SOME PROPERTIES OF THE GAMMA AND PSI FUNCTIONS, WITH APPLICATIONS

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ABSTRACT. In this paper, some monotoneity and concavity properties of the gamma, beta and psi functions are obtained, from which several asymptotically sharp inequalities follow. Applying these properties, the authors improve some well-known results for the volume Ω_n of the unit ball $B^n \subset \mathbb{R}^n$, the surface area ω_{n-1} of the unit sphere S^{n-1} , and some related constants.

1. INTRODUCTION

For real and positive values x and y , the Euler gamma function, the beta and psi (or polygamma) functions are defined as

$$(1.1) \quad \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

respectively. For the extensions to complex variables and for the basic properties of these functions, see [AS], [AAR], [Mi], [T], and [WW]. Over the past half century, many authors have obtained various properties and inequalities for these very important functions (see [A1], [A2], [A5], [G], [Ke], [K2], [L], [MSC] and bibliographies therein).

Formulas for the volumes of geometric bodies sometimes involve the gamma function. This topic and related inequalities have been studied recently in [AQ], [A1]–[A4], [BP], [EL]. Let B^n and S^{n-1} be the unit ball and unit sphere in $\mathbb{R}^n$, respectively, Ω_n be the volume of B^n , and ω_{n-1} denote the surface area of S^{n-1} . Set $\Omega_0 = 1$. It is well known that Ω_n is increasing for $2 \leq n \leq 5$ and decreasing for $n \geq 5$, while ω_{n-1} is increasing for $2 \leq n \leq 7$ and decreasing for $n \geq 7$ (see [BH, pp. 263–264] and [AVV1, p. 38]):

$$(1.2) \quad \Omega_n = \frac{2\pi}{n} \Omega_{n-2} = \frac{\pi^{n/2}}{\Gamma(1+n/2)}, \quad \omega_{n-1} = n\Omega_n = 2 \frac{\pi^{n/2}}{\Gamma(n/2)}.$$

Throughout this paper, we let $\gamma = 0.57721 \dots = -\psi(1)$ denote the Euler-Mascheroni constant, $(2n-1)!! = (2n-1)(2n-3) \dots 1$ for $n \in \mathbb{N} = \{k; k \text{ is a natural number}\}$,

$$(1.3) \quad I_n = \int_0^{\pi/2} \sin^{n-2} t dt, \quad J_n = \int_0^{\pi/2} \sin^{(2-n)/(n-1)} t dt,$$

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(1.4)

$$A_n = \left(\frac{\omega_{n-1}}{2^n c_n} \right)^{1/(n-1)}, \quad B_n = b_n^{n-1}, \quad D_n = \frac{1}{n-1} \left(\frac{\omega_{n-1}}{c_n} \right)^{1/(n-1)},$$

(1.5)

$$b_n = J_n/(n-1) \quad \text{and} \quad c_n = (2J_n)^{1-n} \omega_{n-2},$$

for $n \in \mathbb{N}$ with $n \geq 2$. These constants have applications in some fields of mathematics such as geometry of Grassmannian subspaces of $\mathbb{R}^n$ [KR], optimization theory [B], geometric function theory [AVV1, pp. 234–246], [V], [Vu], as well as geometry of spaces of constant curvature [BH].

These are some examples of the fields where the gamma function is frequently used. The accumulated literature on the gamma function is so vast that it is difficult for someone working chiefly in these specialized areas of research to find the property of the gamma function which, in this particular application, would lead to the desired conclusion. This is exactly how we were led to explore the gamma function, our main research area being geometric function theory; see [V], [Vu], [AVV1], [AQ]. Some of the latest developments on special functions that have been studied in this context are reviewed in [AVV2]. The statement of some of our main results, which we think have independent interest as such, now follows.

Theorem A. For $n = 1, 2, 3, \dots$

$$2\sqrt{\frac{\pi}{2n+\pi-2}} \leq \frac{\Omega_n}{\Omega_{n-1}} < 2\sqrt{\frac{\pi}{2n+1}}.$$

Theorem B. For $x \geq 1$, let $P(x) = \max\{\sqrt{\pi x}, \sqrt{4x-1}\}$. Then for $x \geq 1$

$$x^{(1-\gamma)x-1} P(x) \leq \Gamma(x) P(x) \leq 2\Gamma(x+1/2).$$

Theorem C. For $x > 1$

$$\log x - \frac{1}{2x} - \frac{1}{12x^2} < \psi(x) < \log x - \frac{1}{2x} - \frac{2\gamma-1}{2x^2}.$$

Theorems A, B, and C are special cases of the more technical Theorems 1.17, 1.12, and 2.1, respectively. We next proceed to review some earlier results and to discuss our preliminary and main results.

Various properties of Ω_n and ω_{n-1} have been proved in these contexts. For example, it was shown in [B] that for $n \geq 3$,

$$(1.6) \quad \sqrt{\frac{2\pi}{n+1}} \leq \frac{\Omega_n}{\Omega_{n-1}} \leq \sqrt{\frac{2\pi}{n}}$$

and

$$(1.7) \quad \sqrt{\frac{2\pi}{n-1}} \leq \frac{\omega_{n-1}}{\omega_{n-2}} \leq \sqrt{\frac{2\pi}{n-2}},$$

and a new property of Ω_n was obtained in [AQ]. Several inequalities for A_n, B_n, c_n, D_n, I_n and J_n were shown or stated in [AVV1]. For example, it was proved that for $n \geq 3$,

$$(1.8) \quad \sqrt{\frac{\pi}{2n-2}} < I_n < \sqrt{\frac{\pi}{2n-4}},$$

$$(1.9) \quad -1 < (n-1)(J_n - n + 1 - \log 2) < 1 - (\pi/2) \log 2 < 0,$$

$$(1.10) \quad \lim_{n \rightarrow \infty} A_n/n = 1, \quad \lim_{n \rightarrow \infty} B_n = \lim_{n \rightarrow \infty} D_n = 2,$$

and

$$(1.11) \quad 1 < B_n < 2 \quad \text{for } n \geq 2.$$

(See [AVV1, pp. 42–43].)

It is the purpose of the present paper to show some properties of $\Gamma(x)$, $B(x, y)$ and $\psi(x)$ and then, based on these properties, to improve some well-known results for Ω_n , ω_{n-1} , A_n , B_n , b_n , I_n and J_n , such as (1.6)–(1.11). We now state some of the main results of this paper below.

1.12. Theorem. (1) *The function $f_1(x) \equiv \Gamma(x + 1/2)/[\sqrt{x}\Gamma(x)]$ is strictly increasing and log-concave from $(0, \infty)$ onto $(0, 1)$, $f_2(x) \equiv f_1(1/x)$ is log-convex on $(0, \infty)$, and $f_3(x) \equiv 1/f_1(x)$ is convex on $(0, \infty)$.*

(2) *The function $f_4(x) \equiv x - [\Gamma(x + 1/2)/\Gamma(x)]^2 = x[1 - f_1(x)^2]$ is strictly increasing and concave from $(0, \infty)$ onto $(0, 1/4)$. Moreover, for $x \geq 1$,*

$$(1.13) \quad x^{(1-\gamma)x-1}P(x) \leq \Gamma(x)P(x) \leq 2\Gamma(x + 1/2) \\ \leq \sqrt{\pi x}\Gamma(x)(2/\sqrt{\pi})^{1-1/x} \leq \sqrt{\pi x}(2/\sqrt{\pi})^{1-1/x}x^{x-1/2},$$

with equality in each instance iff $x = 1$, where $P(x) = \max\{\sqrt{\pi x}, \sqrt{4x-1}\}$. The third and the fourth inequalities in (1.13) are asymptotically sharp as x tends to ∞ .

