zeta_2 = 1$. (Actually Sloan and Lyness give a characterization of projection-regular rank r lattice rules.)

Remark 2.4. We note here that Larcher [4] proved the result stated in Theorem 2.1 for any rank 1 lattice rule, which shows that the existence theorems on good rank 1 lattice rules of Hlawka [1], Korobov [2] and Niederreiter [5] are best possible.

For the proof of Theorem 2.1 we need the following generalization of the Chinese remainder theorem:

Lemma 2.5. *Let $a_1, a_2, b_1, b_2, m_1, m_2 \in \mathbb{Z}$ such that $\gcd(a_i, m_i) | b_i$, $1 \le i \le 2$. Then the system of congruences*

$$a_1 x \equiv b_1 \pmod{m_1}, \qquad a_2 x \equiv b_2 \pmod{m_2}$$

has a solution if and only if

$$b_1 a_2 - b_2 a_1 \equiv 0 \pmod{d},$$

where $d := \gcd(m_1 m_2, m_1 a_2, a_1 m_2)$.

Proof. For $1 \le i \le 2$ let $d_i := \gcd(a_i, m_i)$, $a_i = \bar{a}_i d_i$, $b_i = \bar{b}_i d_i$ and $m_i = \bar{m}_i d_i$. Now since $b_i \equiv 0 \pmod{d_i}$, $1 \le i \le 2$, we may divide the first congruence by d_1 and the second one by d_2 and our system of congruences becomes

$$\bar{a}_1 x \equiv \bar{b}_1 \pmod{\bar{m}_1}, \qquad \bar{a}_2 x \equiv \bar{b}_2 \pmod{\bar{m}_2}.$$

Since $\gcd(\bar{a}_i, \bar{m}_i) = 1$, we can find t_i such that $\bar{a}_i t_i \equiv 1 \pmod{\bar{m}_i}$, $1 \le i \le 2$. Now we find that our system of congruences is equivalent to the system

$$x \equiv \bar{b}_1 t_1 \pmod{\bar{m}_1}, \qquad x \equiv \bar{b}_2 t_2 \pmod{\bar{m}_2}.$$

This system has a solution if and only if

$$\bar{b}_1 t_1 - \bar{b}_2 t_2 \equiv 0 \pmod{\gcd(\bar{m}_1, \bar{m}_2)}.$$

From the definition of t_1 and t_2 we find that this congruence is equivalent to the congruence

$$\bar{b}_1 \bar{a}_2 - \bar{b}_2 \bar{a}_1 \equiv 0 \pmod{\gcd(\bar{m}_1, \bar{m}_2)}.$$

Finally from the definition of $\bar{a}_i$ and $\bar{b}_i$, $1 \le i \le 2$, this congruence is equivalent to

$$b_1 a_2 - b_2 a_1 \equiv 0 \pmod{d}$$

with $d := \gcd(m_1, a_1) \gcd(m_2, a_2) \gcd(\bar{m}_1, \bar{m}_2)$. Recalling the definition of $\bar{m}_1$ and $\bar{m}_2$, we have

$$\begin{aligned} d &= \gcd(\gcd(m_2, a_2) m_1, \gcd(m_1, a_1) m_2) \\ &= \gcd(m_1 m_2, m_1 a_2, a_1 m_2) \end{aligned}$$

and we are done. □

Proof of Theorem 2.1. W.l.o.g. we may assume that $z_1 = 1$. In the following let $\bar{n}_1 := n_1/n_2$, $\delta_i := \gcd(z_i, \bar{n}_1)$ and let t_i be defined by $z_i t_i \equiv \delta_i \pmod{\bar{n}_1}$ with $\gcd(t_i, \bar{n}_1) = 1$, $1 \le i \le s$.

(i) Assume that there is an index $2 \le i \le s$ such that $\delta_i > (\log N)^s$. Then we have

$$R_N(\mathbf{z}_1, \mathbf{z}_2) \ge \sum_{\substack{l=1 \\ h_i = l\,(N/\delta_i)}}^{\delta_i - 1} \frac{1}{h_i} \ge \frac{\delta_i}{N} > \frac{(\log N)^s}{N}.$$

So we may assume in the following that $\delta_i \le (\log N)^s$ holds for all $1 \le i \le s$.

