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# SEARCHING FOR KUMMER CONGRUENCES IN AN INFINITE SLOPE FAMILY

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ABSTRACT. We consider powers of a grossencharacter, the corresponding $L$-functions twisted with quadratic Dirichlet characters, and their central critical values. We state several conjectures concerning Kummer-type congruences between these numbers for a ramified prime and describe specific numerical data in support of these conjectures.

## 0. INTRODUCTION

Let $K$ be an imaginary quadratic field of class number 1, and let $E$ be an elliptic curve defined over $K$ with complex multiplication by the ring of integers of $K$. Let $\Omega$ denote a fundamental period of a minimal model of $E$ and $\psi$ the grossencharacter of $K$ attached to $E$. Let $D$ be a fundamental discriminant, and let $\chi_D$ be the quadratic Dirichlet character associated to $\mathbb{Q}(\sqrt{D})$. Then the special values

$$L(k, D, l) = \frac{(2\pi)^{2k-1-l} l!}{\Omega^{2k-1}} \sum_{\mathfrak{a}} \psi(\mathfrak{a})^{2k-1} \chi_D(\mathcal{N}(\mathfrak{a})) \mathcal{N}(\mathfrak{a})^{-l},$$

where the sum is taken over the integral ideals $\mathfrak{a}$ of $K$, are rational numbers [3] if $l$ is an integer and $1 \leq l \leq 2k-1$. (Of course, the series might diverge, in which case one considers the value of the analytic continuation of $L$.)

For a prime number $p$ which splits in $K$ one can construct a two-variable $p$-adic $L$-function that interpolates $L(k, D, l)$ (see [13], [8]). In other words, there exists a $p$-adic analytic function in two variables such that the numbers $L(k, D, l)$, multiplied by an explicit factor, become special values of this function. The existence of this interpolating function is equivalent [9, Proposition 4.0.6] to the fact that certain congruences hold for these numbers; these are the Kummer-type congruences (see below).

The reason for this successful $p$-adic interpolation may be explained as follows. The $L$-series in question are the Mellin transforms of modular forms

$$f = \sum_{\mathfrak{a}} \psi(\mathfrak{a})^{2k-1} q^{\mathcal{N}(\mathfrak{a})}, \qquad q = \exp(2\pi i \tau), \tag{1}$$

of weight $2k$ and a certain level [15, Theorem 4.8.2], twisted with quadratic Dirichlet characters $\chi_D$.

If $p$ splits, these modular forms are specializations of a $p$-ordinary $\Lambda$-adic modular form [6], and one explains the success of the $p$-adic interpolation in the framework of Hida's theory. The situation is more complicated if $p$ is inert; in this case,

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nevertheless, some congruences were found by Hurwitz [7] and reproved by Katz [10] by using Cartier duality. These congruences concern the restriction of the conjectural two-variable $p$-adic $L$-function to the boundary line $l = 2k - 1$ with the variable $k$. On the other hand, if $k$ is fixed (and the variable $l$ is allowed to run over the set of continuous characters of $\mathbb{Z}_p^*$), one gets a $k$-admissible measure according to [19], [1] (see [14] for an exhaustive discussion concerning this case and, in particular, about admissible measures).

The main subject of the present paper is Kummer-type congruences for the primes $p$ which are ramified in $K$. We remind the reader ([6]) that $p$-ordinary modular forms (modular forms of zero slope) are those that survive under the action of Hida's ordinary projector $\mathcal{E}_p = \lim_{n\to\infty} U_p^{n!}$. As the opposite extreme case, we propose the following terminology: we say that a modular form is of *infinite slope* if it belongs to the kernel of the $U_p$-operator. This definition coincides with the definition of slope coming from the consideration of the Newton polygon of the Hecke polynomial at $p$.

If the prime $p$ is ramified in $K$, the modular forms (1) are of infinite slope. One of the motivations for this research was a very naïve approach to a question from [2] about the possibility of completing the eigencurve by including the missing points of infinite slope. The families of infinite slope should live on the boundary of the eigencurve, and it turns out to be easy to produce such a family. The next natural question is about the $p$-adic properties of special values of the $L$-function.