(3) *For each fixed $x > 0$, define functions f_5, f_6 and f_7 on $(0, 1)$ by*

$$f_5(s) = \frac{1}{1-s} \left[\log x - \frac{1}{s} \log \frac{\Gamma(x+s)}{\Gamma(x)} \right], \quad f_6(s) = \frac{\beta(x) - f_5(s)}{s}, \quad f_7(s) = \frac{s[f_5(s) - \alpha(x)]}{1-s},$$

respectively, where $\alpha(x) = \psi(x+1) - \log x$ and $\beta(x) = \log x - \psi(x)$. Then f_5 and f_6 are both strictly decreasing on $(0, 1)$ with ranges $(\alpha(x), \beta(x))$ and $(\xi(x), \eta(x))$, respectively, while f_7 is strictly increasing from $(0, 1)$ onto $(0, \theta(x))$, where

$$\xi(x) = \beta(x) - \alpha(x), \quad \eta(x) = \frac{1}{2}\psi'(x) - \beta(x) \quad \text{and} \quad \theta(x) = \alpha(x) - \frac{1}{2}\psi'(x+1).$$

In particular, for $x > 0$ and $s \in (0, 1)$,

$$(1.14) \quad \exp\{s(s-1)\beta(x)\} < \exp\{s(1-s)\rho(s, x)\} < \frac{\Gamma(x+s)}{x^s\Gamma(x)} \\ < \exp\{s(s-1)\omega(s, x)\} \leq \exp\{s(s-1)\alpha(x)\},$$

where

$$\rho(s, x) = \max\{s\xi(x) - \beta(x), -\alpha(x) + (1-s)\theta(x)/s\}$$

and

$$\omega(s, x) = \min\{s\eta(x) - \beta(x), -\alpha(x)\}.$$

The above theorem yields some inequalities of $\Gamma(x)$. Some similar or related results have been proved recently in [AQ], [BP], [EL], [A5], [K2], [Me1]–[Me3].

1.15. Remarks. (1) For $n \in \mathbb{N}$, let $G(n, k)$ be the so-called *Grassmannian*, and let τ_n denote the invariant measure on $G(n, 1)$, that is, on the set of all straight lines through the origin (cf. [KR, p. 206]). Set $g(n) = \tau_n(G(n, 1))$. Then the functions $f_1(x)$, $f_4(x)$ and $f_5(s)$ in Theorem 1.12 are related to $g(n)$. Therefore, Theorem 1.12 is related to some results in [KR]. For example, it was proved in [KR, p. 214] that $g(n)$ is increasing for $n \in \mathbb{N}$, while Theorem 1.12(1) says that $g(n)/\sqrt{n}$ is also strictly increasing for $n \in \mathbb{N}$ since $g(n) = \sqrt{n\pi/2}f_1(n/2)$ by the expression of $g(n)$ (see [KR, p. 214]).

(2) The double inequality (1.14) and its Corollary 3.2 are related to a result in [LL], which gives the estimates

$$(1-s)\log(x+s/2) < D_s(x) < (1-s)\log\left(x+(\Gamma(s))^{1/(s-1)}\right)$$

for $s \in (0, 1)$ and $x > 0$, where $D_s(x) = \log \Gamma(x+1) - \log \Gamma(x+s)$. For some other related results, see [A3, p. 365], [Me1], [MSC], etc.

1.16. Theorem. (1) *The function $g_1(x) \equiv [xB(x, 1/2)]^{1/x}$ is strictly decreasing and convex from $(0, \infty)$ onto $(1, 4)$.*

(2) *There exists a unique $x_1 \in (1.5, 2)$ such that the function*

$$g_2(x) \equiv \frac{1}{2x+1} \left[\frac{1}{2} B\left(x, \frac{1}{2}\right) \right]^{1/(2x)} B\left(\frac{1}{4x}, \frac{1}{2}\right),$$

for $x \in (0, \infty)$, is strictly decreasing on $(0, x_1]$ and increasing on $[x_1, \infty)$, with $g_2(1/2) = \pi^2/4$ and $\lim_{x \rightarrow \infty} g_2(x) = 2$.

(3) *There exists a unique $x_2 \in (33, 34)$ such that the function*

$$g_3(x) \equiv 2^{1/(2x)} \left(1 + \frac{1}{2x}\right) g_2(x),$$

for $x \in (0, \infty)$, is strictly decreasing on $(0, x_2]$ and increasing on $[x_2, \infty)$, with $g_3(1/2) = \pi^2$ and $\lim_{x \rightarrow \infty} g_3(x) = 2$.

As an example of the applications of the above two theorems, we shall prove the following improvements upon some results for Ω_n , ω_{n-1} , A_n , B_n , b_n , I_n and J_n .

1.17. Theorem. (1) *For $n \in \mathbb{N} \setminus \{1, 2\}$, the function $F_1(n) \equiv 2n - \pi I_n^{-2}$ is strictly increasing with $F_1(3) = 6 - \pi$ and $\lim_{n \rightarrow \infty} F_1(n) = 3$. In particular, for $n \geq 3$,*

$$(1.18) \quad \sqrt{\frac{\pi}{2n - (6 - \pi)}} \leq I_n < \sqrt{\frac{\pi}{2n - 3}},$$

with equality iff $n = 3$.

(2) *The function $F_2(n) \equiv n - 2\pi[\omega_{n-2}/\omega_{n-1}]^2$ is strictly increasing for $n \in \mathbb{N} \setminus \{1, 2\}$, with $F_2(3) = 3 - (\pi/2)$ and $\lim_{n \rightarrow \infty} F_2(n) = 3/2$. In particular, for $n \geq 3$,*

$$(1.19) \quad 2\sqrt{\frac{\pi}{2n - 6 + \pi}} \leq \frac{\omega_{n-1}}{\omega_{n-2}} < 2\sqrt{\frac{\pi}{2n - 3}},$$

with equality iff $n = 3$.

(3) *The function $F_3(n) \equiv n - 2\pi[\Omega_{n-1}/\Omega_n]^2$ is strictly increasing for $n \in \mathbb{N}$, with $F_3(1) = 1 - \pi/2$ and $\lim_{n \rightarrow \infty} F_3(n) = -1/2$. In particular, for $n \in \mathbb{N}$,*

$$(1.20) \quad 2\sqrt{\frac{\pi}{2n + \pi - 2}} \leq \frac{\Omega_n}{\Omega_{n-1}} < 2\sqrt{\frac{\pi}{2n + 1}},$$

with equality iff $n = 1$.

(4) *Let $n \in \mathbb{N}$. Then b_n is strictly decreasing for $n \geq 2$, and B_n is strictly increasing for $n \geq 2$. In particular, for $n \geq 2$,*

$$(1.21) \quad \frac{\pi}{2} \leq B_n < 2, \quad (n-1) \left(\frac{\pi}{2}\right)^{1/(n-1)} \leq J_n \leq (n-1) \min\left\{2^{1/(n-1)}, \frac{\pi}{2}\right\},$$

with equality in each instance iff $n = 2$.

(5) Let $a = 1 - \log 2$, $b = 3 - 0.5B(1/4, 1/2) = 0.37794 \dots$, $c = [\pi^2 - 3(\log 4)^2]/24 = 0.171007 \dots$, and $n \in \mathbb{N}$. Then the function $F_4(n) \equiv (n-1)(n - J_n - a)$ is strictly increasing for $n \geq 3$, with $F_4(3) = d \equiv 2(b - a) = 0.14217 \dots$ and $\lim_{n \rightarrow \infty} F_4(n) = c$. In particular, for $n \geq 3$,

$$(1.22) \quad d \leq (n-1)(n - J_n - a) < c,$$

with equality iff $n = 3$.

(6) Let $n \in \mathbb{N}$. Then the function $F_5(n) \equiv A_n/n$ is strictly decreasing for $2 \leq n \leq 5$ and increasing for $n \geq 5$. In particular,

$$(1.23) \quad \lim_{n \rightarrow \infty} \frac{1}{n} A_n = 1, \quad C \leq \frac{1}{n} A_n \begin{cases} \leq A_2/2 = \pi^2/8, & \text{for } 2 \leq n \leq 5, \\ < 1, & \text{for } n \geq 5, \end{cases}$$

where

$$C = \frac{1}{5} A_5 = \frac{\sqrt{\pi}}{10} \left(\frac{2}{3} \right)^{1/4} \frac{\Gamma(1/8)}{\Gamma(5/8)} = 0.8411396629 \dots$$

(7) Let $n \in \mathbb{N}$. Then D_n is strictly decreasing for $2 \leq n \leq 69$ and increasing for $n \geq 69$. In particular,

$$(1.24) \quad \lim_{n \rightarrow \infty} D_n = 2, \quad D \leq D_n \begin{cases} \leq D_2 = \pi^2, & \text{for } 2 \leq n \leq 69, \\ < 2, & \text{for } n \geq 69, \end{cases}$$

where

$$D = D_{69} = \frac{\sqrt{2\pi}}{68} \left(\frac{33!}{67!!} \right)^{1/68} \frac{\Gamma(1/136)}{\Gamma(69/136)} = 1.9853487779 \dots$$

2. SOME PROPERTIES OF $\psi(x)$

In this section, we obtain some properties of $\psi(x)$, which are also needed in the proofs of the theorems stated in Section 1.

