TAME KERNELS OF CUBIC CYCLIC FIELDS

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ABSTRACT. There are many results describing the structure of the tame kernels of algebraic number fields and relating them to the class numbers of appropriate fields. In the present paper we give some explicit results on tame kernels of cubic cyclic fields. Table 1 collects the results of computations of the structure of the tame kernel for all cubic fields with only one ramified prime p , $7 \leq p < 5,000$.

In particular, we investigate the structure of the 7-primary and 13-primary parts of the tame kernels. The theoretical tools we develop, based on reflection theorems and singular primary units, enable the determination of the structure even of 7-primary and 13-primary parts of the tame kernels for all fields as above. The results are given in Tables 2 and 3.

1. INTRODUCTION

The tame kernels of number fields have been investigated by many authors (see, e.g., [Br1], [Br2], [CH], [Gr], [HK], [Ke], [Ko1]–[Ko4], [Q1]–[Q4], [Ta], and the references therein). In the present paper we give some explicit results for cubic cyclic fields with exactly one ramified prime. Let F be a cubic cyclic field. Basic information on such fields is given, e.g., in [Ha]. We assume, for simplicity, that in F there is only one ramified prime p , $p > 7$. Then $p \equiv 1 \pmod{6}$ and F is the unique cubic subfield of the cyclotomic field $\mathbb{Q}(\zeta_p)$.

We investigate the tame kernel $K_2\mathcal{O}_F$, where $\mathcal{O}_F$ is the ring of integers of F .

The order of the group $K_2\mathcal{O}_F$ is easy to compute using appropriate generalized Bernoulli numbers. First, we discuss the order $\#K_2\mathcal{O}_F$ and its divisibility by small primes. To get information on the structure of the group $K_2\mathcal{O}_F$ we investigate its q -rank for some prime numbers q . For this purpose we develop an appropriate machinery (relations between tame kernels, class groups and groups of singular primary units of appropriate subfields of $F(\zeta_q)$).

We attach tables containing the results of numerical computations of tame kernels and class groups of fields F in question. Our table corrects some data given in Table 1 in [KK]. The computations have been performed using the GP/PARI package, version 2.1.0, and therefore some of the results are guaranteed to be correct only under the assumption of GRH.

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2. ORDERS OF TAME KERNELS

2.1. Basic information on the field F . Let $p > 7$, $p \equiv 1 \pmod{6}$ be a prime number. Let F be the unique cubic subfield of the cyclotomic field $\mathbb{Q}(\zeta_p)$. C.F. Gauss described the field F as follows (see, e.g., [ST]). Let g be a primitive root modulo p . Then the subgroup $H = \langle g^3 \rangle$ of the group $(\mathbb{Z}/p)^* = \langle g \rangle$ has index 3, and there are two nontrivial cosets gH and g^2H . Let

$$\alpha_1 = \sum_{j \in H} \zeta_p^j, \quad \alpha_2 = \sum_{j \in gH} \zeta_p^j, \quad \alpha_3 = \sum_{j \in g^2H} \zeta_p^j$$

be the Gauss periods. Then $F = \mathbb{Q}(\alpha_j)$, $j = 1, 2, 3$, $\alpha_1, \alpha_2, \alpha_3$ are conjugate in F , and the discriminant is p^2 .

It is known that there are unique $A, B \in \mathbb{Z}$ such that

$$(2.1) \quad 4p = A^2 + 27B^2, \quad A \equiv 1 \pmod{3}, \quad B > 0.$$

In the next lemma we collect basic information on p and A .

Lemma 2.1. *If A, B are as in (2.1), then*

- (i) $A + p + 1 \equiv 0 \pmod{9}$,
- (ii) $A + p + 1 \equiv (p - 1)^2 \pmod{27}$,
- (iii) $p(A + 3) \equiv 1 \pmod{27}$.

Proof. This follows easily from (2.1) and $p \equiv 1 \pmod{6}$. □

The minimal polynomial for α_j is

$$f(X) = X^3 + X^2 - \frac{p-1}{3}X - \frac{p(A+3)-1}{27} \in \mathbb{Z}[X].$$

Substituting $X \mapsto \frac{1}{3}(X-1)$ we get another polynomial with the same splitting field:

$$(2.2) \quad g(X) = X^3 - 3pX - Ap,$$

with roots $\beta_j := 3\alpha_j + 1$, $j = 1, 2, 3$, and discriminant $\Delta_g = 27^2 B^2 p^2$.

Now we state some known facts on the class group of the field F . Let $T = \text{Gal}(F/\mathbb{Q})$, and let τ be a generator of T satisfying $\tau(\zeta_p) = \zeta_p^g$.

Theorem 2.2. *For a prime number q let A_q be the Sylow q -subgroup of the class group $Cl(\mathcal{O}_F)$ of F . Then*

- (i) $A_3 = 1$.
- (ii) *If $q \equiv 2 \pmod{3}$, then $A_q = B_q \times \tau(B_q)$, for some subgroup B_q of A_q .*

The same holds if we replace $\mathcal{O}_F$ by the ring $\mathcal{O}_{F,2} = \mathcal{O}_F[\frac{1}{2}]$ of integers of F localized at 2.

Proof. (i) and (ii) follow from [Wa, Theorems 10.4 (b), and 10.8]. The proof of Theorem 10.8 in [Wa] can be easily adapted to get the last statement. □

2.2. Orders of tame kernels. For every abelian totally real field F the order of the group $K_2\mathcal{O}_F$ is known (see [MW] and [Wi]):

$$\#K_2\mathcal{O}_F = w_2(F) |\zeta_F(-1)|,$$

where ζ_F is the Dedekind zeta function of the field F , and $w_2(F)$ is the maximal order of a root of unity belonging to the compositum of all quadratic extensions of F .

For every cubic cyclic field F we have $w_2(F) = 24$, with two exceptions:

$$\begin{aligned} w_2(F) &= 3 \cdot 24 & \text{for } F = \mathbb{Q}(\zeta_9)^+, \\ w_2(F) &= 7 \cdot 24 & \text{for } F = \mathbb{Q}(\zeta_7)^+, \end{aligned}$$

where $\zeta_n = \exp(2\pi i/n)$ is a primitive n -th root of unity, and E^+ denotes the maximal real subfield of a number field E .

Now, let us return to cubic cyclic fields F with only one ramified prime $p > 7$. There are at least two ways to compute $\zeta_F(-1)$.

1) The Dedekind zeta function $\zeta_F(s)$ of the field F can be defined by the Euler product

$$(2.3) \quad \zeta_F(s) = \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{q \text{ splits}} \left(1 - \frac{1}{q^s}\right)^{-3} \prod_{q \text{ is inert}} \left(1 - \frac{1}{q^{3s}}\right)^{-1}.$$

By the functional equation we get

$$\zeta_F(-1) = - \left(\frac{p}{2\pi^2}\right)^3 \zeta_F(2).$$

Consequently,

$$\#K_2\mathcal{O}_F = \frac{3p^3}{\pi^6} \zeta_F(2).$$

From (2.3) it follows that $1 < \zeta_F(2) < \zeta(2)^3$, where $\zeta(s)$ is the Riemann zeta function; hence

$$\frac{3}{\pi^6} p^3 < \#K_2\mathcal{O}_F < \frac{1}{72} p^3$$

or, numerically,

$$(2.4) \quad 0.003120 p^3 < \#K_2\mathcal{O}_F < 0.01389 p^3.$$

2) Now we describe the second method. The generalized Bernoulli number $B_{n,\chi}$ corresponding to a Dirichlet character χ of conductor f is defined by

$$B_{n,\chi} = f^{n-1} \sum_{j=1}^f \chi(j) B_n \left(\frac{j}{f}\right),$$

where $B_n(X)$ is the n -th Bernoulli polynomial. Hence for the complex conjugate character $\bar{\chi}$ we have $B_{n,\bar{\chi}} = \overline{B_{n,\chi}}$.

The Dedekind zeta function of an abelian number field F is the product of L -series:

$$\zeta_F(s) = \prod_{\chi} L(s, \chi),$$

where χ runs over the linear characters of the Galois group $\text{Gal}(F/\mathbb{Q})$.