As in the split case, the modular forms (1) are specializations at weight $2k$ of a $p$-adic family of modular forms, which should be thought of as a $\Lambda$-adic Hecke eigenform of infinite slope. In this case, once again, the restriction of the conjectural two-variable $p$-adic $L$-function for $D = 1$ to the boundary line $l = 2k - 1$ was constructed by Rubin [17]. This is equivalent to saying that the values $L(k, 1, 2k-1)$, where 1 stands for the trivial Dirichlet character, satisfy certain congruences (see [17] for details). Our numerical calculations of the values $L(k, D, l)$ for different $k$ and $l$ did not provide any substantial evidence for the existence of the two-variable $p$-adic $L$-function.

The numbers under consideration in this paper are *central* special values of $L$-series corresponding to the modular forms (1), twisted with quadratic Dirichlet characters. The authors consider the restriction to the *central critical line*, namely $l = k$, to be an interesting setting. The purpose of this paper is to discuss the following conjecture.

**Conjecture 1.** *For any discriminant $D$, and positive integer $k$ fixed modulo $p-1$, there is an Iwasawa function $l_D$ such that*

$$l_D(k)^2 = C(k)L(k, D, k), \tag{2}$$

*where the quantity $C(k)$ does not depend on $D$ and does not vanish.*

In other words we conjecture the existence of a $p$-adic analytic function $\mathcal{F}$ on $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$, where $\mathbb{C}_p$ is the completion of an algebraic closure of $\mathbb{Q}_p$, with the property

$$\mathcal{F}(k) = l_D(k)$$

for a fixed embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ of the field of algebraic numbers into $\mathbb{C}_p$. The existence of such a function is known ([9], 4.0.6) to be equivalent to the following.

**Kummer-type congruences.** *For a nonnegative integer $A$ and positive integers $k_1, k_2$ the congruence*

$$k_1 \equiv k_2 \bmod (p-1)p^A$$

*implies the congruence*

$$l_D(k_1) \equiv l_D(k_2) \bmod p^{A+1}.$$

This equivalence allows us to check numerically a conjecture about the existence of an interpolating Iwasawa function just by checking whether the Kummer congruences hold. This is precisely what we are doing in this paper.

There are two questions to discuss in connection with Conjecture 1.

The first, and the most important one, is that the "square" of the conjecture is true, namely, that the right-hand side of (2) satisfies Kummer congruences. In the case when the prime $p$ splits in the imaginary quadratic field $K$ this statement is a special case of results from [8] (see [11] for a detailed discussion).

Our setting differs slightly from the usual one. We consider only the central special values $L(k, D, k)$ of the $L$-function associated to different modular forms of different weights $2k$ and we do not take into account other special values and cyclotomic twists (this is because we have no idea of how to interpolate all the numbers $L(k, D, l)$ by a two-variable $p$-adic $L$-function). We are allowed to multiply the values $L(k, D, k)$ by a nonzero algebraic number $C(k)$. Now there are two ways to make a nonvacuous claim. The first one is to introduce an additional discrete argument, namely to consider twists by Dirichlet characters, and to demand that the *normalization constant* $C(k)$, a common nonzero factor that depends on $k$, does not depend on the Dirichlet character. Another way is to produce an explicit expression for $C(k)$. We try to do the latter in a very special case in Section 3 of the paper, and we stick to the former approach here. We consider only quadratic twists; thus $C(k)$ becomes a normalization constant (a common nonzero factor) for all the values $L(k, D, k)$, independent of the discriminant $D$. As to the discriminants $D$, we consider all of them, including those which are divisible by $p$. In other words, we first view Conjecture 1 as a way to say that the restriction of $L(k, D, l)$ to the central critical line is a $p$-adic analytic function even though the two-variable $p$-adic $L$-function may not exist.