2.1. Theorem. (1) The function $h_1(x) \equiv \psi(x + 1/2) - \psi(x) - [1/(2x)]$ is strictly decreasing and convex from $(0, \infty)$ onto $(0, \infty)$. Moreover, $h_2(x) \equiv h_1(1/x)$ is convex on $(0, \infty)$.

(2) The function $h_3(x) \equiv xh_1(x)$ is strictly decreasing from $(0, \infty)$ onto $(0, 1/2)$.

(3) The function $h_4(x) \equiv x^2h_1(x)$ is strictly increasing from $(0, \infty)$ onto $(0, 1/8)$. In particular, for $x > 1$,

$$(2.2) \quad \psi(x) + \frac{1}{2x} + \frac{1.5 - \log 4}{x^2} < \psi\left(x + \frac{1}{2}\right) < \psi(x) + \frac{1}{2x} + \frac{1}{8x^2}.$$

(4) The function $h_5(x) \equiv x^2[\psi(x) - \log x] + (x/2)$ is strictly decreasing and convex from $(0, \infty)$ onto $(-1/12, 0)$. In particular, for $x > 1$,

$$(2.3) \quad \log x - \frac{1}{2x} - \frac{1}{12x^2} < \psi(x) < \log x - \frac{1}{2x} - \frac{2\gamma - 1}{2x^2}.$$

(5) Let η and θ be functions defined in Theorem 1.12(3). Then η and θ are both strictly decreasing from $(0, \infty)$ onto $(0, \infty)$.

Proof. (1) Differentiation gives

$$h'_1(x) = -\frac{h_6(x)}{x^2} \equiv \psi'\left(x + \frac{1}{2}\right) - \psi'(x) + \frac{1}{2x^2} = \frac{1}{2x^2} - \int_0^\infty \frac{te^{-xt}}{1 + e^{-t/2}} dt,$$

where

$$h_6(x) = \int_0^\infty \frac{ue^{-u} du}{1 + e^{-u/(2x)}} - \frac{1}{2},$$

since [AS, 6.4.1]

$$(2.4) \quad \psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-xt}}{1 - e^{-t}} dt.$$

Clearly, h_6 is strictly decreasing from $(0, \infty)$ onto $(0, 1/2)$. Hence h'_1 is strictly increasing from $(0, \infty)$ onto $(-\infty, 0)$, and the monotoneity and convexity of h_1 follow.

By [AS, 6.3.5], $h_1(x) = \psi(x + 1/2) - \psi(x + 1) + [1/(2x)]$ so that $h_1(0^+) = \infty$. The limiting value $\lim_{x \rightarrow \infty} h_1(x) = 0$ follows from [AS, 6.3.18].

The convexity of h_2 follows from the monotoneity of h_6 , since

$$h'_2(x) = -h'_1(1/x)/x^2 = h_6(1/x).$$

(2) Put $y = 1/x$. Then

$$h_3(x) = \frac{h_2(y)}{y} \quad \text{and} \quad \frac{h'_2(y)}{\frac{d}{dy}(y)} = h'_2(y),$$

and hence, the monotoneity of h_3 follows from the convexity of h_2 and the Monotone l'Hôpital's Rule [AVV1, Theorem 1.25].

By [AS, 6.3.5], $h_3(x)$ can be written as

$$h_3(x) = \frac{x}{2} \left[2\psi\left(x + \frac{1}{2}\right) - \psi(x + 1) - \psi(x) \right],$$

and hence, $h_3(0^+) = 1/2$. It follows from [AS, 6.3.18] that $\lim_{x \rightarrow \infty} h_3(x) = 0$.

(3) Write $h_4(x) = h_2(y)/y^2$, where $y = 1/x$, and let

$$h_7(y) = h_6\left(\frac{1}{y}\right) = \int_0^\infty \frac{ue^{-u} du}{1 + e^{-uy/2}} - \frac{1}{2}, \quad h_8(y) = y.$$

Then $h_7(0) = h_8(0) = 0$, and by (2.4),

$$(2.5) \quad \frac{h'_2(y)}{\frac{d}{dy}(y^2)} = \frac{h_7(y)}{2h_8(y)},$$

$$(2.6) \quad \frac{h'_7(y)}{h'_8(y)} = h_9(y) \equiv \frac{1}{2} \int_0^\infty \frac{u^2 e^{-u(1+y/2)}}{(1 + e^{-uy/2})^2} du.$$

Clearly, h_9 is strictly decreasing in y on $(0, \infty)$. Hence the monotoneity of h_4 follows from the Monotone l'Hôpital's Rule [AVV1, Theorem 1.25].

Clearly, $h_4(0^+) = \lim_{x \rightarrow 0} xh_3(x) = 0$. By l'Hôpital's Rule, (2.5) and (2.6),

$$\lim_{x \rightarrow \infty} h_4(x) = \lim_{y \rightarrow 0} \frac{h_2(y)}{y^2} = h_9(0) = \frac{1}{16} \int_0^\infty u^2 e^{-u} du = \frac{1}{8}.$$

Inequality (2.2) is clear.

(4) Using [AS, 6.3.21], we obtain by differentiation

$$\begin{aligned}
 h'_5(x) &= h_{10}(x) \equiv 2x[\psi(x) - \log x] + x^2 \left[\psi'(x) - \frac{1}{x} \right] + \frac{1}{2} \\
 &= -2x \left[\frac{1}{2x} + \int_0^\infty \frac{2tdt}{(t^2 + x^2)(e^{2\pi t} - 1)} \right] + \frac{1}{2} \\
 &\quad + x^2 \left[\frac{1}{2x^2} + \int_0^\infty \frac{4xt dt}{(t^2 + x^2)^2(e^{2\pi t} - 1)} \right] \\
 &= 4x^3 \int_0^\infty \frac{tdt}{(t^2 + x^2)^2(e^{2\pi t} - 1)} - 4x \int_0^\infty \frac{tdt}{(t^2 + x^2)(e^{2\pi t} - 1)} \\
 &= -4x \int_0^\infty \frac{t^3 dt}{(t^2 + x^2)^2(e^{2\pi t} - 1)} = -4x \int_0^\infty \frac{u^3 du}{(1 + u^2)^2(e^{2\pi ux} - 1)},
 \end{aligned}$$

where $u = t/x$. Since $x \mapsto x/(e^{2\pi ux} - 1)$ is strictly decreasing on $(0, \infty)$, h_{10} is strictly increasing on $(0, \infty)$. Hence the monotoneity and convexity of h_5 follow.

Clearly, $h_5(0^+) = 0$, $h_5(1) = (1/2) - \gamma$. By [AS, 6.3.18], $h_5(\infty) = -1/12$.

(5) By differentiation and (2.4), we get

$$\begin{aligned}
 2x\eta'(x) &= 2x \left[\frac{1}{2}\psi''(x) - \frac{1}{x} + \psi'(x) \right] = \int_0^\infty \frac{xt(2-t)e^{-xt}}{1 - e^{-t}} dt - 2 \\
 &= \frac{1}{x} \int_0^\infty \frac{u(2-u/x)e^{-u}}{1 - e^{-u/x}} du - 2,
 \end{aligned}$$

which is strictly increasing from $(0, \infty)$ onto $(-\infty, 0)$ since the function $v \mapsto v(2-v)/(1 - e^{-v})$ is strictly decreasing from $(0, \infty)$ onto $(-\infty, 2)$. This yields the monotoneity of η .