In our case there are two nontrivial cubic Dirichlet characters χ and $\overline{\chi}$, where

$$\chi(a) = \begin{cases} \zeta_3^k, & \text{if } a \pmod{p} \in g^k H, \ k = 0, 1, 2, \\ 0, & \text{if } p|a. \end{cases}$$

Hence

$$\zeta_F(s) = \zeta(s) L(s, \chi) L(s, \overline{\chi}).$$

Applying the formula (see [Wa, Theorem 4.2])

$$L(-1, \chi) = -\frac{B_{2, \chi}}{2}$$

and $\zeta(-1) = -\frac{1}{12}$ we get

$$\zeta_F(-1) = \zeta(-1) \frac{B_{2, \chi} \cdot B_{2, \overline{\chi}}}{4} = -\frac{1}{48} |B_{2, \chi}|^2.$$

Hence

$$\#K_2\mathcal{O}_F = \frac{1}{2} |B_{2, \chi}|^2.$$

It is easy to compute $B_{2, \chi}$ (see [Wa, Exercise 4.2 (a)]):

$$B_{2, \chi} = \frac{1}{p} \sum_{j=1}^{p-1} \chi(j) j^2.$$

For $k = 0, 1, 2$, define $T_k := \{j : 1 \leq j \leq p-1, j \pmod{p} \in g^k H\}$ and

$$S_k := \frac{1}{p} \sum_{j \in T_k} j^2.$$

Since $j \in T_k$ iff $j \equiv g^{3r+k} \pmod{p}$ for some r , $0 \leq r < (p-1)/3$, we get

$$\sum_{j \in T_k} j^2 \equiv g^{2k} \sum_{r=0}^{(p-1)/3} g^{6r} = g^{2k} \frac{g^{2(p-1)} - 1}{g^6 - 1} \equiv 0 \pmod{p},$$

so the S_k are integers. Moreover,

$$S_0 + S_1 + S_2 = \frac{1}{p} \sum_{j=1}^{p-1} j^2 = \frac{p-1}{6} \cdot (2p-1).$$

Now,

$$B_{2, \chi} = \frac{1}{p} \sum_{j=1}^{p-1} \chi(j) j^2 = S_0 + \zeta_3 S_1 + \zeta_3^2 S_2 = S_0 - \frac{1}{2}(S_1 + S_2) + \frac{\sqrt{3}}{2}i(S_1 - S_2).$$

Consequently,

$$\begin{aligned} \#K_2\mathcal{O}_F &= \frac{1}{2} |B_{2, \chi}|^2 = \frac{1}{2} \left((S_0 - \frac{1}{2}(S_1 + S_2))^2 + \frac{3}{4}(S_1 - S_2)^2 \right) \\ &= \frac{1}{4} ((S_0 - S_1)^2 + (S_1 - S_2)^2 + (S_2 - S_0)^2) \\ (2.5) \quad &= \frac{1}{2} ((S_0 + S_1 + S_2)^2 - 3(S_0 S_1 + S_1 S_2 + S_2 S_0)) \\ &= \frac{1}{2} \left(\left(\frac{p-1}{6} \cdot (2p-1) \right)^2 - 3(S_0 S_1 + S_1 S_2 + S_2 S_0) \right). \end{aligned}$$

2.3. Results on the order of $K_2\mathcal{O}_F$.

Theorem 2.3.

$$\lim_{p \rightarrow \infty} \#K_2\mathcal{O}_F = \infty.$$

Proof. This follows from (2.4). $\square$

Theorem 2.4. *Let $v_p(m)$ be the p -adic valuation of m . Then*

- (i) $v_2(\#K_2\mathcal{O}_F)$ is odd.
- (ii) For every prime number $q \equiv -1 \pmod{6}$, the number $v_q(\#K_2\mathcal{O}_F)$ is even. Moreover, $q \mid \#K_2\mathcal{O}_F$ iff $S_0 \equiv S_1 \equiv S_2 \pmod{q}$.
- (iii) $3 \mid \#K_2\mathcal{O}_F$ iff $p \equiv 1 \pmod{18}$.
- (iv) If $p \equiv 1 \pmod{18}$, then

$$3 \parallel \#K_2\mathcal{O}_F \quad \text{iff} \quad S_1, S_2, S_3 \text{ are distinct modulo } 3.$$

In this case we have $\#K_2\mathcal{O}_F \equiv 6 \pmod{9}$. Moreover,

$$3^2 \mid \#K_2\mathcal{O}_F \quad \text{iff} \quad S_0 \equiv S_1 \equiv S_2 \pmod{3}.$$

Proof. In (2.5) put $U = S_0 - S_1$, $V = S_1 - S_2$; then $-U - V = S_2 - S_0$. Consequently,

$$(2.6) \quad \#K_2\mathcal{O}_F = \frac{1}{2}(U^2 + UV + V^2).$$

For every prime number $q \equiv 2 \pmod{3}$ and $U, V \in \mathbb{Z}$ we have $q \mid U^2 + UV + V^2$ iff $q \mid U$ and $q \mid V$. Hence $v_q(U^2 + UV + V^2)$ is even. Then from (2.6) we get (i) and (ii).

To prove (iii) we apply the last formula of (2.5). From $p \equiv 1 \pmod{6}$, it follows that $2p - 1 \equiv 1 \not\equiv 0 \pmod{3}$. Hence $3 \mid \#K_2\mathcal{O}_F$ iff $3 \mid \frac{p-1}{6}$, i.e., iff $p \equiv 1 \pmod{18}$.

For $U, V \in \mathbb{Z}$ we have $3 \parallel U^2 + UV + V^2$ iff $U \equiv V \not\equiv 0 \pmod{3}$, and $9 \mid U^2 + UV + V^2$ iff $U \equiv V \equiv 0 \pmod{3}$. Moreover, if $U \equiv V \equiv \pm 1 \pmod{3}$, then $U^2 + UV + V^2 \equiv 3 \pmod{9}$.

Therefore (iv) follows from (2.6). $\square$

Remark. Part (iii) of Theorem 2.4 follows also from a more general result of M. Kolster and A. Movahhedi ([KM], Example 2 on pp. 51–52), since a prime $p \equiv 1 \pmod{6}$ is inert in the cyclotomic $\mathbb{Z}_3$ -extension of $\mathbb{Q}$ iff $p \not\equiv 1 \pmod{18}$.

3. THE 2-PRIMARY PART OF THE TAME KERNEL

3.1. The 2-rank of $K_2\mathcal{O}_F$. We apply the general formula for the 2-rank of $K_2\mathcal{O}_F$, for an arbitrary number field F (see [Ta]):

$$(3.1) \quad 2\text{-rank } K_2\mathcal{O}_F = r_1 + g_2 - 1 + 2\text{-rank } Cl(\mathcal{O}_{F,2}),$$

where r_1 (resp. g_2) is the number of real (resp. dyadic) places of F .

For a cubic cyclic field F we have

$$g_2 = \begin{cases} 1, & \text{if } 2 \text{ is inert in } F, \\ 3, & \text{if } 2 \text{ splits in } F. \end{cases}$$

Then the formula (3.1) takes the form

$$(3.2) \quad 2\text{-rank } K_2\mathcal{O}_F = 2\text{-rank } Cl(\mathcal{O}_{F,2}) + \begin{cases} 3, & \text{if } 2 \text{ is inert in } F, \\ 5, & \text{if } 2 \text{ splits in } F. \end{cases}$$

The following lemma gives conditions equivalent to the splitting of 2 in the cubic cyclic field F defined in Section 2.1.

Lemma 3.1. *The following conditions are equivalent:*

- (i) 2 splits in F ,
- (ii) A is even,
- (iii) $p = X^2 + 27Y^2$ for some $X, Y \in \mathbb{Z}$,
- (iv) 2 is a cubic residue modulo p .

Proof. The polynomial defined by (2.2) satisfies $g(X) \equiv X^3 + X + A \pmod{2}$. Hence by the Hensel lemma $g(X)$ splits in $\mathbb{Q}_2[X]$ iff A is even iff 2 splits in F .

The equivalence of (ii) and (iii) is obvious; for the equivalence of (iii) and (iv) see [IR, Proposition 9.6.2]. $\square$

Lemma 3.2. *We have*

$$2\text{-rank } K_2\mathcal{O}_F = 2\text{-rank } Cl(\mathcal{O}_{F,2}) + \begin{cases} 3, & \text{if } A \text{ is odd,} \\ 5, & \text{if } A \text{ is even.} \end{cases}$$

Moreover, the 2-rank of $K_2\mathcal{O}_F$ is odd.

Proof. The first assertion follows from (3.2) and Lemma 3.1. To prove the second it is sufficient to apply Theorem 2.2 to (3.2). $\square$

Lemma 3.3. *Let $W_2(F)$ be the wild kernel of the field F . Then*

$$K_2\mathcal{O}_F/W_2(F) = (\mathbb{Z}/2)^r,$$

where

$$r = \begin{cases} 3, & \text{if } 2 \text{ is inert in } F, \\ 5, & \text{if } 2 \text{ splits in } F. \end{cases}$$

Proof. This follows from the description of the group $K_2\mathcal{O}_F/W_2(F)$ given in [Br1], Theorem 2. $\square$

From the above lemmas we get immediately

Corollary 3.4. *For every cubic cyclic field F as above we have*

$$2\text{-rank } K_2\mathcal{O}_F = 2\text{-rank } Cl(\mathcal{O}_{F,2}) + 2\text{-rank } (K_2\mathcal{O}_F/W_2(F)).$$

3.2. Elements of order 2 in $K_2\mathcal{O}_F$. Elements of order 2 in $K_2\mathcal{O}_F$ can be determined explicitly as follows. By a theorem of Tate ([Ta, Theorem 6.3]), the group $\Delta_F := \{a \in F^* : \{-1, a\} = 1\}$ satisfies $2\text{-rank}(\Delta_F/F^{*2}) = 1$; hence $\Delta_F = F^{*2} \cup 2F^{*2}$.

Let $\varepsilon_1, \varepsilon_2$ be fundamental units of F . Changing sign, if necessary, we may assume that $N\varepsilon_1 = 1$ and $\varepsilon_2 = \tau(\varepsilon_1)$, where τ is a generator of the Galois group $T = \text{Gal}(F/\mathbb{Q})$. Then $\{-1, -1\}, \{-1, \varepsilon_1\}, \{-1, \tau(\varepsilon_1)\} \in K_2\mathcal{O}_F$.

Additional elements of order 2 in $K_2\mathcal{O}_F$ can be determined as follows. In view of the last statement of Theorem 2.2 we can take independent generators of the group ${}_2Cl(\mathcal{O}_{F,2})$ of the form $Cl(\mathfrak{p}_j), Cl(\tau(\mathfrak{p}_j))$, $j = 1, \dots, t$, where $2t = 2\text{-rank } Cl(\mathcal{O}_{F,2})$, and $\mathfrak{p}_j$ are prime ideals satisfying $\mathfrak{p}_j \nmid 2$. Then the ideals $\mathfrak{p}_j^2$ are principal, $\mathfrak{p}_j^2 = (\gamma_j)$, for $j = 1, \dots, t$. We may assume that $N\gamma_j > 0$. Then $N\gamma_j = N\mathfrak{p}_j^2 = (N\mathfrak{p}_j)^2 \in F^{*2}$.