The second question is more delicate and concerns the choice of the square root. The numbers $L(k, D, k)$ are known to be essentially perfect squares of the Fourier coefficients of modular forms of half-integral weight connected with the initial modular forms by the Shimura correspondence. Thus the question about the canonical choice of the square roots can be put as the question of whether the corresponding set of half-integral weight modular forms is, in a sense, a $p$-adic family. Such a type of conjecture was first formulated in [11] in the case when $p$ splits on the basis of numerical data. Shortly after that the conjecture was reformulated in a much more appropriate framework of Jacobi forms instead of modular forms of half-integral weight and partially proven by Kohnen [12]. The rest of the conjecture (still in the case when $p$ splits) was recently proved by the second author [5] under an additional condition. The authors hope that our Conjecture 1 may be proved in the same framework, by an explicit construction of the corresponding family of Jacobi forms.

The authors expect that a statement similar to Conjecture 1 holds whenever one has a $p$-adic family of modular Hecke eigenforms which are newforms of infinite slope. In this context complex multiplication can be considered as a convenient and arithmetically interesting way to produce such a family [6]. Also, we expect that the assertion should still be true if one considers the twists by arbitrary tame Dirichlet characters and not only by quadratic characters.

We tried to test numerically some special cases of Conjecture 1. We state them in the subsequent sections of the paper as separate conjectures, showing numerical data in support. For the numerical calculations we pick $\mathbb{Q}(\sqrt{-7})$ and $\mathbb{Q}(\sqrt{-11})$ for the imaginary quadratic field $K$ and denote by $\psi_7$ and $\psi_{11}$ the corresponding grossencharacters.

Short tables of the special values under consideration for $\psi = \psi_7$ are computed in [4], but they are not of sufficient length to observe the congruences.

Let us mention here some consequences of our computations which seem to be especially interesting.

The results of the calculations do not suggest the existence of a smooth two-variable $p$-adic $L$-function when the prime $p$ is ramified in $K$. Nevertheless, the "restriction to the center of the critical line" seems to be an Iwasawa function (see Section 1), as is the case if the prime $p$ splits, but the calculations show that this is probably *not* true when the prime $p$ is inert.

The quantities in the right-hand side of Conjecture 1 are essentially perfect squares, and therefore Conjecture 1 provides a "canonical" choice of the square roots. A statement similar to Conjecture 1 is true for primes which split (see [12], [5]). The calculations show that these two choices of the square root are *different*, though closely related (see Section 2).

Another canonical choice of the square root, at least in the case $\psi = \psi_7$ and $D = 1$, follows from [16]. This setting has almost nothing to do with Conjecture 1. However, the simple link, found experimentally (see Section 3), seems interesting. Note that we have not succeeded in finding anything similar for an inert prime. Instead we record some strange congruences, which do not fit into our framework (see Section 4).

All the numerical calculations were executed with the use of the Number Theoretic package PARI-GP.

## 1. Calculations in support of Conjecture 1

Assume that Conjecture 1 is true. Then the right-hand side of (2), namely, the central special values of the $L$-function twisted with Dirichlet character, satisfies Kummer congruences. One can test numerically the following statement.

**Conjecture 2.** *There exists a nonzero normalization constant $C(k)$ such that if*

$$k_1 \equiv k_2 \bmod (p-1)p^A$$

*with an integer $A \geq 0$, then*

$$C(k_1)L(k_1, D, k_1) \equiv C(k_2)L(k_2, D, k_2) \bmod p^{A+1}. \tag{3}$$

For the numerical experiments we pick $C(k)$ such that both sides of (3) are not divisible by $p$. The computations show that the statement of Conjecture 2 is true in the following cases:

$$\psi = \psi_7, \quad 1 \le k \le 16, \quad 1 \le d \le 78, \quad p = 7.$$
$$\psi = \psi_{11}, \quad 1 \le k \le 19, \text{ odd}, \quad 1 \le d \le 78, \quad p = 11.$$

Here and in the following $d = |D|$ if $D$ is odd, and $d = |D|/4$ if $D \equiv 0 \bmod 4$. In all the cases listed above one finds the predicted congruences modulo $p$ if $k_1 \equiv k_2 \bmod (p-1)$. Additionally, we checked the congruences modulo 49 when $\psi = \psi_7$ for the pair $k_1 = 1$ and $k_2 = 43$ and $1 \le d \le 41$.