By [AS, 6.3.5], we can rewrite $\eta(x)$ as

$$\eta(x) = \frac{1}{2}\psi'(x+1) + \frac{1}{2x^2} - \log x + \psi(x+1) - \frac{1}{x},$$

from which we see that $\eta(0^+) = \infty$. The limiting value $\lim_{x \rightarrow \infty} \eta(x) = 0$ follows from [AS, 6.3.18 and 6.4.12].

Next, similarly, we have

$$2x\theta'(x) = \int_0^\infty \frac{xt(2+t)e^{-(x+1)t}}{1 - e^{-t}} dt - 2 = \frac{1}{x} \int_0^\infty \frac{u(2+u/x)e^{-u}}{e^{u/x} - 1} du - 2,$$

which is strictly increasing from $(0, \infty)$ onto $(-2, 0)$ since $v \mapsto v(2+v)/(e^v - 1)$ is strictly decreasing from $(0, \infty)$ onto $(0, 2)$. Hence the monotoneity of θ follows.

Clearly, $\theta(0^+) = \infty$. By [AS, 6.3.18 and 6.4.12], we obtain $\lim_{x \rightarrow \infty} \theta(x) = 0$. $\square$

2.7. Remarks. (1) The double inequality (2.3) improves the inequality obtained in [ABRVV, Theorem 3.1].

(2) It is well known that [AS, 6.1.3]

$$(2.8) \quad \gamma = \lim_{n \rightarrow \infty} d_n, \quad d_n = \sum_{k=1}^n \frac{1}{k} - \log n,$$

and by [AS, 6.3.2],

$$d_n - \gamma = \psi(n+1) - \log n.$$

S. R. Tims and J. A. Tyrrell [TT] obtained the bounds

$$(2.9) \quad \frac{1}{2(n+1)} < d_n - \gamma < \frac{1}{2(n-1)}, \quad n \geq 2,$$

for $d_n - \gamma$, which was improved by R. M. Young [Y] as

$$(2.10) \quad \frac{1}{2(n+1)} < d_n - \gamma < \frac{1}{2n}, \quad \text{for } n \in \mathbb{N},$$

while G. D. Anderson et al. [ABRVV] proved

$$(2.11) \quad \frac{1-\gamma}{n} < d_n - \gamma < \frac{1}{2n}, \quad \text{for } n \in \mathbb{N}.$$

Recently, H. Alzer [A3, Theorem 3] established the double inequality

$$(2.12) \quad \frac{1}{2(n+\alpha)} \leq d_n - \gamma < \frac{1}{2(n+\beta)}, \quad \text{for } n \in \mathbb{N},$$

where $\alpha = \{1/[2(1-\gamma)]\} - 1 = 0.1826 \dots$ and $\beta = 1/6$. Let h_5 be as in Theorem 2.1(4). Then

$$h_5(n) = n^2 \left\{ \psi(n+1) - \log n - \frac{1}{n} \right\} + \frac{n}{2} = n^2(d_n - \gamma) - \frac{n}{2},$$

and hence our Theorem 2.1(4) implies the following estimates for $d_n - \gamma$:

2.13. Corollary. *For $n \in \mathbb{N}$,*

$$(2.14) \quad \frac{1}{2n} - \frac{\alpha}{n^2} < d_n - \gamma \leq \frac{1}{2n} - \frac{\beta}{n^2},$$

with equality iff $n = 1$, where the constants $\alpha = 1/12 = 0.08333 \dots$ and $\beta = \gamma - 1/2 = 0.07721 \dots$ are best possible.

Some further results on the approximation of γ are given in [K1].

3. PROOFS OF THE MAIN THEOREMS

In this section we prove the theorems stated in Section 1.

3.1. *Proof of Theorem 1.12.* (1) Logarithmic differentiation gives

$$f_1'(x)/f_1(x) = h_1(x),$$

where h_1 is as in Theorem 2.1(1), and hence the monotoneity and log-concavity of f_1 follow from Theorem 2.1(1). By [AS, 6.1.15 and 6.1.37], we have

$$f_1(0^+) = \lim_{x \rightarrow 0} \sqrt{x} \frac{\Gamma(x+1/2)}{\Gamma(x+1)} = 0$$

and

$$\lim_{x \rightarrow \infty} f_1(x) = \lim_{x \rightarrow \infty} \frac{e^{-(x+1/2)}(x+1/2)^x}{\sqrt{x}e^{-x}x^{x-1/2}} = 1.$$

Since $f_2'(x)/f_2(x) = -h_4(1/x)$, where h_4 is as in Theorem 2.1(3), the log-convexity of f_2 follows from Theorem 2.1(3).

Next, by differentiation,

$$f_3'(x) = -f_1'(x)/f_1(x)^2 = -h_1(x)/f_1(x),$$

which is strictly increasing on $(0, \infty)$ by Theorem 2.1(1), and hence the convexity of f_3 follows.

(2) Set $y = \Gamma(x + 1/2)/\Gamma(x)$. Then

$$f_4'(x) = 1 - 2y^2 \left[\psi\left(x + \frac{1}{2}\right) - \psi(x) \right]$$

and

$$\frac{1}{2y^2} f_4''(x) = - \left[\psi\left(x + \frac{1}{2}\right) - \psi(x) \right]^2 f_8(x),$$

where

$$f_8(x) = f_9'(x) = 2 + \frac{\psi'(x + 1/2) - \psi'(x)}{[\psi(x + 1/2) - \psi(x)]^2},$$

$$f_9(x) = 2x - \left[\psi\left(x + \frac{1}{2}\right) - \psi(x) \right]^{-1} = 2 \frac{h_4(x)}{h_3(x) + 1/2}$$

and h_3 and h_4 are as in Theorem 2.1. By Theorem 2.1(2)-(3), f_9 is strictly increasing from $(0, \infty)$ onto $(0, 1/2)$, so that $f_8(x) > 0$ for $x \in (0, \infty)$, and hence $f_4''(x) < 0$ for $x \in (0, \infty)$. This yields the concavity of f_4 . Furthermore, by part (1) and Theorem 2.1(2),

$$\begin{aligned} f_4'(x) &> \lim_{x \rightarrow \infty} f_4'(x) \\ &= 1 - 2 \lim_{x \rightarrow \infty} f_1(x)^2 \cdot x \left[\psi\left(x + \frac{1}{2}\right) - \psi(x) \right] \\ &= 1 - 2 \left[\lim_{x \rightarrow \infty} f_1(x) \right]^2 \left[\lim_{x \rightarrow \infty} h_3(x) + \frac{1}{2} \right] = 0 \end{aligned}$$

for $x \in (0, \infty)$, and hence the monotoneity of f_4 follows.

By part (1), $f_4(0^+) = \lim_{x \rightarrow 0} [x - x f_1(x)^2] = 0$. Applying l'Hôpital's Rule, we obtain

$$\begin{aligned} \lim_{x \rightarrow \infty} f_4(x) &= \lim_{x \rightarrow \infty} \frac{1 - f_1(x)^2}{1/x} = \lim_{t \rightarrow 0} \frac{1 - f_2(t)^2}{t} \\ &= -2 \lim_{t \rightarrow 0} f_2(t) f_2'(t) = 2 \lim_{t \rightarrow 0} f_2(t)^2 h_4(1/t) \\ &= 2 \lim_{x \rightarrow \infty} f_1(x)^2 h_4(x) = 1/4. \end{aligned}$$

The second and third inequalities in (1.13) follow from the monotoneity of f_4 on $(0, \infty)$ and the log-convexity of f_2 on $(0, 1)$, while the first and fourth inequalities in (1.13) hold by [AQ, Theorem 1.5]. The equality case is clear. The asymptotic sharpness of the third and fourth inequalities in (1.13) follows from the monotoneity of f_1 and [AQ, Theorem 1.5].