It follows that $\{-1, \gamma_j\}, \{-1, \tau(\gamma_j)\} \in K_2\mathcal{O}_F$ for $j = 1, \dots, t$.

Finally, if 2 splits in F , $(2) = \mathfrak{p} \cdot \tau(\mathfrak{p}) \cdot \tau^2(\mathfrak{p})$, and the class $Cl(\mathfrak{p})$ in $Cl(\mathcal{O}_F)$ has order r , then the ideal $\mathfrak{p}^r$ is principal, $\mathfrak{p}^r = (\gamma)$ and $N\gamma = N(\mathfrak{p}^r) = 2^r$. Consequently, $\{-1, \gamma\}, \{-1, \tau(\gamma)\} \in K_2\mathcal{O}_F$.

Consider the elements

$$(3.3) \quad -1, \varepsilon_1, \tau(\varepsilon_1), \gamma_1, \tau(\gamma_1), \dots, \gamma_t, \tau(\gamma_t)$$

if 2 is inert in F , and

$$(3.4) \quad -1, \varepsilon_1, \tau(\varepsilon_1), \gamma, \tau(\gamma), \gamma_1, \tau(\gamma_1), \dots, \gamma_t, \tau(\gamma_t)$$

if 2 splits in F . Then the elements (3.3), respectively (3.4), are multiplicatively independent modulo $\Delta_F = F^{*2} \cup 2F^{*2}$. It follows that the elements

$$\{-1, -1\}, \{-1, \varepsilon_1\}, \{-1, \tau(\varepsilon_1)\}, \{-1, \gamma_j\}, \{-1, \tau(\gamma_j)\},$$

where $j = 1, \dots, t$, are independent generators of the group ${}_2K_2\mathcal{O}_F$ if 2 is inert in F . Similarly, the same elements and $\{-1, \gamma\}, \{-1, \tau(\gamma)\}$ are independent generators of ${}_2K_2\mathcal{O}_F$ if 2 splits in F .

We have

$$\text{2-rank } Cl(\mathcal{O}_F) = \text{2-rank } Cl(\mathcal{O}_{F,2}),$$

unless 2 splits in F , $(2) = \mathfrak{p} \cdot \tau(\mathfrak{p}) \cdot \tau^2(\mathfrak{p})$, and the class $Cl(\mathfrak{p})$ generates in $Cl(\mathcal{O}_F)$ a direct summand of an even order.

Hence $\text{2-rank } Cl(\mathcal{O}_{F,2}) = \text{2-rank } Cl(\mathcal{O}_F) - 2$ in the latter case, since in view of Theorem 2.2 both 2-ranks are even.

Theorem 3.5. *For $r \geq 2$ the 2^r -rank of $K_2\mathcal{O}_F$ is even.*

Proof. Let $c \in K_2\mathcal{O}_F$ be an element of order 2^r , $r \geq 2$. Then $b := c^{2^{r-1}}$ is a nontrivial element of ${}_2K_2\mathcal{O}_F$. Consequently, $b = \{-1, a\}$, where a is the product of some elements (3.3), respectively (3.4),

$$(3.5) \quad a = (-1)^{\alpha_0} \cdot \varepsilon_1^{\alpha_1} \cdot \tau(\varepsilon_1)^{\alpha'_1} \cdot \gamma^{\alpha_2} \cdot \tau(\gamma)^{\alpha'_2} \cdot \prod_{j=1}^t \gamma_j^{\beta_j} \cdot \tau(\gamma_j)^{\beta'_j},$$

where the exponents $\alpha_0, \alpha_1, \alpha'_1, \alpha_2, \alpha'_2, \beta_j, \beta'_j$ are 0 or 1.

Then also $\tau(c) \in K_2\mathcal{O}_F$ has order 2^r . It is sufficient to prove that $\tau(b) \neq b$, or equivalently, that $\tau(a)$ and a are multiplicatively independent modulo Δ_F .

We have proved above that the norms $N\varepsilon_1, N\gamma, N\gamma_j$ belong to Δ_F . Consequently, $\tau^2(\xi) \equiv (\xi \cdot \tau(\xi))^{-1} \pmod{\Delta_F}$, where $\xi = \varepsilon_1, \gamma$ or γ_j . Hence from (3.5) we get

$$(3.6) \quad \begin{aligned} \tau(a) &\equiv (-1)^{\alpha_0} \cdot \tau(\varepsilon_1)^{\alpha_1} \cdot (\varepsilon_1 \tau(\varepsilon_1))^{-\alpha'_1} \cdot \tau(\gamma)^{\alpha_2} \cdot (\gamma \tau(\gamma))^{-\alpha'_2} \\ &\cdot \prod_{j=1}^t \left(\tau(\gamma_j)^{\beta_j} \cdot (\gamma_j \tau(\gamma_j))^{-\beta'_j} \right) \pmod{\Delta_F}. \end{aligned}$$

Assume that $\tau(a) \equiv a \pmod{\Delta_F}$. In view of the multiplicative independence of the elements (3.3) and (3.4) modulo Δ_F , they should appear in (3.5) and in (3.6) with exponents of the same parity.

Therefore,

$$\alpha_1 \equiv -\alpha'_1 \pmod{2}, \quad \alpha'_1 \equiv \alpha_1 - \alpha'_1 \pmod{2};$$

hence $\alpha_1 \equiv \alpha'_1 \equiv 0 \pmod{2}$. In a similar way we get $\alpha_2 \equiv \alpha'_2 \equiv 0 \pmod{2}$, $\beta_j \equiv \beta'_j \equiv 0 \pmod{2}$, i.e., $a \in \Delta_F$, $b = 1$. We have a contradiction since b is an element of order 2. $\square$

Corollary 3.6. *We have*

$$4\text{-rank } K_2\mathcal{O}_F \leq 2\text{-rank } K_2\mathcal{O}_F - 1,$$

and if an element ε_1, γ or some γ_j in (3.3), respectively in (3.4), is not totally positive, then

$$4\text{-rank } K_2\mathcal{O}_F \leq 2\text{-rank } K_2\mathcal{O}_F - 3.$$

Proof. The first inequality follows from the fact that $4\text{-rank } K_2\mathcal{O}_F \leq 2\text{-rank } K_2\mathcal{O}_F$ and the two numbers are of different parities.

If an element $\alpha \in F^*$ is not totally positive, then applying the three real Hilbert symbols of F to $\{-1, \alpha\}$ we see that $\{-1, \alpha\}$ is not a square in K_2F . In particular, $\{-1, -1\}$ is not a square. It follows that if ε_1, γ or some γ_j is not totally positive, then $4\text{-rank } K_2\mathcal{O}_F \leq 2\text{-rank } K_2\mathcal{O}_F - 2$, and the corollary follows since the two ranks are of different parities. $\square$

4. THE q -RANK OF THE TAME KERNEL FOR q ODD

4.1. Notation. Let q be an odd prime number, ζ_q a primitive q -th root of unity, and $G := \text{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q})$. Then

$$G = \{\sigma_a : 1 \leq a \leq q-1\},$$

where $\sigma_a(\zeta_q) = \zeta_q^a$. The mapping $(\mathbb{Z}/q)^* \rightarrow G$, $a \mapsto \sigma_a$ is an isomorphism. For a fixed primitive root h modulo q the automorphism $\sigma := \sigma_h$ generates G .

Let ω be the q -adic Teichmüller character of the group $(\mathbb{Z}/q)^*$. Then, for $1 \leq a \leq q-1$, the value $\omega(a) \in \mathbb{Z}_q^*$ is determined uniquely by the conditions $\omega(a)^{q-1} = 1$ and $\omega(a) \equiv a \pmod{q}$. All irreducible characters of $G = (\mathbb{Z}/q)^*$ are ω^j , $0 \leq j \leq q-2$. The corresponding primitive idempotents of the group ring $\mathbb{Z}_q[G]$ are

$$(4.1) \quad \varepsilon_j = \frac{1}{q-1} \sum_{a=1}^{q-1} \omega(a)^j \sigma_a^{-1} = \frac{1}{q-1} \sum_{k=0}^{q-2} \omega(h)^{kj} \sigma^{-k}, \quad 0 \leq j \leq q-2.$$

In particular, $\varepsilon_0 = \frac{1}{q-1} N$, where $N = 1 + \sigma + \sigma^2 + \dots + \sigma^{q-2} = N_{\mathbb{Q}(\zeta_q)/\mathbb{Q}}$ is the norm element in the group ring $\mathbb{Z}_q[G]$.