(ii) Assume that $n_2 > (\log N)^s$. Then we have

$$R_N(\mathbf{z}_1, \mathbf{z}_2) \ge \sum_{\substack{l=1 \\ h_1 = l\,(N/n_2)}}^{n_2 - 1} \frac{1}{h_1} \ge \frac{n_2}{N} > \frac{(\log N)^s}{N}.$$

So we may assume in the following that $n_2 \le (\log N)^s$.

(iii) Assume that there is an index $2 \leq i \leq s$ such that one of the rationals $\frac{\delta_i t_i}{\bar{n}_1}$ has a continued fraction coefficient $a_k^i > (\log N)^s$. W.l.o.g. assume that $i = 2$. Then we have

$$\begin{aligned} R_N(\mathbf{z}_1, \mathbf{z}_2) &\geq \sideset{}{^*}\sum_{\substack{-N<h_1,h_2<N \\ h_1+h_2z_2\equiv 0 \pmod{n_1} \\ h_1\zeta_1+h_2\zeta_2\equiv 0 \pmod{n_2}}} \frac{1}{r(h_1)r(h_2)} \\ &\geq \sideset{}{^*}\sum_{\substack{-n_1<h_1,h_2<n_1 \\ h_1+h_2z_2\equiv 0 \pmod{\bar{n}_1}}} \frac{1}{n_2 n_2 r(h_1)r(h_2)} \\ &= \sideset{}{^*}\sum_{\substack{-n_1<h_1,h_2<n_1 \\ h_1t_2+h_2\delta_2\equiv 0 \pmod{\bar{n}_1}}} \frac{1}{n_2 n_2 r(h_1)r(h_2)} \\ &\geq \sideset{}{^*}\sum_{\substack{-n_1<h_1,h_2<n_1 \\ h_1\equiv 0 \pmod{\delta_2} \\ h_1t_2+h_2\delta_2\equiv 0 \pmod{\bar{n}_1}}} \frac{1}{n_2 n_2 r(h_1)r(h_2)} \\ &\geq \sideset{}{^*}\sum_{\substack{-\bar{n}_1/\delta_2<h_1,h_2<\bar{n}_1/\delta_2 \\ h_1t_2+h_2\equiv 0 \pmod{\bar{n}_1/\delta_2}}} \frac{1}{n_2 n_2 \delta_2 r(h_1)r(h_2)}. \end{aligned}$$

For $h_1 \in \mathbb{Z}$ let

$$H(h_1) := \begin{cases} -\frac{\bar{n}_1}{\delta_2}\left\{h_1\frac{\delta_2 t_2}{\bar{n}_1}\right\}, & \text{if } \left\{h_1\frac{\delta_2 t_2}{\bar{n}_1}\right\} \leq \frac{1}{2}, \\ \frac{\bar{n}_1}{\delta_2}\left(1-\left\{h_1\frac{\delta_2 t_2}{\bar{n}_1}\right\}\right), & \text{if } \left\{h_1\frac{\delta_2 t_2}{\bar{n}_1}\right\} > \frac{1}{2}. \end{cases}$$

Then we have $h_1 t_2 + H(h_1) \equiv 0 \pmod{\bar{n}_1/\delta_2}$ and

$$|H(h_1)| = \frac{\bar{n}_1}{\delta_2}\left\|h_1\frac{\delta_2 t_2}{\bar{n}_1}\right\|.$$

(Here and in the following $\|.\|$ denotes the distance to the nearest integer function, i.e., $\|x\| = \min(\{x\}, 1-\{x\})$.) Now let

$$\frac{\delta_2 t_2}{\bar{n}_1} = [0; a_1, a_2, \ldots, a_m]$$

and let $q_{-1}, q_0, q_1, \ldots, q_m$ be the denominators of the convergents of $\frac{\delta_2 t_2}{\bar{n}_1}$, $q_{-1} = 0, q_0 = 1$ and $q_l = a_l q_{l-1} + q_{l-2}$ for $1 \leq l \leq m$. Assume that $a_k > (\log N)^s$. Let $h_1 := q_{k-1}$, then we have

$$R_N(\mathbf{z}_1, \mathbf{z}_2) \geq \frac{1}{n_2 n_2 \delta_2 q_{k-1}|H(q_{k-1})|}.$$