(3) For $x > 0$, let

$$f_{10}(s) = \log x - \frac{1}{s} \log \frac{\Gamma(x+s)}{\Gamma(x)}, \quad f_{11}(s) = 1 - s,$$

$$f_{12}(s) = s\psi(x+s) - \log \Gamma(x+s) + \log \Gamma(x) \quad \text{and} \quad f_{13}(s) = s^2.$$

Then $f_5(s) = f_{10}(s)/f_{11}(s)$, $f_{10}(1) = f_{11}(1) = f_{12}(0) = f_{13}(0) = 0$, and

$$\frac{f_{10}'(s)}{f_{11}'(s)} = \frac{f_{12}(s)}{f_{13}(s)}, \quad \frac{f_{12}'(s)}{f_{13}'(s)} = \frac{1}{2} \psi'(x+s).$$

Since $\psi'(t)$ is strictly decreasing on $(0, \infty)$, the monotoneity of f_5 follows from the Monotone l'Hôpital's Rule [AVV1, Theorem 1.25].

By l'Hôpital's Rule,

$$f_5(1^-) = \lim_{s \rightarrow 1} \frac{f_{12}(s)}{f_{13}(s)} = \psi(x+1) - \log x = \alpha(x),$$

and

$$\begin{aligned} f_5(0^+) &= f_{10}(0^+) = \log x - \lim_{s \rightarrow 0} \frac{\log \Gamma(x+s) - \log \Gamma(x)}{s} \\ &= \log x - \psi(x) = \beta(x). \end{aligned}$$

Let $f_{14}(s) = s\beta(x) - sf_5(s)$, $f_{15}(s) = s^2$,

$$\begin{aligned} f_{16}(s) &= \psi'(x+s)(1-s)^2 - 2\{\log x - \psi(x+s)\}(1-s) \\ &\quad + s \log x - \log \Gamma(x+s) + \log \Gamma(x), \end{aligned}$$

and $f_{17}(s) = (1-s)^3$. Then $f_6(s) = f_{14}(s)/f_{15}(s)$, $f_{14}(0) = f_{15}(0) = f'_{14}(0) = f'_{15}(0) = f_{16}(1) = f_{17}(1) = 0$ and

$$\frac{f''_{14}(s)}{f''_{15}(s)} = \frac{f_{16}(s)}{2f_{17}(s)}, \quad \frac{f'_{16}(s)}{f'_{17}(s)} = -\frac{1}{3}\psi''(x+s).$$

It is well known that ψ'' is strictly increasing on $(0, \infty)$. Hence the monotoneity of f_6 follows from the Monotone l'Hôpital's Rule [AVV1, Theorem 1.25].

By l'Hôpital's Rule, we get

$$f_6(0^+) = \lim_{s \rightarrow 0} \frac{f'_{14}(s)}{f'_{15}(s)} = \lim_{s \rightarrow 0} \frac{f''_{14}(s)}{f''_{15}(s)} = \frac{f_{16}(0)}{2f_{17}(0)} = \frac{1}{2}\psi'(x) - \beta(x) = \eta(x).$$

The limiting value $f_6(1^-)$ follows from the result for f_5 .

Next, let

$$f_{18}(s) = s \log x - \log \frac{\Gamma(x+s)}{\Gamma(x)} - s(1-s)\alpha(x) \quad \text{and} \quad f_{19}(s) = (1-s)^2.$$

Then $f_7(s) = f_{18}(s)/f_{19}(s)$, $f_{18}(1) = f_{19}(1) = f'_{18}(1) = f'_{19}(1) = 0$ and

$$\frac{f''_{18}(s)}{f''_{19}(s)} = \alpha(x) - \frac{1}{2}\psi'(x+s),$$

which is strictly increasing in s on $(0, 1)$ by the well-known monotoneity of ψ' . Hence the monotoneity of f_7 follows from the Monotone l'Hôpital's Rule [AVV1, Theorem 1.25].

Clearly, $f_7(0^+) = 0$. By l'Hôpital's Rule, $f_7(1^-) = \theta(x)$.

The inequalities in (1.14) are clear.

3.2. Corollary. (1) For $x > 1$ and $s \in (0, 1)$,

$$x^s \exp\left\{\frac{s(s-1)}{2x}\left(1 + \frac{1}{6x}\right)\right\} < \frac{\Gamma(x+s)}{\Gamma(x)} < x^s \exp\left\{\frac{s(s-1)}{2x}\left(1 - \frac{1}{6x}\right)\right\}.$$

(2) For $s \in [0, 1]$, let $\lambda(s) = 1 + 1/[3(1-s)]$ and let β be as in Theorem 1.12(3). Then for $x > (1-s)/2$ and $s \in (0, 1)$,

$$(3.3) \quad \frac{\Gamma(x+s)}{\Gamma(x)} > x^s \exp\left\{-s(1-s)\beta\left(\frac{1-s}{2}\right)\right\} > x^s \exp\{-s\lambda(s)\}.$$

Proof. By (2.3) and [AS, 6.3.5], we have

$$\beta(x) < \frac{1}{2x} \left(1 + \frac{1}{6x}\right) \quad \text{and} \quad \alpha(x) > \frac{1}{2x} \left(1 - \frac{1}{6x}\right),$$

and hence part (1) follows from (1.14)

Next, by (2.4),

$$x\beta'(x) = 1 - x\psi'(x) = 1 - \int_0^\infty \frac{xte^{-xt}}{1 - e^{-t}} dt = 1 - \frac{1}{x} \int_0^\infty \frac{ue^{-u} du}{1 - e^{-u/x}},$$

which is strictly increasing from $(0, \infty)$ onto $(-\infty, 0)$ since $v \mapsto v/(1 - e^{-v})$ is strictly increasing from $(0, \infty)$ onto $(1, \infty)$. Hence β is strictly decreasing and convex from $(0, \infty)$ onto itself. This, together with the first inequality in (2.3), yields (3.3). $\square$

3.4. *Proof of Theorem 1.16.* (1) By (1.1), we can rewrite $g_1(x)$ as

$$g_1(x) = \left[\frac{\sqrt{\pi}x\Gamma(x)}{\Gamma(x+1/2)} \right]^{1/x} = \left[\frac{\sqrt{\pi}\Gamma(x+1)}{\Gamma(x+1/2)} \right]^{1/x}.$$

Logarithmic differentiation gives

$$(3.5) \quad g_1'(x)/g_1(x) = g_4(x) \equiv g_5(x)/g_6(x),$$

where $g_6(x) = x^2$ and

$$g_5(x) = x \left[\psi(x+1) - \psi\left(x + \frac{1}{2}\right) \right] + \log \frac{\Gamma(x+1/2)}{\sqrt{\pi}\Gamma(x+1)},$$

with $g_5(0) = g_6(0) = 0$, $\lim_{x \rightarrow \infty} g_5(x) = -\infty$, and

$$(3.6) \quad \frac{g_5'(x)}{g_6'(x)} = g_7(x) \equiv \frac{1}{2} \left[\psi'(x+1) - \psi'\left(x + \frac{1}{2}\right) \right].$$

Since $\psi''(x)$ is strictly increasing, $g_7'(x) > 0$. Hence g_4 is strictly increasing on $(0, \infty)$ by the Monotone l'Hôpital's Rule [AVV1, Theorem 1.25]. By l'Hôpital's Rule, (3.5), (3.6), and [AS, 6.4.2, 6.4.4, 6.4.10 and 23.2.24],

$$g_4(0^+) = g_7(0^+) = \frac{1}{2} \left[\psi'(1) - \psi'\left(\frac{1}{2}\right) \right] = -\zeta(2) = -\frac{\pi^2}{6}$$

and

$$\lim_{x \rightarrow \infty} g_4(x) = \lim_{x \rightarrow \infty} g_7(x) = 0.$$

Here $\zeta(x)$ is the Riemann zeta function. Hence g_4 is strictly increasing from $(0, \infty)$ onto $(-\pi^2/6, 0)$, so that the monotonicity of g_1 follows from (3.5). Since $g_1'(x) = -g_1(x) \cdot [-g_4(x)]$, g_1' is strictly increasing on $(0, \infty)$ and the convexity of g_1 follows.