Since $\sigma_a \varepsilon_j = \omega(a)^j \varepsilon_j = \varepsilon_j \sigma_a$ in $\mathbb{Z}_q[G]$, for a $\mathbb{Z}_q[G]$ -module M we have

$$\varepsilon_j M = \{m \in M : \sigma_a(m) = \omega(a)^j m\}$$

and we get a decomposition of M into a direct sum of $\mathbb{Z}_q[G]$ -submodules:

$$M = \bigoplus_{j=0}^{q-2} \varepsilon_j M = NM \oplus \bigoplus_{j=1}^{q-2} \varepsilon_j M.$$

The group μ_q of q -th roots of unity has the natural structure of a $\mathbb{Z}_q[G]$ -module. We define the action of G on $\mu_q \otimes M$ by

$$(\zeta \otimes m)^\tau = \zeta^\tau \otimes m^\tau, \quad \text{where } \zeta \in \mu_q, m \in M, \tau \in G.$$

Obviously,

$$(4.2) \quad (\mu_q \otimes M)^G = \varepsilon_0(\mu_q \otimes M).$$

Since $\omega(a) \equiv a \pmod{q}$, we get $\zeta^{\omega(a)} = \zeta^a$, so

$$\sigma_a(\zeta \otimes m) = \zeta^a \otimes \sigma_a(m) = \omega(a)(\zeta \otimes \sigma_a(m)).$$

Consequently, in view of (4.1),

$$\begin{aligned} \varepsilon_{j+1}(\zeta \otimes m) &= \frac{1}{q-1} \sum_{a=1}^{q-1} \omega(a)^{j+1} \sigma_a^{-1}(\zeta \otimes m) = \frac{1}{q-1} \sum_{a=1}^{q-1} \omega(a)^j \sigma_a^{-1}(\zeta^a \otimes m) \\ &= \frac{1}{q-1} \sum_{a=1}^{q-1} \omega(a)^j (\zeta \otimes \sigma_a^{-1}(m)) = \zeta \otimes \varepsilon_j m. \end{aligned}$$

Hence $\varepsilon_{j+1}(\mu_q \otimes M) = \mu_q \otimes \varepsilon_j M$ for every G -module M . In particular,

$$(4.3) \quad \varepsilon_0(\mu_q \otimes M) = \mu_q \otimes \varepsilon_{q-2} M,$$

since $\varepsilon_{q-1} = \varepsilon_0$.

4.2. The q -rank of $K_2\mathcal{O}_F$. Let $E = F(\zeta_q)$, where F is the cubic cyclic field defined above, and $q \neq p$. Denote by $\lambda : Cl(\mathcal{O}_E) \rightarrow Cl(\mathcal{O}_E[1/q])$ the homomorphism of the class groups induced by the imbedding $\mathcal{O}_E \rightarrow \mathcal{O}_E[1/q]$, and let $A = A_E$ be the Sylow q -subgroup of $Cl(\mathcal{O}_E)$. Then $\lambda(A)$ is the Sylow q -subgroup of $Cl(\mathcal{O}_E[1/q])$ by the surjectivity of λ .

Since A is a q -group on which $G = \text{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q}) = \text{Gal}(E/F)$ acts, we have

$$A = \bigoplus_{j=0}^{q-2} \varepsilon_j A.$$

Lemma 4.1. *For $j \neq 0$ the mapping $\lambda : \varepsilon_j A \rightarrow \varepsilon_j \lambda(A)$ is an isomorphism.*

Proof. Since λ commutes with the action of G , we have $\lambda(\varepsilon_j A) = \varepsilon_j \lambda(A) = \varepsilon_j A$, i.e., λ is surjective.

Moreover, the group $\ker \lambda$ is generated by the classes containing prime ideals $\mathfrak{Q}$ of E that divide q . Since q does not ramify in F , and prime divisors $\mathfrak{q}$ of q in F totally ramify in E , it follows that $\sigma(\mathfrak{Q}) = \mathfrak{Q}$ for every $\sigma \in G$.

Consequently, $\ker \lambda \cap A \subset A^G = \varepsilon_0 A$. Therefore $\ker \lambda \cap \varepsilon_j A = 0$ for $j \neq 0$, and the lemma follows. $\square$

Let S' be the set of prime ideals of F dividing q that split completely in E . For cubic cyclic fields the set S' is empty as follows from the next more general lemma.

Lemma 4.2. *If F is any Galois extension of $\mathbb{Q}$ of odd degree d and q is an odd prime number, then the set S' of ideals of F that divide q and split completely in $E = F(\zeta_q)$ is empty.*

Proof. By assumptions E is a Galois extension of $\mathbb{Q}$. For fields K, L , where $\mathbb{Q} \subset K \subset L \subset E$, denote by $e_q(L/K)$ the ramification degree in the extension L/K of a prime ideal of K that divides q . Then

$$q-1 = e_q(\mathbb{Q}(\zeta_q)/\mathbb{Q}) | e_q(E/\mathbb{Q}) = e_q(F/\mathbb{Q}) \cdot e_q(E/F).$$

Since $e_q(F/\mathbb{Q}) | (F:\mathbb{Q}) = d$, where d is odd, and $2 | q-1$, it follows that $2 | e_q(E/F)$. Hence every prime ideal of F dividing q ramifies in E , so it cannot split completely in E . $\square$

Theorem 4.3. *Let $E = F(\zeta_q)$, where F is the cubic cyclic field defined above. Then*

$$q\text{-rank } K_2\mathcal{O}_F = q\text{-rank } \varepsilon_{q-2}A_E.$$

Proof. There is an exact sequence

$$(4.4) \quad 0 \longrightarrow (\mu_q \otimes Cl(\mathcal{O}_E[1/q]))^G \longrightarrow K_2\mathcal{O}_F/q \longrightarrow \mu_q^{S'} \longrightarrow 0$$

(see [Ke, Theorem 5.4], and [Ge]). Therefore by (4.2), (4.3) and Lemma 4.1 we get

$$(4.5) \quad \begin{aligned} (\mu_q \otimes Cl(\mathcal{O}_E[1/q]))^G &= (\mu_q \otimes \lambda(A))^G = \varepsilon_0(\mu_q \otimes \lambda(A)) \\ &= \mu_q \otimes \varepsilon_{q-2}\lambda(A) = \mu_q \otimes \varepsilon_{q-2}A. \end{aligned}$$

Since all terms in (4.4) are elementary abelian q -groups, (4.5) yields

$$q\text{-rank}(K_2\mathcal{O}_F/q) = q\text{-rank}(\mu_q \otimes \varepsilon_{q-2}A) + |S'|.$$

Tensoring by μ_q and dividing by q do not change the q -rank, and $S' = \emptyset$ by Lemma 4.2. Hence

$$q\text{-rank } K_2\mathcal{O}_F = q\text{-rank}(\varepsilon_{q-2}A). \quad \square$$

Let $E' = \mathbb{Q}(\zeta_q)$, and denote by A' the Sylow q -subgroup of $Cl(\mathcal{O}_{E'}) = Cl(\mathbb{Z}[\zeta_q])$. Then $A' = 1$ iff q is a regular prime.

By the theorems of Herbrand and Ribet (see [IR, Chapter 15, §3]) for $q > 2$ we have $\varepsilon_{q-2}A' \neq 1$ iff q divides the numerator of the Bernoulli number $B_2 = \frac{1}{6}$. Hence $\varepsilon_{q-2}A' = 1$ for every odd prime number q .

Theorem 4.4. *If $q \equiv -1 \pmod{6}$ is a prime, and $\varepsilon_j A' = 1$ for some j , $0 \leq j \leq q-2$, then the q -rank of $\varepsilon_j A$ is even. In particular, the q -rank of $\varepsilon_{q-2}A$ is even.*

Proof. (cf. [Wa, Proof of Theorem 10.8]). Let τ be a generator of the Galois group $T := \text{Gal}(F/\mathbb{Q}) = \text{Gal}(E/\mathbb{Q}(\zeta_q))$. Since $q \neq p$, it follows that σ and τ commute, and consequently T acts on the group $\varepsilon_j A$ for $0 \leq j \leq q-2$.

If $a \in \varepsilon_j A$ is fixed by τ , then

$$N_{E/\mathbb{Q}(\zeta_q)}(a) = a \cdot a^\tau \cdot a^{\tau^2} = a^3 \quad \text{and} \quad N_{E/\mathbb{Q}(\zeta_q)}(\varepsilon_j A) \subseteq \varepsilon_j A' = 1,$$

by assumption. Hence $a^3 = 1$, and so $a = 1$. Thus every nontrivial orbit in $\varepsilon_j A$ under the action of T has three elements. The same holds for orbits contained in the maximal elementary abelian q -subgroup B_j of $\varepsilon_j A$. We have $r_j := q\text{-rank } B_j = q\text{-rank } \varepsilon_j A$. Hence $\#B_j = q^{r_j} \equiv 1 \pmod{3}$. Then from $q \equiv -1 \pmod{3}$ it follows that r_j is even.

The last assertion follows from $\varepsilon_{q-2}A' = 1$. $\square$

Corollary 4.5. *If $q \equiv 5 \pmod{6}$ is a prime, then the q -rank of $K_2\mathcal{O}_F$ is even.*

By analogy with the class group of F (see Theorem 2.3 (ii)) we state the following conjecture.

Conjecture 4.6. *If $q \equiv 5 \pmod{6}$ is a prime, then*

$$\text{Syl}_q(K_2\mathcal{O}_F) = A' \times \tau(A')$$

for some subgroup A' of the Sylow q -subgroup of $K_2\mathcal{O}_F$.

4.3. The case $q = 3$. If $q = 3$, we can give a more precise description of the fields in question. From

$$4p = A^2 + 27B^2$$

it follows that in the field $\mathbb{Q}(\zeta_3)$ the prime p splits into two principal ideals, $(p) = (\pi)(\bar{\pi})$, where

$$\pi = \frac{1}{2}(A + 3B\sqrt{-3}) = \frac{1}{2}(A + 3B) + 3B\zeta_3,$$

and $\bar{\pi}$ is the complex conjugate of π .