## 2. Noncompatibility of square roots

In this section, we test whether our choice of the square root $l_D(k)$ is compatible with another natural choice, which appears when one considers the splitting primes.

We say that a prime $q$ satisfies the condition (*) if $q$ splits in $K$ and there exists a fundamental discriminant $D_0 < 0$ and $k_0 > 0$ such that $q | D_0$ and $L(k_0, D_0, k_0) \neq 0$. We note that there are infinitely many primes $q$ that satisfy the condition (*).

The following proposition is proved in [5] and [12].

**Proposition.** *There is a normalization constant $c(k) \neq 0$ and a choice of square root*

$$c(k, D)^2 = c(k) L(k, D, k)$$

*for $D < 0$ with the following property.*

*For any prime $q$ which satisfies (*) and for any $D$ which is not divisible by $q$ the congruence*

$$k_1 \equiv k_2 \bmod (q-1)q^A \quad \textit{with} \quad k_1, k_2 > A \ge 0$$

*implies*

$$c(k_1, D) \equiv c(k_2, D) \bmod q^{A+1}.$$

We stress that the square root $c(k, D)$ and the normalization constant $c(k)$ may be chosen to be the same for different splitting primes $q$. In other words, there exist specific choices of the normalization constant and of the square root, which are suitable for most of (conjecturally for all) the splitting primes.

We are now in a position to make the choice of square root in Conjecture 1 more precise.

**Conjecture 3.** *One can choose*

$$c(k) = C(k) \quad \textit{and} \quad c(k, D) = \epsilon_D l_D(k),$$

*where $\epsilon_D = \pm 1$ depends only on the Kronecker symbol $\left(\frac{D}{p}\right)$.*

Note that this conjecture means that, in a sense, the choices of the square root for ramified and split primes are *not* compatible though they may be somehow related.

Let $d$ be as in Section 1. The conjecture above is supported by the numerical calculation of the following special values:

$$\psi = \psi_7, \quad L(2, D, 2) \text{ and } L(32, D, 32) \text{ for } 2 \le d \le 3 \quad (p = 7, \quad q = 11),$$
$$\psi = \psi_7, \quad L(3, D, 3) \text{ and } L(33, D, 33) \text{ for } 1 \le d \le 30 \quad (p = 7, \quad q = 11),$$
$$\psi = \psi_{11}, \quad L(3, D, 3) \text{ and } L(23, D, 23) \text{ for } 2 \le d \le 35 \quad (p = 11, \quad q = 5).$$

Note that $k_1$ and $k_2$ in the above table are chosen so that $k_1 \equiv k_2 \bmod$ g.c.d. $((p-1),(q-1))$, the prime $q > 3$ splits and $p$ ramifies in $K$.

In order to illustrate what kind of numerical data support our conjectures, we present a small table below. Here $\psi = \psi_7$; we took two values of $k$, and we pick $C(2) = 1$ and $C(32) = 1/7^4$.

| $D$ | $L(2, D, 2)$ | $L(32, D, 32)/7^4$ |
|---|---|---|
| $-3$ | $2^5$ | $2^{65}\ 3^{26}\ 5^2\ 31^2\ 5953^2\ 15489079^2$ |
| $-8$ | $2$ | $2^1\ 3^{14}\ 5^2\ 31^2\ 103^2\ 125131^2\ 30860858002939^2$ |
| $-11$ | $2^8$ | $2^{68}\ 3^{12}\ 5^2\ 31^2\ 6336319321700454216698 3^2$ |
| $-15$ | $2^6$ | $2^{66}\ 3^{26}\ 5^{12}\ 31^2\ 1741^2\ 1203127^2\ 2597689387^2$ |
| $-19$ | $2^5\ 3^2$ | $2^{65}\ 3^{12}\ 5^2\ 19^4\ 31^2\ 172330350416196851413880 9^2$ |
| $-24$ | $2^2$ | $2^2\ 3^{26}\ 5^2\ 31^2\ 189213645789042009123835 9^2$ |
| $-40$ | $2^6$ | $2^6\ 3^{12}\ 5^{12}\ 31^2\ 43^2\ 1747^2\ 2575277^2\ 62908696006594601^2$ |
| $-88$ | $2^3\ 3^2$ | $2^3\ 3^{14}\ 5^2\ 29^2\ 31^2\ 379^2\ 327570798975776305389589672159 1^2$ |