Since

$$\frac{\delta_2 t_2}{\bar{n}_1} - \frac{p_{k-1}}{q_{k-1}} = \frac{\theta_k}{a_k q_{k-1}^2}$$

with $|\theta_k| < 1$, it follows that

$$q_{k-1}\frac{\delta_2 t_2}{\bar{n}_1} = p_{k-1} + \frac{\theta_k}{a_k q_{k-1}}$$

and hence we have

$$|H(q_{k-1})| = \frac{\bar{n}_1}{\delta_2} \left\| \frac{\theta_k}{a_k q_{k-1}} \right\| \leq \frac{\bar{n}_1}{\delta_2 a_k q_{k-1}}.$$

From this we get

$$R_N(\mathbf{z}_1, \mathbf{z}_2) \quad \geq \quad \frac{\delta_2 a_k q_{k-1}}{n_2 n_2 \delta_2 q_{k-1} \bar{n}_1} = \frac{a_k}{N} > \frac{(\log N)^s}{N}.$$

So we may assume in the following that all continued fraction coefficients of the rationals $\frac{\delta_i t_i}{\bar{n}_1}$, $2 \leq i \leq s$, are less than or equal to $(\log N)^s$.

Moreover we assume N so large that

$$\log N < 2 \log \left(\frac{N}{(\log N)^{3s}} \right).$$

For the finitely many N that do not satisfy the last inequality, the assertion of the theorem is trivially true with $c_s > 0$ small enough.

(iv) Define $d_1 := n_2$ and for $2 \leq k \leq s$ define $d_k := \gcd(z_k \zeta_1 - \zeta_k, d_{k-1})$. For $2 \leq k \leq s$ and for $v, w \in \mathbb{Z}$ define

$$R_N^k(\mathbf{z}_1, \mathbf{z}_2, v, w) := \sideset{}{^*}\sum_{\substack{-N < h_1, \ldots, h_k < N \\ h_1 + h_2 z_2 + \ldots + h_k z_k \equiv v \pmod{n_1} \\ h_1 \zeta_1 + h_2 \zeta_2 + \ldots + h_k \zeta_k \equiv w \pmod{n_2}}} \frac{1}{r(h_1) \ldots r(h_k)}.$$

We shall prove that for $v\zeta_1 \equiv w \pmod{d_k}$ we have

$$R_N^k(\mathbf{z}_1, \mathbf{z}_2, v, w) \geq c(s,k) d_k \frac{(\log N)^k}{N}, \tag{3}$$

where $c(s,k) > 0$ is a constant depending only on s and k (but not on N). We do this by induction on k.

$k = 2$: Let $v, w \in \mathbb{Z}$ with $v\zeta_1 \equiv w \pmod{d_2}$ and define

$$R^2 := R_N^2(\mathbf{z}_1, \mathbf{z}_2, v, w) = \sideset{}{^*}\sum_{\substack{-N < h_1, h_2 < N \\ h_1 + h_2 z_2 \equiv v \pmod{n_1} \\ h_1 \zeta_1 + h_2 \zeta_2 \equiv w \pmod{n_2}}} \frac{1}{r(h_1) r(h_2)}.$$

For $h_2 \in \mathbb{Z}$ the system

$$\begin{aligned} h_1 + h_2 z_2 &\equiv v \pmod{n_1}, \\ h_1 \zeta_1 + h_2 \zeta_2 &\equiv w \pmod{n_2} \end{aligned} \tag{4}$$

has a solution h_1 iff

$$h_2 \zeta_2 \equiv w \pmod{\sigma_1} \tag{5}$$

and

$$h_2 (z_2 \zeta_1 - \zeta_2) \equiv v\zeta_1 - w \pmod{n_2}. \tag{6}$$

(Here $\sigma_1 := \gcd(\zeta_1, n_2)$. The second congruence is obtained with Lemma 2.5.) Let h be a solution of congruence (6). Then we have

$$\zeta_2 h \equiv w + \zeta_1 (z_2 h - v) \pmod{n_2}.$$

Now from the definition of σ_1 we obtain $\zeta_2 h \equiv w \pmod{\sigma_1}$ and so h is also a solution of congruence (5). Hence in the following we only have to consider congruence (6).