Moreover,

$$(4.6) \quad E = F(\zeta_3) = \mathbb{Q}(\zeta_3, \Pi), \quad \text{where} \quad \Pi = \sqrt[3]{\pi/\bar{\pi}}.$$

In fact, one can easily verify that

$$\beta := \pi\Pi + \bar{\pi}/\Pi = \sqrt[3]{p\pi} + \sqrt[3]{p\bar{\pi}} \in \mathbb{Q}(\zeta_3, \Pi)$$

is a root of $g(X) = X^3 - 3pX - Ap$. Equivalently, one can use the Cardano formulas.

Hence $E = F(\zeta_3) = \mathbb{Q}(\beta, \zeta_3) \subseteq \mathbb{Q}(\zeta_3, \Pi)$ and (4.6) follows, since both fields in question are of degree 6 over $\mathbb{Q}$.

Theorem 4.7. *We have*

$$3\text{-rank } K_2\mathcal{O}_F = 3\text{-rank } A_E.$$

Proof. For $q = 3$ we have

$$\varepsilon_0 = \frac{1}{2}(1 + \sigma) = \frac{1}{2}N, \quad \varepsilon_1 = \frac{1}{2}(1 - \sigma).$$

Hence $\varepsilon_0 A_E = N_{E/F}(A_E) = 1$, since the class number of F is not divisible by 3 (by Theorem 2.2 (i)). Consequently, $\varepsilon_1 A_E = (\varepsilon_0 + \varepsilon_1)A_E = A_E$, and the assertion follows from Theorem 4.3. $\square$

Now we can give a direct proof of a result obtained earlier.

Theorem 4.8. *If $p \equiv 1 \pmod{9}$, then $3\text{-rank } K_2\mathcal{O}_F \geq 1$.*

Proof. Set $\gamma = \sqrt[3]{p}$ for simplicity. We shall prove that $E(\gamma)$ is an abelian unramified extension of E of degree 3. Then $3\text{-rank } A_E \geq 1$, and the assertion follows from Theorem 4.7.

Since the extension $E/\mathbb{Q}$ is abelian, and $\gamma = \sqrt[3]{p}$ cannot belong to any abelian extension of $\mathbb{Q}$ (the Galois group of the polynomial $X^3 - p \in \mathbb{Q}[X]$ is S_3), it follows that $(E(\gamma) : E) = 3$. Moreover, $E(\gamma)$ is a cyclic extension of E , since $\zeta_3 \in E$.

Evidently $(p) = (\gamma)^3$ is the cube of an ideal in $\mathcal{O}_E$, and the congruence $X^3 \equiv p \pmod{(1 - \zeta_3)3}$ has a solution $X = 1$ in $\mathcal{O}_E$, since $p \equiv 1 \pmod{9}$. Hence p is a singular primary element in E with respect to the prime number 3. From [Wa, Exercise 9.3 (b)] it follows that $E(\gamma)$ is an unramified extension of E , as claimed. $\square$

5. REFLECTION THEOREMS

We apply reflection theorems to prove some estimates of the q -rank $K_2\mathcal{O}_F$; for more details see [Br2]. We extend the above notation as follows.

Let L be the maximal unramified and elementary abelian q -extension of E with the Galois group $H := \text{Gal}(L/E)$. Then the Artin reciprocity map gives an isomorphism of G -modules $A/q \longrightarrow H$.

By Kummer theory, $L = E(B^{1/q})$, where B is a subgroup of E^* containing E^{*q} . Set $B_0 := B/E^{*q}$. Then every principal ideal (b) , where $b \in B$, is the q -th power of an ideal in E , since L/E is unramified. Moreover, B_0 is isomorphic to the dual $\widehat{H}$ of H as a G -module.

Define ${}_q A = \{a \in A : a^q = 1\}$. Then there is a homomorphism of G -modules

$$\varphi : B_0 \longrightarrow {}_q A$$

such that $\varphi(bE^{*q}) = Cl(\mathfrak{a})$, where the ideal $\mathfrak{a}$ of $\mathcal{O}_E$ is defined by the condition $(b) = \mathfrak{a}^q$.

Theorem 5.1 (See [Br2, Theorem 3.1]).

$$q\text{-rank}(\varepsilon_j A) = q\text{-rank}(\varepsilon_{q-j} B_0). \quad \square$$

Let U_E be the group of units of $\mathcal{O}_E$, and denote by U'_E its subgroup of units u satisfying

$$u \equiv x^q \pmod{q(1 - \zeta_q)}$$

for some $x \in \mathcal{O}_E$. Such a u is called a *singular primary unit*. It is easy to see that $U_E^q \subseteq U'_E$.

Theorem 5.2 (See [Br2] (3.1)).

$$\ker \varphi = U'_E / U_E^q. \quad \square$$

Theorem 5.3. *We have*

$$q\text{-rank}(\varepsilon_2(U'_E / U_E^q)) \leq q\text{-rank}(K_2 \mathcal{O}_F) \leq q\text{-rank}(\varepsilon_2 A_E) + q\text{-rank}(\varepsilon_2(U'_E / U_E^q)).$$

Proof. Since $\varphi : B_0 \longrightarrow {}_q A$ is a homomorphism of G -modules, it induces homomorphisms of the corresponding direct summands:

$$\varphi_j : \varepsilon_j B_0 \longrightarrow \varepsilon_j({}_q A) \quad \text{for } 0 \leq j \leq q-2.$$

Hence

$$q\text{-rank}(\ker \varphi_j) \leq q\text{-rank}(\varepsilon_j B_0) \leq q\text{-rank}(\varepsilon_j({}_q A)) + q\text{-rank}(\ker \varphi_j).$$

Since $\ker \varphi_j = \varepsilon_j \ker \varphi$ and $q\text{-rank}(A) = q\text{-rank}({}_q A)$, the assertion follows from Theorems 5.1, 5.2 for $j = 2$, and 4.3. $\square$

Theorem 5.3 gives some estimates from below and from above of the q -rank of $K_2 \mathcal{O}_F$ by means of the q -ranks of some subgroups of the class group and of the group of singular primary units (modulo q -th powers) of the field $E = F(\zeta_q)$.

For large prime numbers q , the degree $(E : \mathbb{Q}) = q - 1$ is large, and it is difficult to determine its class group and the group of units, and the action of the Galois group $\text{Gal}(E/\mathbb{Q})$ on them. We prove that sometimes E can be replaced by its proper subfields, where the necessary computations are much easier to perform.

For a fixed primitive root h modulo q set $\omega(h) = \zeta_{q-1} \in \mathbb{Z}_q^*$, and $t = \frac{q-1}{2}$. Then σ^t is the complex conjugation on E . The maximal real subfield E^+ of E is the subfield fixed by σ_t . Then $N_{E/E^+} = 1 + \sigma^t$.

Lemma 5.4. *Under the above notation we have*

$$\varepsilon_2 = \rho \cdot N_{E/E^+} \quad \text{for some } \rho \in \mathbb{Z}_q[G].$$

Proof. By the definition of ε_2 we get

$$\varepsilon_2 = \frac{1}{q-1} \sum_{k=0}^{q-2} \zeta_{q-1}^{2k} \sigma^{-k} = \frac{1}{q-1} \sum_{k=0}^{t-1} \zeta_t^k \sigma^{-k} (1 + \sigma^t) = \frac{1}{q-1} \sum_{k=0}^{t-1} \zeta_t^k \sigma^{-k} \cdot N_{E/E^+}. \quad \square$$

Assume that $q \equiv 1 \pmod{6}$, and for $j = 0, 1, 2$, let

$$\eta_j := \frac{1}{3} \sum_{\ell=0}^2 \zeta_3^{\ell j} \tau^{-\ell},$$

where $\zeta_3 \in \mathbb{Z}_q$ and τ is a generator of $T := \text{Gal}(F/\mathbb{Q}) = \text{Gal}(E/\mathbb{Q}(\zeta_q))$. Then η_0, η_1, η_2 are primitive idempotents of the group ring $\mathbb{Z}_q[T]$. Hence

$$(5.1) \quad \varepsilon_2 = \sum_{j=0}^2 \varepsilon_2 \eta_j$$

in the group ring $\mathbb{Z}_q[G \times T]$. Set $r = (q-1)/6$. For $j = 0, 1, 2$, let T_j be the subgroup of $G \times T$ generated by σ^t and $\sigma^{rj} \tau^{-1}$, and denote by E_j the subfield of E fixed by T_j . Then $\#T_j = 6$, $E_j \subset E^+$, and $(E_j : \mathbb{Q}) = (q-1)/2 = t$ for $j = 0, 1, 2$. In particular, $E_0 = E^{\langle \sigma^t, \tau^{-1} \rangle} = \mathbb{Q}(\zeta_q)^+$ is the maximal real subfield of $\mathbb{Q}(\zeta_q)$.

Lemma 5.5. *Under the above notation we have*

$$\varepsilon_2 \eta_j = \rho_j \cdot N_{E/E_j}$$

for $j = 0, 1, 2$ and some $\rho_j \in \mathbb{Z}_q[G \times T]$.