By l'Hôpital's Rule and [AS, 6.3.2, 6.3.3 and 6.3.18],

$$\begin{aligned} g_1(0^+) &= \exp \left\{ \lim_{x \rightarrow 0} \frac{\log \Gamma(x+1) - \log \Gamma(x+1/2) + (\log \pi)/2}{x} \right\} \\ &= \exp(\psi(1) - \psi(1/2)) = 4 \end{aligned}$$

and

$$\lim_{x \rightarrow \infty} g_1(x) = \exp \left\{ \lim_{x \rightarrow \infty} \left[\psi(x+1) - \psi\left(x + \frac{1}{2}\right) \right] \right\} = 1.$$

(2) By (1.1) and [AS, 6.1.15], $g_2(x)$ can be rewritten as

$$g_2(x) = \frac{\sqrt{\pi}\Gamma(1+1/(4x))}{\Gamma((3/2)+1/(4x))} \left[\frac{\sqrt{\pi}}{2} \frac{\Gamma(x)}{\Gamma(x+1/2)} \right]^{1/(2x)}.$$

Logarithmic differentiation gives

$$(3.7) \quad \frac{4x^2 g_2'(x)}{g_2(x)} = g_8(x) \equiv 2x \left[\psi(x) - \psi\left(x + \frac{1}{2}\right) \right] - 2 \log \frac{\sqrt{\pi}\Gamma(x)}{2\Gamma(x+1/2)} + \psi\left(\frac{1}{4x} + \frac{3}{2}\right) - \psi\left(\frac{1}{4x} + 1\right).$$

Since

$$g_8'(x) = 2x \left[\psi'(x) - \psi'\left(x + \frac{1}{2}\right) \right] + \frac{1}{4x^2} \left[\psi'\left(\frac{1}{4x} + 1\right) - \psi'\left(\frac{1}{4x} + \frac{3}{2}\right) \right],$$

which is positive for $x \in (0, \infty)$ by the monotonicity of ψ' on $(0, \infty)$, g_8 is strictly increasing on $(0, \infty)$.

Let h_3 be as in Theorem 2.1. Then g_8 can be rewritten as

$$(3.8) \quad g_8(x) = -2h_3(x) - 1 - 2 \log \frac{\sqrt{\pi}\Gamma(x)}{2\Gamma(x+1/2)} + \psi\left(\frac{1}{4x} + \frac{3}{2}\right) - \psi\left(\frac{1}{4x} + 1\right).$$

Therefore, by [AS, 6.3.18 and 6.1.37],

$$g_8(0^+) = -2 + \log 4 + \lim_{x \rightarrow 0} \left\{ 2 \log x + \log \left[\left(\frac{1}{4x} + \frac{3}{2} \right) / \left(\frac{1}{4x} + 1 \right) \right] - \frac{1}{[1/(2x)] + 3} + \frac{1}{[1/(2x)] + 2} \right\} = -\infty$$

and

$$\begin{aligned} \lim_{x \rightarrow \infty} g_8(x) &= -1 + \psi\left(\frac{3}{2}\right) - \psi(1) - 2 \log \frac{\sqrt{\pi}}{2} - 2 \lim_{x \rightarrow \infty} \log \frac{\Gamma(x)}{\Gamma(x+1/2)} \\ &= 1 - \log \pi - 2 \lim_{x \rightarrow \infty} \log \left[\sqrt{\frac{e}{x}} \left(1 + \frac{1}{2x} \right)^{-x} \right] = \infty. \end{aligned}$$

Hence g_8 has a unique zero $x_1 \in (0, \infty)$ such that $g_8(x) < 0$ for $x \in (0, x_1)$ and $g_8(x) > 0$ for $x > x_1$, so that the piecewise monotonicity of g_2 follows from (3.7).

By [AS, 6.3.7 and 6.3.8], we have

$$\begin{aligned} \psi\left(\frac{2}{3}\right) - \psi\left(\frac{1}{6}\right) &= \psi\left(1 - \frac{1}{3}\right) - \psi\left(\frac{1}{6}\right) \\ &= \pi \cot \frac{\pi}{3} + \psi\left(\frac{1}{3}\right) - \psi\left(\frac{1}{6}\right) \\ &= \pi \cot \frac{\pi}{3} + \left[\frac{1}{2} \psi\left(\frac{1}{6}\right) + \frac{1}{2} \psi\left(\frac{1}{6} + \frac{1}{2}\right) + \log 2 \right] - \psi\left(\frac{1}{6}\right) \\ &= \pi \cot \frac{\pi}{3} + \frac{1}{2} \left[\psi\left(\frac{2}{3}\right) - \psi\left(\frac{1}{6}\right) \right] + \log 2, \end{aligned}$$

so that

$$\psi\left(\frac{2}{3}\right) - \psi\left(\frac{1}{6}\right) = 2\pi \cot \frac{\pi}{3} + \log 4.$$

Consequently, by [AS, 6.3.2, 6.3.4 and 6.3.5],

$$\begin{aligned}
 g_8\left(\frac{3}{2}\right) &= 3\left[\psi\left(\frac{3}{2}\right) - \psi(2)\right] - 2\log\frac{\sqrt{\pi}\Gamma(3/2)}{2\Gamma(2)} + \psi\left(\frac{10}{6}\right) - \psi\left(\frac{7}{6}\right) \\
 &= 3(1 - \log 4) - 2\log\frac{\pi}{4} + \psi\left(\frac{2}{3} + 1\right) - \psi\left(\frac{1}{6} + 1\right) \\
 &= -\frac{3}{2} - \log 4 - 2\log\pi + \psi\left(\frac{2}{3}\right) - \psi\left(\frac{1}{6}\right) \\
 &= -\frac{3}{2} - 2\log(2\pi) + 2\pi\cot\frac{\pi}{3} + \log 4 = -0.16186\cdots < 0.
 \end{aligned}$$

On the other hand, by [AS, 6.3.2, 6.3.4, 6.1.12, 6.3.5 and 6.3.8],

$$\begin{aligned}
 g_8(2) &= 4\left(\log 4 - \frac{5}{3}\right) - 2\log\frac{2}{3} + \left[\psi\left(\frac{5}{8}\right) + \frac{8}{5}\right] - \left[8 + \psi\left(\frac{1}{8}\right)\right] \\
 &= 3\log 4 + 2\log 3 - \frac{196}{15} + \psi\left(\frac{5}{8}\right) - \psi\left(\frac{1}{8}\right) \\
 &= 3\log 4 + 2\log 3 - \frac{196}{15} + \psi\left(\frac{1}{8} + \frac{1}{2}\right) - \psi\left(\frac{1}{8}\right) \\
 &= 2\log 12 - \frac{196}{15} + 2\left[\psi\left(\frac{1}{4}\right) - \psi\left(\frac{1}{8}\right)\right].
 \end{aligned}$$

It follows from [GR, 8.366.4 and 8.363.6, pp. 944–945] that

$$\psi\left(\frac{1}{4}\right) = -\gamma - \frac{\pi}{2} - 3\log 2$$

and

$$\begin{aligned}
 \psi\left(\frac{1}{8}\right) &= -\gamma - 3\log 2 - \frac{\pi}{2}\cot\frac{\pi}{8} + \sum_{k=1}^7\left(\cos\frac{k\pi}{4}\right)\log\left(2\sin\frac{k\pi}{8}\right) \\
 &= -\gamma - 4\log 2 - \frac{\pi}{2}\cot\frac{\pi}{8} + \sqrt{2}\log\frac{\sin(\pi/8)}{\sin(3\pi/8)} \\
 &= -\gamma - 4\log 2 - \frac{\pi}{2}\cot\frac{\pi}{8} + \sqrt{2}\log\tan\frac{\pi}{8},
 \end{aligned}$$

so that

$$\psi\left(\frac{1}{4}\right) - \psi\left(\frac{1}{8}\right) = -\frac{\pi}{2} + \log 2 + \frac{\pi}{2}\cot\frac{\pi}{8} + \sqrt{2}\log\cot\frac{\pi}{8},$$

and hence,

$$g_8(2) = 3\log 4 + 2\log 3 - \frac{196}{15} - \pi + \pi\cot\frac{\pi}{8} + 2\sqrt{2}\log\cot\frac{\pi}{8} = 0.225224\cdots > 0.$$

Consequently, $x_1 \in (3/2, 2)$.