The fact that the entries in the table are perfect squares or twice perfect squares is expected in view of [12, Proposition 1], [11, remarks after Conjecture A], though still surprising. Of course, this fact makes it simple to check the incompatibility claim even with this short table. Notice that $2 \equiv 32 \bmod \text{g.c.d.}\ (7-1, 11-1)$, the prime 11 splits in $\mathbb{Q}(\sqrt{-7})$, and, according to our Conjecture 2 and Proposition above, the columns of the table are congruent modulo 11 and 7. Meanwhile, in order to preserve these congruences, one has to choose two *different* values of the square roots (no choice of the square root preserves the modulo 77 congruence between the two columns).

## 3. Central special values without character twists

In this section we do not consider twists with Dirichlet characters and therefore we omit index $D$ from the notation: put $D = 1$, assume $k$ to be odd, and write

$$l(k)^2 = \left( \frac{((k-3)/2)!}{(k-2)!} \right)^2 L(k, 1, k). \tag{4}$$

We restrict the consideration to the grossencharacter $\psi_7$ corresponding to $\mathbb{Q}(\sqrt{-7})$. Thus, we put $p = 7$. In this case the canonical choice of the square root for the central special values was discovered in [16], and its $p$-adic interpolation for a splitting prime was constructed in [18]. We use *the same* choice of square root as in [16, 18]; with this choice the numbers $l(k)$ are well defined by (4). Moreover, we can make use of the remarkable recursive formula from [16]. Making use of this formula, it is easy to compute hundreds of values $l(k)$, and the results of this computation support the following precise analogue of Kummer congruences.

**Conjecture 4.** *If $k_1 \equiv k_2 \bmod p^A(p-1)$ for an integer $A \geq 0$ and if $k_1$ and $k_2$ are odd numbers greater than 3, then*

$$l(k_1) \equiv l(k_2) \bmod p^{A+1}.$$

*In other words, $l(k)$ is an Iwasawa function.*

We mention also the following congruence, found experimentally.

**Conjecture 5.** *If $k \geq 5$ is odd, then $l(k) \equiv 1 \bmod 7$.*

The statements of this section, while they indirectly support Conjecture 1, are quite different. We do not consider twists with quadratic characters, and we stipulate very specific values for $C(k)$. Neither an affirmative nor negative answer to Conjecture 4 will guarantee the same answer to Conjecture 1.

## 4. A remark about an inert prime

We continue with $D = 1$, odd $k$ and $\psi = \psi_7$. The prime $q = 5$ is inert in $K = \mathbb{Q}(\sqrt{-7})$. Let $r(k)$ be the canonical square root for $L(k,1,k)/2$ as in [16]. We failed to find a nice and natural normalization for the numbers $r(k)$ so that the Kummer congruences modulo $q$ hold. Instead of this, within the range of our computations, the following statement is true.

**Conjecture 6.** *For a positive integer $n$ one has*

$$d(n) := \operatorname{ord}_q r(1+24n) = \operatorname{ord}_q r(1+120n)$$

*and*

$$q^{-d(n)} r(1+24n) \equiv q^{-d(n)} r(1+120n) \bmod q.$$

Note that $24 = q^2 - 1$ and $120 = q(q^2-1)$. We have no general framework to explain this phenomenon.

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