From $v\zeta_1 - w \equiv 0 \pmod{d_2}$ and $d_2 = \gcd(z_2\zeta_1 - \zeta_2, n_2)$ we find that congruence (6) has d_2 incongruent $(\mathrm{mod}\ n_2)$ solutions $x_1, \ldots, x_{d_2} \in \mathbb{Z}$ with $0 \le x_i < n_2$. Now let $i \in \{1, \ldots, d_2\}$ and let $h_2 = x_i + \bar{h}_2 n_2$. Then system (4) becomes

$$h_1 + (x_i + \bar{h}_2 n_2) z_2 \equiv v \pmod{n_1}, \tag{7}$$

$$h_1\zeta_1 + (x_i + \bar{h}_2 n_2)\zeta_2 \equiv w \pmod{n_2}. \tag{8}$$

From congruence (8) we get

$$h_1\zeta_1 \equiv w - x_i\zeta_2 \pmod{n_2}. \tag{9}$$

Since x_i is a solution of congruence (6) (and hence of congruence (5)), we have $w - x_i\zeta_2 \equiv 0 \pmod{\sigma_1}$. Now define $\alpha := \zeta_1/\sigma_1$, $\omega_i := (w - x_i\zeta_2)/\sigma_1$, $\bar{n}_2 := n_2/\sigma_1$. Then congruence (9) may be rewritten as

$$h_1\alpha \equiv \omega_i \pmod{\bar{n}_2}. \tag{10}$$

Let $\tau_1 \in \mathbb{Z}$ be defined by $\zeta_1\tau_1 \equiv \sigma_1 \pmod{n_2}$ with $\gcd(\tau_1, n_2) = 1$ and define $s_i := \omega_i\tau_1$. Then we obtain from (10) the congruence $h_1 \equiv s_i \pmod{\bar{n}_2}$ and hence h_1 is of the form

$$h_1 = s_i + \bar{h}_1\bar{n}_2$$

(w.l.o.g. assume that $0 \le s_i < \bar{n}_2$). Substituting this in congruence (7), we get

$$\bar{h}_1\bar{n}_2 + \bar{h}_2 n_2 z_2 \equiv v - s_i - x_i z_2 \pmod{n_1}. \tag{11}$$

Once again we note that x_i is a solution of congruence (6), i.e.,

$$v\zeta_1 - w - \zeta_1 z_2 x_i + x_i\zeta_2 \equiv 0 \pmod{n_2}.$$

By the definition of τ_1 we obtain

$$v\sigma_1 - (w - x_i\zeta_2)\tau_1 - \sigma_1 z_2 x_i \equiv 0 \pmod{n_2}$$

and hence we have $v - s_i - z_2 x_i \equiv 0 \pmod{\bar{n}_2}$. So we get an integer a_i such that $v - s_i - z_2 x_i = a_i\bar{n}_2$. Therefore congruence (11) becomes

$$\bar{h}_1 + \bar{h}_2\sigma_1 z_2 \equiv a_i \pmod{\sigma_1\bar{n}_1}. \tag{12}$$

(Recall that $n_1 = \bar{n}_1 n_2$.) Now we have

$$R^2 \ge \sum_{i=1}^{d_2} \sideset{}{^*}\sum_{\substack{-N < h_1, h_2 < N \\ h_2 = x_i + \bar{h}_2 n_2 \\ h_1 = s_i + \bar{h}_1\bar{n}_2 \\ \bar{h}_1 + \bar{h}_2\sigma_1 z_2 \equiv a_i \pmod{\sigma_1\bar{n}_1}}} \frac{1}{r(s_i + \bar{h}_1\bar{n}_2) r(x_i + \bar{h}_2 n_2)}. \tag{13}$$

Denote the inner sum in inequality (13) by $\sum(i)$ for $1 \le i \le d_2$.