Proof. In the definition of ε_2 (see (4.1)) there is a summation over k , where $0 \leq k \leq q-2$. Since $q-1 = 6r$, we can substitute $k = ir + m$, where $0 \leq i \leq 5$, $0 \leq m \leq r-1$. Then

$$\begin{aligned} \varepsilon_2 \eta_j &= \frac{1}{3(q-1)} \sum_{\ell=0}^2 \zeta_3^{\ell j} \tau^{-\ell} \sum_{i=0}^5 \sum_{m=0}^{r-1} \zeta_{6r}^{2ir+2m} \sigma^{-ir-m} \\ &= \frac{1}{3(q-1)} \left(\sum_{m=0}^{r-1} \zeta_{3r}^m \sigma^{-m} \right) \cdot \left(\sum_{\ell=0}^2 \sum_{i=0}^5 \zeta_3^{\ell j+i} \sigma^{-ir} \tau^{-\ell} \right). \end{aligned}$$

Denote the first sum in brackets by ξ_1 , and the second by ξ_2 . For fixed ℓ put $s = \ell j + i$; then for $0 \leq i \leq 5$, s runs over all residues modulo 6. Hence

$$\xi_2 = \sum_{\ell=0}^2 \sum_{s=0}^5 \zeta_3^s \sigma^{-r(s-\ell j)} \tau^{-\ell} = \left(\sum_{s=0}^5 \zeta_3^s \sigma^{-rs} \right) \cdot \left(\sum_{\ell=0}^2 (\sigma^{rj} \tau^{-1})^\ell \right).$$

Denote the first factor of the last expression by ξ_3 , and the second by ξ_4 . Then, since $3r = t$ and $\sigma^{-t} = \sigma^t$, we get

$$\xi_3 = 1 + \zeta_3 \sigma^{-r} + \zeta_3^2 \sigma^{-2r} + \sigma^t + \zeta_3 \sigma^{-r} \sigma^t + \zeta_3^2 \sigma^{-2r} \sigma^t = \xi_5 \cdot (1 + \sigma^t) = \xi_5 \cdot N_{E/E^+},$$

where $\xi_5 = 1 + \zeta_3 \sigma^{-r} + \zeta_3^2 \sigma^{-2r}$.

Similarly, for $j = 0, 1, 2$,

$$\xi_4 = \sum_{\ell=0}^2 (\sigma^{rj} \tau^{-1})^\ell$$

is the norm from E to the subfield fixed by $\sigma^{rj} \tau^{-1}$. Hence $(1 + \sigma^t) \xi_4$ is the norm from E to the subfield of E^+ fixed by $\sigma^{rj} \tau^{-1}$, i.e., $(1 + \sigma^t) \xi_4 = N_{E/E_j}$.

Finally we get

$$\varepsilon_2 \eta_j = \xi_1 \xi_2 = \xi_1 \xi_3 \xi_4 = \xi_1 \xi_5 (1 + \sigma^t) \xi_4 = \xi_1 \xi_5 N_{E/E_j}. \quad \square$$

For every subfield L of E we define U'_L to be the group of singular primary units in L ,

$$U'_L = U'_E \cap L.$$

Since $\text{Gal}(E/\mathbb{Q})$ acts on U'_E , it follows that

$$N_{E/L} U'_E \subseteq U'_L.$$

Lemma 5.6. *The natural inclusion $U'_L \longrightarrow U'_E$ induces an injection*

$$U'_L/U_L^q \longrightarrow U'_E/U_E^q \quad \text{for } q > 3.$$

Proof. Assume that $\varepsilon \in U'_L \cap U_E^q$. Then $\varepsilon = \eta^q$ for some $\eta \in U_E$. Consequently, $\eta = \sqrt[q]{\varepsilon}$, and so $(L(\eta) : L) = 1$ or q . On the other hand, $(L(\eta) : L) \mid (E : \mathbb{Q}) = 3(q-1)$, which is not divisible by q . Hence $(L(\eta) : L) = 1$, i.e., $\eta \in U_L$, $\varepsilon \in U_L^q$, and the lemma follows. $\square$

From the above we get inclusions of elementary abelian q -groups

$$N_{E/L}(U'_E/U_E^q) \subseteq U'_L/U_L^q \subseteq U'_E/U_E^q;$$

hence, acting by ε_2 , we obtain

$$(5.2) \quad q\text{-rank}(\varepsilon_2 N_{E/L}(U'_E/U_E^q)) \leq q\text{-rank}(\varepsilon_2(U'_L/U_L^q)) \leq q\text{-rank}(\varepsilon_2(U'_E/U_E^q)).$$

Lemma 5.7. *Let L_1, L_2 be subfields of E , and set $L_0 = L_1 \cap L_2$. Assume that L_1, L_2 are proper subfields of $L_1 L_2$, and $U'_{L_0}/U_{L_0}^q = 1$. Then the mapping*

$$U'_{L_1}/U_{L_1}^q \times U'_{L_2}/U_{L_2}^q \longrightarrow U'_{L_1 L_2}/U_{L_1 L_2}^q,$$

given by $(u_1, u_2) \mapsto u_1 u_2$, is injective.

Proof. Suppose that $u_1 \in U'_{L_1}$, $u_2 \in U'_{L_2}$ satisfy

$$(5.3) \quad u_1 u_2 = u^q, \quad \text{for some } u \in U_{L_1 L_2}.$$

We have

$$N_{L_1 L_2/L_1}(u_1) = u_1^r, \quad \text{where } r = (L_1 L_2 : L_1) \mid (E : \mathbb{Q}) = 3(q-1),$$

$$N_{L_1 L_2/L_1}(u_2) = N_{L_2/L_0}(u_2) \in U'_{L_0} = U_{L_0}^q \subseteq U_{L_1}^q,$$

$$N_{L_1 L_2/L_1}(u^q) = (N_{L_1 L_2/L_1}(u))^q \subseteq U_{L_1}^q.$$

Consequently, from (5.3), it follows that $u_1^r \in U_{L_1}^q$, so $u_1 \in U_{L_1}^q$, since $q \nmid r$. By symmetry we get $u_2 \in U_{L_2}^q$; hence the mapping in question is injective. $\square$

From Lemma 5.7 it also follows that the restricted mapping

$$\varepsilon_2(U'_{L_1}/U_{L_1}^q) \times \varepsilon_2(U'_{L_2}/U_{L_2}^q) \longrightarrow \varepsilon_2(U'_{L_1 L_2}/U_{L_1 L_2}^q)$$

is injective. Consequently, under the assumptions of Lemma 5.7 we have that

$$(5.4) \quad \sum_{j=1}^2 q\text{-rank}\left(\varepsilon_2\left(U'_{L_j}/U_{L_j}^q\right)\right) \leq q\text{-rank}\left(\varepsilon_2\left(U'_{L_1 L_2}/U_{L_1 L_2}^q\right)\right) \\ \leq q\text{-rank}\left(\varepsilon_2\left(U'_E/U_E^q\right)\right).$$

Theorem 5.8. *Let E_j be the subfield of E fixed by the group $T_j = \langle \sigma^{rj} \tau^{-1}, \sigma^t \rangle$. Then*

$$(5.5) \quad \begin{aligned} \max_{0 \leq j \leq 2} \left(q\text{-rank}(\varepsilon_2(U'_{E_j}/U^q_{E_j})) \right) &\leq q\text{-rank}(K_2\mathcal{O}_F) \\ &\leq \sum_{j=0}^2 q\text{-rank}(\varepsilon_2 A_{E_j}) + \sum_{j=0}^2 q\text{-rank}(\varepsilon_2(U'_{E_j}/U^q_{E_j})). \end{aligned}$$

Moreover, if the class number of the field $\mathbb{Q}(\zeta_q)$ is not divisible by q , and in the field $L_0 := E_1 \cap E_2$ we have $U'_{L_0}/U^q_{L_0} = 1$, then

$$(5.6) \quad \begin{aligned} \sum_{j=1}^2 q\text{-rank} \left(\varepsilon_2 \left(U'_{E_j}/U^q_{E_j} \right) \right) &\leq q\text{-rank}(K_2\mathcal{O}_F) \\ &\leq \sum_{j=1}^2 q\text{-rank}(\varepsilon_2 A_{E_j}) + \sum_{j=1}^2 q\text{-rank} \left(\varepsilon_2 \left(U'_{E_j}/U^q_{E_j} \right) \right). \end{aligned}$$

Proof. The first inequality in (5.5) follows from (5.2) and Theorem 5.3.

By Lemma 5.5, from (5.1) we get

$$\begin{aligned} q\text{-rank}(\varepsilon_2 A_E) &= \sum_{j=0}^2 q\text{-rank}(\varepsilon_2 \eta_j A_E) \leq \sum_{j=0}^2 q\text{-rank}(\varepsilon_2 N_{E/E_j}(A_E)) \\ &\leq \sum_{j=0}^2 q\text{-rank}(\varepsilon_2 A_{E_j}). \end{aligned}$$

Similarly, by (5.2) we obtain

$$q\text{-rank}(\varepsilon_2(U'_E/U^q_E)) \leq \sum_{j=0}^2 q\text{-rank}(\varepsilon_2(U'_{E_j}/U^q_{E_j})).$$

Now the second inequality in (5.5) follows from Theorem 5.3.

Let us observe that E_0 is the field fixed by the group $\langle \tau^{-1}, \sigma^t \rangle$; hence $E_0 = \mathbb{Q}(\zeta_q)^+$. From the assumption $q\text{-rank}(A_{\mathbb{Q}(\zeta_q)}) = 0$ it follows that $q\text{-rank}(A_{E_0}) = 0$ and that there is no abelian unramified extension of $\mathbb{Q}(\zeta_q)$ of degree q . Then from [Wa, Exercise 9.3] we deduce that every singular primary unit in $\mathbb{Q}(\zeta_q)$ is a q -th power of a unit. By Lemma 5.6 the same holds for the field $\mathbb{Q}(\zeta_q)^+$.

Thus we have proved that $q\text{-rank}(\varepsilon_2(U'_{E_0}/U^q_{E_0})) = 0$.