Clearly,

$$g_2\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \cdot \frac{\pi}{2} \cdot \frac{\Gamma(1/2)}{\Gamma(1)} = \frac{\pi^2}{4}.$$

It follows from [AS, 6.1.37] that $\lim_{x \rightarrow \infty} g_2(x) = 2$.

(3) Logarithmic differentiation gives

(3.9)

$$\begin{aligned} \frac{4x^2 g'_3(x)}{g_3(x)} &= g_9(x) \equiv g_8(x) + \frac{2}{2x+1} - 2 - \log 4 \\ &= 2x \left[\psi(x) - \psi\left(x + \frac{1}{2}\right) \right] + 2 \log \frac{\Gamma(x+1/2)}{\sqrt{\pi}\Gamma(x)} + \psi\left(\frac{1}{4x} + \frac{1}{2}\right) - \psi\left(\frac{1}{4x}\right) - 4x. \end{aligned}$$

By [AS, 6.4.1], for each $x > 0$,

$$\begin{aligned} g'_8(x) &= 2x \int_0^\infty \frac{te^{-xt}dt}{1+e^{-t/2}} + \frac{1}{4x^2} \int_0^\infty \frac{te^{-[1+1/(4x)]t}dt}{1+e^{-t/2}} \\ &> 2x \int_0^\infty \frac{te^{-xt}dt}{1+e^{-t/2}} = \frac{2}{x} \int_0^\infty \frac{(xt)e^{-xt}d(xt)}{1+e^{-t/2}} \\ &= \frac{2}{x} \int_0^\infty \frac{ue^{-u}du}{1+e^{-u/(2x)}} > \frac{1}{x} \int_0^\infty ue^{-u}du = \frac{1}{x}, \end{aligned}$$

and hence,

$$g'_9(x) = g'_8(x) - \frac{4}{(2x+1)^2} > \frac{1}{x} - \frac{4}{(2x+1)^2} = \frac{4x^2+1}{x(2x+1)^2} > 0,$$

so that g_9 is strictly increasing on $(0, \infty)$. Since $g_9(0^+) = -\infty$ and $\lim_{x \rightarrow \infty} g_9(x) = \infty$, g_9 has a unique zero x_2 such that $g_9(x) < 0$ for $x \in (0, x_2)$ and $g_9(x) > 0$ for $x \in (x_2, \infty)$. This, together with (3.9), yields the piecewise monotonicity of g_3 .

Applying the formulas [AS, 6.3.2, 6.3.4, 6.3.5 and 6.1.12]

$$(3.10) \quad \psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k}, \quad \psi\left(n + \frac{1}{2}\right) = -\gamma - \log 4 + 2 \sum_{k=1}^n \frac{1}{2k-1},$$

(3.11)

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} \sqrt{\pi},$$

and [GR, 8.370 and 8.372]

$$(3.12) \quad \psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right) = 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{x+k},$$

we get

$$\begin{aligned} \frac{1}{2n} g_9(n) &= g_{10}(n) \equiv \log 2 - 2 + \frac{1}{n} \log \frac{(2n-1)!!}{(n-1)!} \\ &+ 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{2nk+1} - \sum_{k=1}^n \frac{1}{k(2k-1)} - \frac{1}{n} \end{aligned}$$

for $n \in \mathbb{N}$, and hence, by the properties of the Leibniz series and by computation,

$$\begin{aligned} g_{10}(33) &= \log 2 - 2 + \frac{1}{33} \log \frac{65!!}{32!} + 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{66k+1} - \sum_{k=1}^{33} \frac{1}{k(2k-1)} - \frac{1}{33} \\ &< \log 2 - 2 + \frac{1}{33} \log \frac{65!!}{32!} + 2 \sum_{k=0}^{30000} \frac{(-1)^k}{66k+1} - \sum_{k=1}^{33} \frac{1}{k(2k-1)} - \frac{1}{33} \\ &= -0.000381 \cdots < 0 \end{aligned}$$

and

$$\begin{aligned}
 g_{10}(34) &= \log 2 - 2 + \frac{1}{34} \log \frac{67!!}{33!} + 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{68k+1} - \sum_{k=1}^{34} \frac{1}{k(2k-1)} - \frac{1}{34} \\
 &> \log 2 - 2 + \frac{1}{34} \log \frac{67!!}{33!} + 2 \sum_{k=0}^{30001} \frac{(-1)^k}{68k+1} - \sum_{k=1}^{34} \frac{1}{k(2k-1)} - \frac{1}{34} \\
 &= 0.000065 \dots > 0.
 \end{aligned}$$

Hence $x_2 \in (33, 34)$.

Finally, it is clear that $g_3(1/2) = 4g_2(1/2) = \pi^2$ and $\lim_{x \rightarrow \infty} g_3(x) = \lim_{x \rightarrow \infty} g_2(x) = 2$. $\square$

3.13. *Proof of Theorem 1.17.* (1) By the expression

$$I_n = \frac{1}{2} B\left(\frac{n-1}{2}, \frac{1}{2}\right),$$

$F_1(n)$ can be rewritten as

$$F_1(n) = 2n - 4 \left[\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \right]^2 = 4f_4\left(\frac{n-1}{2}\right) + 2$$

by (1.1), where f_4 is as in Theorem 1.12. Hence the monotoneity of F_1 and the limiting value $\lim_{n \rightarrow \infty} F_1(n) = 3$ follow from Theorem 1.12(2). Clearly, $F_1(1) = 6 - \pi$.

Thus, for $n \geq 3$, $6 - \pi \leq F_1(n) < 3$, so that (1.18) holds.

(2) By (1.2),

$$F_2(n) = 2 \left\{ \frac{n-1}{2} - \left[\frac{\Gamma\left(\frac{n-1}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \right]^2 + \frac{1}{2} \right\} = 2f_4\left(\frac{n-1}{2}\right) + 1,$$

which is strictly increasing for $n \geq 3$ with $F_2(3) = 3 - \pi/2$ and $\lim_{n \rightarrow \infty} F_2(n) = 3/2$ by Theorem 1.12(2). Hence the result follows.

(3) It follows from (1.2) that

$$\frac{\Omega_{n-1}}{\Omega_n} = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma\left(\frac{n+1}{2}\right)} = \frac{\omega_n}{\omega_{n+1}},$$

so that

$$F_3(n) = n - 2\pi \left(\frac{\omega_n}{\omega_{n+1}} \right)^2 = F_2(n+2) - 2.$$

Hence part (3) follows from part (2).

(4) It follows from (1.2), the expression [AVV1, (2.32), p. 41]

$$(3.14) \quad J_n = \frac{1}{2} B\left(\frac{1}{2(n-1)}, \frac{1}{2}\right)$$

and [AS, 6.1.15] that

$$(3.15) \quad b_n = \frac{\sqrt{\pi}}{2(n-1)} \frac{\Gamma\left(\frac{1}{2(n-1)}\right)}{\Gamma\left(\frac{1}{2(n-1)} + \frac{1}{2}\right)} = \sqrt{\pi} F_6\left(\frac{1}{2(n-1)}\right),$$

where $F_6(x) = \Gamma(x+1)/\Gamma(x+1/2)$. Since

$$F_6'(x) = F_6(x)[\psi(x+1) - \psi(x+1/2)] > 0$$

by the monotonicity of $\psi(x)$, F_6 is strictly increasing on $(0, \infty)$. Hence the monotonicity of b_n follows from (3.15), and

$$(3.16) \quad 1 = \sqrt{\pi}F_6(0) < b_n \leq b_2 = \pi/2.$$

Next, let g_1 be as in Theorem 1.16(1). Then by (1.4) and (3.14),

$$B_n = \left[\frac{1}{2(n-1)} B\left(\frac{1}{2(n-1)}, \frac{1}{2}\right) \right]^{n-1} = \sqrt{g_1\left(\frac{1}{2(n-1)}\right)},$$

and hence the monotonicity of B_n follows from Theorem 1.16(1). Moreover, for $n \geq 2$,

$$(3.17) \quad \frac{\pi}{2} = B_2 \leq B_n < \lim_{n \rightarrow \infty} B_n = \sqrt{g_1(0^+)} = 2.$$

The second double inequality in (1.21) follows from (3.16) and (3.17).