Define $\delta := \sigma_1 \gcd(z_2, \bar{n}_1) = \sigma_1\delta_2$. From $\bar{h}_1 + \bar{h}_2\sigma_1 z_2 \equiv a_i \pmod{\sigma_1\bar{n}_1}$ it follows that $\bar{h}_1 = b + l\delta$ for a b with $0 \le b < \delta$, and $a_i - b \equiv 0 \pmod{\delta}$; furthermore, $\bar{h}_2\sigma_1 z_2 \equiv a_i - b - l\delta \pmod{\sigma_1\bar{n}_1}$. Let $u := \frac{a_i - b}{\delta} t_2$. Then $\bar{h}_2 \equiv u - lt_2 \pmod{m}$, where $m := \bar{n}_1/\delta_2$, and so $\bar{h}_2$ is of the form

$$\bar{h}_2 = m\left(\frac{u - lt_2}{m} + k\right),$$

where $k \in \mathbb{Z}$. It follows that for every $l \in \mathbb{Z}$ there is a solution $\bar{h}_1$ and $\bar{h}_2$ of congruence (12) with

$$\begin{aligned} \bar{h}_1 &= b + l\delta, \\ |\bar{h}_2| &= m \left\| \frac{u}{m} - l\frac{t_2}{m} \right\|. \end{aligned}$$

Hence we have

$$\begin{aligned} \sum(i) &\geq \sum_{l=0}^{m-1} \frac{1}{\frac{n_2}{\sigma_1}(b + l\delta + 1)n_2\left(1 + m\left\|\frac{u}{m} - l\frac{t_2}{m}\right\|\right)} \\ &\geq \sum_{l=0}^{m-1} \frac{1}{n_2\frac{n_2}{\sigma_1}(\delta(1+l)+1)m\left(\frac{1}{m} + \left\|\frac{u}{m} - l\frac{t_2}{m}\right\|\right)} \\ &\geq \sum_{l=0}^{m-1} \frac{1}{n_2\frac{n_2}{\sigma_1}\frac{\bar{n}_1}{\delta_2}2\delta_2\sigma_1(l+1)\left(\frac{1}{m} + \left\|\frac{u}{m} - l\frac{t_2}{m}\right\|\right)} \\ &\geq \frac{1}{4N}\sum_{l=0}^{m-1} \frac{1}{(l+1)\max\left(\frac{1}{m}, \left\|\frac{u}{m} - l\frac{t_2}{m}\right\|\right)}. \end{aligned}$$

Since $\gcd(t_2, \bar{n}_1) = 1$, it follows that $\gcd(t_2, m) = 1$. By our assumptions on n_2, N and δ_2 we get $N = n_2 n_2 \delta_2 m < (\log N)^{3s} m$ and hence $\log N \leq 2\log m$. Furthermore, we have that $\frac{t_2}{m} = \frac{\delta_2 t_2}{\bar{n}_1}$ has continued fraction coefficients $a_i < (\log N)^s \leq 2^s(\log m)^s$. But under these assumptions G. Larcher proved in [4, p. 48, inequality $(*)$] that

$$\sum_{l=0}^{m-1} \frac{1}{(l+1)\max\left(\frac{1}{m}, \left\|a - l\frac{t_2}{m}\right\|\right)} \geq c(s)(\log m)^2$$

holds for every $a \in [0, 1)$. (Here $c(s) > 0$ is a constant depending only on s.) So we get

$$\sum(i) \quad \geq \quad \frac{1}{4N}c(s)(\log m)^2 \geq \frac{c(s)}{8}\frac{(\log N)^2}{N}.$$

Inserting this in inequality (13), we get

$$R^2 \geq c(s, 2)d_2\frac{(\log N)^2}{N},$$

such that the case $k = 2$ is proved.

$k - 1 \longrightarrow k$: For short we write $R^k(v, w)$ instead of $R_N^k(\mathbf{z}_1, \mathbf{z}_2, v, w)$. Let $v\zeta_1 \equiv w \pmod{d_k}$. Then we have

$$\begin{aligned} R^k(v, w) &\geq \widetilde{\sum_l}\frac{1}{r(l)} \sideset{}{^*}\sum_{\substack{-N < h_1, \ldots, h_{k-1} < N \\ h_1 + h_2 z_2 + \ldots + h_{k-1}z_{k-1} \equiv v - lz_k \pmod{n_1} \\ h_1\zeta_1 + h_2\zeta_2 + \ldots + h_{k-1}\zeta_{k-1} \equiv w - l\zeta_k \pmod{n_2}}} \frac{1}{r(h_1)\ldots r(h_{k-1})} \\ &= \widetilde{\sum_l}\frac{1}{r(l)}R^{k-1}(v - lz_k, w - l\zeta_k), \end{aligned}$$