Now we proceed as in the proof of (5.5). To prove the first inequality in (5.6) we use (5.4) with $L_j = E_j$ for $j = 1, 2$, and Theorem 5.3.

To prove the second inequality it is sufficient to observe that on the right-hand side of (5.5) the summands corresponding to $j = 0$ vanish, as proved above. $\square$

6. THE 7-RANK OF $K_2\mathcal{O}_F$

We apply Theorem 5.8 with $q = 7$ to prove some estimates for the 7-rank of $K_2\mathcal{O}_F$. In many cases these estimates suffice to determine the structure of the 7-primary part of the group $K_2\mathcal{O}_F$. Our arguments can be extended *mutatis mutandis* to other primes $q \equiv 1 \pmod{6}$.

We are going to give an effective description of the fields E_j to look for singular primary units in these fields, and to describe the action of ε_2 on these units. For this purpose we have to fix some notation.

6.1. Notation. For a fixed primitive root g modulo p , where $p \equiv 1 \pmod{6}$ is a prime, we have defined the Gauss periods

$$\alpha_1 = \sum_{k=1}^{(p-1)/3} \zeta_p^{g^{3k}}, \quad \alpha_2 = \sum_{k=1}^{(p-1)/3} \zeta_p^{g^{3k+1}}, \quad \alpha_3 = \sum_{k=1}^{(p-1)/3} \zeta_p^{g^{3k+2}}.$$

Obviously, α_1 does not depend on the primitive root g chosen, but if we replace g by g^{-1} , then α_2 and α_3 permute; hence the number $(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)$ changes sign.

We shall assume henceforth that a primitive root g modulo p is chosen in such a way that

$$(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1) > 0.$$

In particular, for the prime 7, we choose the primitive root 3. Then the Gauss periods are

$$\gamma_1 = \zeta_7 + \zeta_7^{-1} = 1.24, \quad \gamma_2 = \zeta_7^3 + \zeta_7^{-3} = -1.80, \quad \gamma_3 = \zeta_7^2 + \zeta_7^{-2} = -0.44;$$

hence $\gamma_1 > \gamma_3 > \gamma_2$, and so $(\gamma_1 - \gamma_2)(\gamma_2 - \gamma_3)(\gamma_3 - \gamma_1) > 0$.

Since $\gamma_1, \gamma_2, \gamma_3$ are the roots of the polynomial $f(X) = X^3 + X^2 - 2X - 1$ with discriminant

$$\Delta_f = ((\gamma_1 - \gamma_2)(\gamma_2 - \gamma_3)(\gamma_3 - \gamma_1))^2 = 49,$$

it follows that

$$(\gamma_1 - \gamma_2)(\gamma_2 - \gamma_3)(\gamma_3 - \gamma_1) = 7.$$

On the other hand,

$$(\gamma_1 - \gamma_2)(\gamma_2 - \gamma_3)(\gamma_3 - \gamma_1) = U_\gamma - V_\gamma,$$

where

$$U_\gamma = \gamma_1\gamma_2^2 + \gamma_2\gamma_3^2 + \gamma_3\gamma_1^2, \quad V_\gamma = \gamma_1^2\gamma_2 + \gamma_2^2\gamma_3 + \gamma_3^2\gamma_1.$$

Moreover, by the Vieta formulas,

$$U_\gamma + V_\gamma = \sum \gamma_1\gamma_2 \cdot \sum \gamma_1 - 3\gamma_1\gamma_2\gamma_3 = -1.$$

Solving the system $U_\gamma - V_\gamma = 7$, $U_\gamma + V_\gamma = -1$ we get $U_\gamma = 3$, $V_\gamma = -4$.

We denote by σ the automorphism of the field $\mathbb{Q}(\zeta_7)$ satisfying $\sigma(\zeta_7) = \zeta_7^3$. Then $\sigma(\gamma_i) = \gamma_{i+1}$, where the indices are taken modulo 3. We repeat the same for the cubic subfield F of the cyclotomic field $\mathbb{Q}(\zeta_p)$, where $p \equiv 1 \pmod{6}$, $p > 7$. The field F is generated by any root β_i of the polynomial

$$g(X) = X^3 - 3pX - Ap,$$

with discriminant $\Delta_g = (27Bp)^2$. Since $\beta_i = 3\alpha_i + 1$, where α_i are the Gauss sums given above, from our assumption on the primitive root g modulo p it follows that

$$(\beta_1 - \beta_2)(\beta_2 - \beta_3)(\beta_3 - \beta_1) = 27(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1) > 0.$$

Moreover, the automorphism $\tau \in \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ given by $\tau(\zeta_p) = \zeta_p^g$ satisfies $\tau(\beta_i) = \beta_{i+1}$, where the indices are taken modulo 3.

As above we set

$$U_\beta = \beta_1\beta_2^2 + \beta_2\beta_3^2 + \beta_3\beta_1^2, \quad V_\beta = \beta_1^2\beta_2 + \beta_2^2\beta_3 + \beta_3^2\beta_1.$$

Then

$$U_\beta - V_\beta = 27Bp, \quad U_\beta + V_\beta = -3Ap;$$

hence

$$U_\beta = -\frac{3}{2}(A - 9B), \quad V_\beta = -\frac{3}{2}(A + 9B).$$

Now let us consider the compositum of the cubic fields $\mathbb{Q}(\zeta_7)^+ = \mathbb{Q}(\gamma_i)$ and $F = \mathbb{Q}(\beta_i)$. It is the maximal real subfield E^+ of $E = F(\zeta_7)$. Since

$$\text{Gal}(E^+/\mathbb{Q}) = \mathbb{Z}/3 \times \mathbb{Z}/3 = \langle \sigma \rangle \times \langle \tau \rangle,$$

where the automorphisms σ and τ are restricted to E^+ , there are four cubic subfields of E^+ . Namely, $F = \mathbb{Q}(\beta_i)$, $E_0 = \mathbb{Q}(\zeta_7)^+ = \mathbb{Q}(\gamma_i)$, E_1 , the subfield fixed by $\sigma\tau^{-1}$, and E_2 , the subfield fixed by $\sigma\tau$.

Hence, for $a \in E^+$ we have

$$a \in E_1 \quad \text{iff} \quad \sigma(a) = \tau(a), \quad \text{and} \quad a \in E_2 \quad \text{iff} \quad \sigma(a) = \tau^{-1}(a).$$

Define

$$\rho_1 = \gamma_1\beta_1 + \gamma_2\beta_3 + \gamma_3\beta_2, \quad \rho_2 = \gamma_1\beta_1 + \gamma_2\beta_2 + \gamma_3\beta_3.$$

It is easy to see that $\rho_1, \rho_2 \in E^+$, and $\sigma(\rho_1) = \tau(\rho_1)$, $\sigma(\rho_2) = \tau^{-1}(\rho_2)$. Moreover, $\rho_1, \rho_2 \notin \mathbb{Q}$. Therefore,

$$E_1 = \mathbb{Q}(\rho_1), \quad E_2 = \mathbb{Q}(\rho_2).$$

Applying the Vieta formulas one can verify that the minimal polynomials for ρ_1 and ρ_2 are, respectively,

$$g_1(X) = X^3 - 21pX - 7p\frac{A - 27B}{2},$$

$$g_2(X) = X^3 - 21pX - 7p\frac{A + 27B}{2},$$

with discriminants

$$\Delta_{g_1} = 3^6 \cdot 7^2 \cdot p^2 \cdot \left(\frac{A+B}{2}\right)^2 \quad \text{and} \quad \Delta_{g_2} = 3^6 \cdot 7^2 \cdot p^2 \cdot \left(\frac{A-B}{2}\right)^2.$$

6.2. The action of ε_2 . Let a cyclic group $G = \langle \sigma \rangle$ of order 6 act on an elementary abelian 7-group A . In the following lemma we describe the action on A of the primitive idempotents

$$\varepsilon_j = \frac{1}{6} \sum_{k=0}^5 \omega(3)^{kj} \sigma^{-k}, \quad 0 \leq j < 6,$$

of the group ring $\mathbb{Z}_7[G]$, where $\omega(3) \in \mathbb{Z}_7$ satisfies $\omega(3)^6 = 1$, $\omega(3) \equiv 3 \pmod{7}$.

We apply this lemma in the case when $\varepsilon_j = \varepsilon_2$, $G = \text{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q})$, and A is either the maximal elementary abelian 7-subgroup of the class group of E_j , or the group of classes of singular primary units of E_j modulo seventh powers.

Lemma 6.1. *Let a cyclic group $G = \langle \sigma \rangle$ of order 6 act on an elementary abelian 7-group A , and let $a \in A$.*

(i) *If*

$$(6.1) \quad (1 + \sigma + \sigma^2)a = 1,$$

then $\varepsilon_j a = 1$ for $j = 0, 1, 3, 5$.

(ii) *Moreover,*

$$\begin{aligned} \sigma(a) = a^2 & \quad \text{iff} \quad (\varepsilon_2 a = a \quad \text{and} \quad (6.1) \text{ holds}), \\ \sigma(a) = a^4 & \quad \text{iff} \quad (\varepsilon_4 a = a \quad \text{and} \quad (6.1) \text{ holds}). \end{aligned}$$

Proof. (i) Let us observe that (6.1) implies $\sigma^3 a = a$, so

$$\begin{aligned} \varepsilon_j a &= \frac{1}{6} \sum_{k=0}^5 3^{kj} \sigma^{-k} \cdot a = \frac{1}{6} \sum_{k=0}^2 (3^{kj} + 3^{(k+3)j}) \sigma^{-k} \cdot a \\ (6.2) \quad &= -(1 + 3^{3j}) \sum_{k=0}^2 3^{kj} \sigma^{-k} \cdot a. \end{aligned}$$

Since $3^{3j} \equiv (-1)^j \pmod{7}$, it follows that $\varepsilon_j a = 1$ for j odd. Moreover, $\varepsilon_0 a = \frac{1}{6}(1 + \sigma + \sigma^2)a = 1$, by (6.1). This proves (i).