(5) We know that $n - J_n - a$ has the expression

$$n - J_n - a = \int_0^{\pi/2} \left[1 - (\sin t)^{1/(n-1)} \right] \tan \frac{t}{2} dt.$$

(Cf. [AVV1, p. 43].) Hence

$$(3.18) \quad F_4(n) = \int_0^{\pi/2} F_7\left(\frac{1}{n-1}, t\right) \tan \frac{t}{2} dt,$$

where $F_7(x, t) = [1 - (\sin t)^x]/x$. Since the function

$$F_8(x, t) \equiv \frac{\frac{d}{dx}[1 - (\sin t)^x]}{\frac{d}{dx}(x)} = (\sin t)^x \log \frac{1}{\sin t}$$

is strictly decreasing in x on $(0, \infty)$ for each fixed $t \in (0, \pi/2)$, it follows from the Monotone l'Hôpital's Rule [AVV1, Theorem 1.25] that F_7 is strictly decreasing in x on $(0, \infty)$. Hence F_4 is strictly increasing for $n \geq 3$.

Clearly, $F_4(3) = 2(3 - J_3 - a) = 2(b - a)$. Set $x = 1/[2(n-1)]$. Then, by (3.14),

$$\begin{aligned} F_4(n) &= \frac{1}{2x} \left[\frac{1}{2x} + 1 - \frac{1}{2} B\left(x, \frac{1}{2}\right) - a \right] \\ &= \frac{1}{4x^2} \left[x \log 4 - \sqrt{\pi} \frac{\Gamma(x+1)}{\Gamma\left(x + \frac{1}{2}\right)} + 1 \right]. \end{aligned}$$

Hence, by l'Hôpital's Rule,

$$\begin{aligned}
 F_4(\infty) &\equiv \lim_{n \rightarrow \infty} F_4(n) \\
 &= \frac{1}{4} \lim_{x \rightarrow 0} \frac{x \log 4 + 1 - [\sqrt{\pi} \Gamma(x+1)/\Gamma(x+1/2)]}{x^2} \\
 &= \frac{1}{8} \lim_{x \rightarrow 0} \frac{1}{x} \left\{ \log 4 - \sqrt{\pi} \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \left[\psi(x+1) - \psi\left(x+\frac{1}{2}\right) \right] \right\} \\
 &= -\frac{\sqrt{\pi}}{8} \lim_{x \rightarrow 0} \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \\
 &\quad \times \left\{ \left[\psi(x+1) - \psi\left(x+\frac{1}{2}\right) \right]^2 + \left[\psi'(x+1) - \psi'\left(x+\frac{1}{2}\right) \right] \right\} \\
 &= -\frac{1}{8} \left[(\log 4)^2 + \frac{\pi^2}{6} - \psi'\left(\frac{1}{2}\right) \right] = \frac{1}{8} \left[\frac{\pi^2}{3} - (\log 4)^2 \right] = c.
 \end{aligned}$$

The inequality (1.22) is clear.

(6) By (1.2), (1.4), (1.5) and [AS, 6.1.15], we have

$$A_n = \left[\frac{\pi^{n/2} \Gamma\left(\frac{n-1}{2}\right)}{2^n \Gamma\left(\frac{n}{2}\right)} \right]^{1/(n-1)} \frac{\Gamma\left(\frac{1}{2(n-1)}\right)}{\Gamma\left(\frac{1}{2(n-1)} + \frac{1}{2}\right)}.$$

Putting $x = (n-1)/2$ and applying [AS, 6.1.15], we get

(3.19)

$$F_5(n) = \frac{\sqrt{\pi}}{2} \frac{1}{2x+1} \left[\frac{\sqrt{\pi}}{2} \frac{\Gamma(x)}{\Gamma\left(x+\frac{1}{2}\right)} \right]^{1/(2x)} \frac{\Gamma\left(\frac{1}{4x}\right)}{\Gamma\left(\frac{1}{4x} + \frac{1}{2}\right)} = \frac{1}{2} g_2(x),$$

where g_2 is as in Theorem 1.16(2).

Next, computation gives

(3.20)

$$F_5(4) = \frac{\sqrt{\pi}}{2} \left[\frac{\sqrt{\pi}}{2} \frac{\Gamma(3/2)}{\Gamma(2)} \right]^{1/3} \frac{\Gamma(1+1/16)}{\Gamma(1+9/16)} = 0.8890941475 \dots$$

and

(3.21)

$$F_5(5) = \frac{\sqrt{\pi}}{2} \left[\frac{\sqrt{\pi}}{2} \frac{\Gamma(2)}{\Gamma(2+1/2)} \right]^{1/4} \frac{\Gamma(1+1/8)}{\Gamma(1+5/8)} = 0.8411396629 \dots$$

The result for F_5 now follows from (3.19), (3.20), (3.21) and Theorem 1.16(2).

The limiting value $\lim_{n \rightarrow \infty} (1/n) A_n = 1$ follows from Theorem 1.16(2), and the inequality in (1.23) is clear.

(7) Put $x = (n-1)/2$. Then, by (3.19),

$$(3.22) \quad D_n = 2^{n/(n-1)} \frac{1}{n-1} A_n = 2^{n/(n-1)} \frac{n}{n-1} \cdot \frac{A_n}{n} = g_3(x),$$

where g_3 is as in Theorem 1.16(3). Hence, by Theorem 1.16(3), D_n is strictly decreasing for $2 \leq n \leq 67$ and strictly increasing for $n \geq 69$.

By computation, we obtain

$$\begin{aligned}
 D_{67} &= \frac{2^{67/66}}{66} A_{67} = \frac{1}{66} \left\{ \pi^{67/2} \frac{\Gamma(33)}{\Gamma(33+1/2)} \right\}^{1/66} \frac{\Gamma(1/132)}{\Gamma(67/132)} \\
 &= \frac{\sqrt{2\pi}}{66} \left(\frac{32!}{65!!} \right)^{1/66} \frac{\Gamma(1/132)}{\Gamma(67/132)} = 1.98535339227 \dots, \\
 D_{68} &= \frac{2^{68/67}}{67} A_{68} = \frac{1}{67} \left\{ \pi^{68/2} \frac{\Gamma(33+1/2)}{\Gamma(34)} \right\}^{1/67} \frac{\Gamma(1/134)}{\Gamma(68/134)} \\
 &= \frac{1}{67} \frac{\pi^{69/134}}{2^{33/67}} \left(\frac{65!!}{33!} \right)^{1/67} \frac{\Gamma(1/134)}{\Gamma(68/134)} = 1.98534941481 \dots, \\
 D_{69} &= \frac{2^{69/68}}{68} A_{69} = \frac{1}{68} \left\{ \pi^{69/2} \frac{\Gamma(34)}{\Gamma(34+1/2)} \right\}^{1/68} \frac{\Gamma(1/136)}{\Gamma(69/136)} \\
 &= \frac{\sqrt{2\pi}}{68} \left(\frac{33!}{67!!} \right)^{1/68} \frac{\Gamma(1/136)}{\Gamma(69/136)} = 1.98534877792 \dots, \\
 D_{70} &= \frac{2^{70/69}}{69} A_{70} = \frac{1}{69} \left\{ \pi^{70/2} \frac{\Gamma(34+1/2)}{\Gamma(35)} \right\}^{1/69} \frac{\Gamma(1/138)}{\Gamma(70/138)} \\
 &= \frac{1}{69} \frac{\pi^{71/138}}{2^{34/69}} \left(\frac{67!!}{34!} \right)^{1/69} \frac{\Gamma(1/138)}{\Gamma(70/138)} = 1.98535124396 \dots.
 \end{aligned}$$

These values, together with the conclusion just proved above, yield the piecewise monotonicity of D_n .

Clearly, $D_2 = \pi^2$. By (3.22) and (1.23), $\lim_{n \rightarrow \infty} D_n = 2$. Hence (1.24) holds. $\square$

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