where $\widetilde{\sum_l}$ denotes summation over all integers $-N < l < N$ such that

$$(v - lz_k)\zeta_1 \equiv w - l\zeta_k \pmod{d_{k-1}}. \tag{14}$$

Now we get from the induction hypothesis that

$$R^k(v, w) \geq c(s, k-1) d_{k-1} \frac{(\log N)^{k-1}}{N} \widetilde{\sum_l} \frac{1}{r(l)}. \tag{15}$$

Since by our assumption $d_k = \gcd(z_k\zeta_1 - \zeta_k, d_{k-1})$ is a divisor of $v\zeta_1 - w$, we find d_k incongruent solutions $x_1, \ldots, x_{d_k}$ of congruence (14), $0 \leq x_i < d_{k-1}$. Now we have

$$\begin{aligned} \widetilde{\sum_l} \frac{1}{r(l)} &\geq \sum_{i=1}^{d_k} \sum_{\substack{l=0 \\ l = x_i + \bar{l} d_{k-1}}}^{N-1} \frac{1}{r(x_i + \bar{l} d_{k-1})} \geq \sum_{i=1}^{d_k} \sum_{\bar{l}=0}^{N/d_{k-1}-1} \frac{1}{(\bar{l}+1) d_{k-1}} \\ &\geq \frac{d_k}{d_{k-1}} \log \frac{N}{d_{k-1}} \geq \frac{1}{2} \frac{d_k}{d_{k-1}} \log N, \end{aligned}$$

since $d_{k-1} \leq d_1 = n_2$ and hence

$$\log \frac{N}{d_{k-1}} \geq \log \frac{N}{n_2} = \log n_1 \geq \frac{1}{2} \log N.$$

Inserting this result in (15) will finish our induction proof of inequality (3). The result follows.

□

Problem 2.6. (1) It remains an open question whether Theorem 2.1 holds without the existence of an index $1 \leq i_0 \leq s$ such that $\gcd(z_{i_0}, n_1) = 1$.

(2) Is the lower bound from Theorem 2.1 also true for rank r lattice rules, $2 < r \leq s$?

References

[1] Hlawka, E.: *Zur angenäherten Berechnung mehrfacher Integrale.* Monatsh. Math. **66**: 140-151, 1962. MR **26:**888

[2] Korobov, N.M.: *Numbertheoretical Methods in Approximate Analysis.* Moscow: Fizmatgiz. 1963. (In Russian.) MR **28:**716

[3] Kuipers, L., Niederreiter, H.: *Uniform Distribution of Sequences.* John Wiley, New York, 1974. MR **54:**7415

[4] Larcher, G.: *A Best Lower Bound for Good Lattice Points.* Monatsh. Math. **104**: 45-51, 1987. MR **89f:**11103

[5] Niederreiter, H.: *Existence of Good Lattice Points in the Sense of Hlawka.* Monatsh. Math. **86**: 203-219, 1978. MR **80e:**10039

[6] Niederreiter, H.: *The Existence of Efficient Lattice Rules for Multidimensional Numerical Integration.* Math. Comp. **58**: 305-314 and S7-S16, 1992. MR **92e:**65023

[7] Niederreiter, H.: *Random Number Generation and Quasi-Monte Carlo Methods.* No. 63 in CBMS-NSF Series in Applied Mathematics. SIAM, Philadelphia, 1992. MR **93h:**65008

[8] Niederreiter, H., Sloan, I.H.: *Lattice rules for multiple integration and discrepancy.* Math. Comp. **54**: 303-312, 1990. MR **90f:**65036

[9] Sloan, I.H., Joe, S.: *Lattice Methods for Multiple Integration.* Oxford Univ. Press, New York and Oxford, 1994. MR **98a:**65026

[10] Sloan, I.H., Lyness, J.N.: *Lattice rules: Projection regularity and unique representations.* Math. Comp. **54**: 649-660, 1990. MR **91a:**65062

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