(ii) Assume that $\sigma(a) = a^2$. Then $(1 + \sigma + \sigma^2)a = a \cdot a^2 \cdot a^4 = a^7 = 1$. Thus (6.1) holds.

Now, by (6.2) and (6.1),

$$\varepsilon_2 a = -(1 + 3^6)(1 + 3^2 \sigma^{-1} + 3^4 \sigma^{-2})a = (2 + 3\sigma)a = a^2 \cdot \sigma(a)^3.$$

Therefore, $\varepsilon_2 a = a$ iff $a \cdot \sigma(a)^3 = 1$ iff $\sigma(a) = a^2$.

The proof of the last statement is similar. $\square$

6.3. Estimates of the 7-rank of $K_2 \mathcal{O}_F$. The class group of the field $\mathbb{Q}(\zeta_7)$ is trivial, and there are no nontrivial singular primary units with respect to $q = 7$ in the field $E_1 \cap E_2 = \mathbb{Q}$. Therefore, the assumptions of the “moreover” part of Theorem 5.8 are satisfied, and from (5.6) we get

$$\begin{aligned} (6.3) \quad &\sum_{j=1}^2 7\text{-rank}(\varepsilon_2(U'_{E_j}/U_{E_j}^7)) \leq 7\text{-rank}(K_2 \mathcal{O}_F) \\ &\leq \sum_{j=1}^2 7\text{-rank}(\varepsilon_2 A_{E_j}) + \sum_{j=1}^2 7\text{-rank}(\varepsilon_2(U'_{E_j}/U_{E_j}^7)). \end{aligned}$$

Lemma 6.2. *Assume that the class numbers of the fields E_1 and E_2 are not divisible by 7. Then equality holds in (6.3) :*

$$(6.4) \quad 7\text{-rank}(K_2 \mathcal{O}_F) = \sum_{j=1}^2 7\text{-rank}(\varepsilon_2(U'_{E_j}/U_{E_j}^7)).$$

Proof. From the assumption it follows that $7\text{-rank}(\varepsilon_2 A_{E_j}) = 0$ for $j = 1, 2$, so (6.3) yields the assertion. $\square$

6.4. Computation of the 7-rank of $K_2 \mathcal{O}_F$. We give some details concerning the computation of the 7-rank of $K_2 \mathcal{O}_F$ via the 7-ranks of other groups appearing in (6.3).

Since the field E_j , $j = 1, 2$, is cubic and totally real, there are two fundamental units ξ_1, ξ_2 in E_j , and we can assume that $\xi_2 = \sigma(\xi_1)$ and $N_{E_j/\mathbb{Q}}(\xi_1) = 1$. Therefore $U_{E_j}/U_{E_j}^7 = \mathbb{F}_7^2$, and the characteristic polynomial of the action of σ on the linear space $U_{E_j}/U_{E_j}^7$ over $\mathbb{F}_7$ is $X^2 + X + 1$. Since the polynomial has distinct roots 2, 4 in $\mathbb{F}_7$, it follows that the corresponding eigenspaces $V_j^{(2)}$ and $V_j^{(4)}$ are 1-dimensional.

From Lemma 6.1 it follows that

$$\varepsilon_2(U'_{E_j}/U_{E_j}^7) = V_j^{(2)} \cap (U'_{E_j}/U_{E_j}^7).$$

Consequently,

$$7\text{-rank}(U'_{E_j}/U_{E_j}^7) \leq \dim V_j^{(2)} = 1,$$

and to determine this 7-rank it is sufficient to find a unit $u \in U_{E_j}$ such that the corresponding coset uU_{E_j} is an eigenvector corresponding to the eigenvalue 2, i.e., $\sigma(u) \equiv u^2 \pmod{U_{E_j}^7}$, and to verify whether $u \in U'_{E_j}$.

This last verification can be done as follows. The field E_j equals $\mathbb{Q}(\rho_j)$, where ρ_j is a root of the polynomial $g_j(X)$ defined above. Let us observe that, for $p > 7$,

$$A^2 - B^2 \equiv A^2 + 27B^2 = 4p \not\equiv 0 \pmod{7};$$

hence $A \not\equiv \pm B \pmod{7}$. Consequently, $g_j(X)$ is an Eisenstein polynomial with respect to 7. Hence the root $x := \rho_j$ of $g_j(X)$ satisfies $v_7(x) = \frac{1}{3}$, where v_7 is the normalized 7-adic valuation, $v_7(7) = 1$.

Consequently, every unit $u \in U_{E_j}$ can be written in the form

$$(6.5) \quad u = a_0 + a_1x + a_2x^2, \quad \text{where } a_i \in \mathbb{Q}, \quad v_7(a_0) = 0.$$

Raising to the sixth power induces an isomorphism of the group $U_{E_j}/U_{E_j}^7$; hence we can replace u by u^6 . In that case $a_0 \equiv 1 \pmod{7}$.

So let us assume that in (6.5) we have $a_0 \equiv 1 \pmod{7}$. Then u is a singular primary unit iff $u \equiv 1 \pmod{7x}$, or, equivalently,

$$(6.6) \quad a_0 \equiv 1 \pmod{49}, \quad a_1 \equiv a_2 \equiv 0 \pmod{7}.$$

We proceed similarly to determine the 7-rank of A_{E_j} . We look for generators of the maximal elementary abelian 7-subgroup of the class group A_{E_j} . It is a linear space over $\mathbb{F}_7$. Then we describe the action of the automorphism σ on it and determine the eigenspace $W_j^{(2)}$ corresponding to the eigenvalue 2. In view of Lemma 6.1,

$$7\text{-rank}(\varepsilon_2 A_{E_j}) = \dim W_j^{(2)} \quad \text{for } j = 1, 2.$$

The results of the computations are given in Table 2 (at the end of the paper), where we used the following shorthand notation, for $j = 1, 2$:

$$v_7 := v_7(\#K_2\mathcal{O}_F), d_j := \dim \left(V_j^{(2)} \cap \left(U'_{E_j}/U_{E_j}^7 \right) \right), h_j := 7\text{-rank}(\varepsilon_2 A_{E_j}),$$

and $(K_2\mathcal{O}_F)_7$ is the 7-primary part of $K_2\mathcal{O}_F$.

Thus if a coset $uU_{E_j}^7$ generates the linear space $V_j^{(2)}$, then $d_j = 1$ or 0 according as u is a singular primary unit or not.

It turns out that the 7-primary part A_{E_j} of the class group of the field E_j is nontrivial in three cases only:

- For $p = 2707$ we have $A_{E_1} = \mathbb{Z}/7$. We have computed that $\varepsilon_2 A_{E_1} = 1$ (fortunately!), so the class group does not contribute to the 7-rank of $K_2\mathcal{O}_F$. Consequently, by (6.4),

$$7\text{-rank}(K_2\mathcal{O}_F) = \sum_{j=1}^2 7\text{-rank}(\varepsilon_2(U'_{E_j}/U_{E_j}^7)) = 1.$$

- For $p = 3313$ we have $A_{E_2} = \mathbb{Z}/7$. Since

$$\sum_{j=1}^2 7\text{-rank}(\varepsilon_2(U'_{E_j}/U_{E_j}^7)) = 0,$$

it follows that $1 \leq 7\text{-rank}(K_2\mathcal{O}_F) \leq 7\text{-rank}(\varepsilon_2 A_{E_2}) \leq 7\text{-rank}(A_{E_2}) = 1$; hence equality holds everywhere, and $7\text{-rank } K_2\mathcal{O}_F = 1$.

- For $p = 4513$ we have $A_{E_1} = \mathbb{Z}/7$. Since $7^2 \parallel \#K_2\mathcal{O}_F$ and

$$\sum_{j=1}^2 7\text{-rank} \left(\varepsilon_2(U'_{E_j}/U_{E_j}^7) \right) = 2,$$

it follows that $7\text{-rank}(K_2\mathcal{O}_F) = 2$, and the knowledge of the 7-rank of $\varepsilon_2 A_{E_1}$ is not necessary.

7. THE 13-RANK OF $K_2\mathcal{O}_F$

Now we apply Theorem 5.8 with $q = 13$. The arguments are similar to those in the case $q = 7$ considered above, so we omit some details.

For the prime number 13 we choose the primitive root 2 and we consider the Gauss periods

$$\gamma_j = \zeta_{13}^{2^j} + \zeta_{13}^{-2^j}, \quad 1 \leq j \leq 6.$$

Then the automorphism σ such that $\sigma(\zeta_{13}) = \zeta_{13}^2$ satisfies $\sigma(\gamma_j) = \gamma_{j+1}$, $1 \leq j \leq 6$, where the indices are taken modulo 6.

Evidently $\mathbb{Q}(\gamma_j) = \mathbb{Q}(\zeta_{13})^+$ for $1 \leq j \leq 6$. Moreover, the field $\mathbb{Q}(\gamma_j)$ contains a unique cubic subfield $F_0 := \mathbb{Q}(\gamma_j + \gamma_{j+3})$ and a unique quadratic subfield $\mathbb{Q}(\sqrt{13}) = \mathbb{Q}(\gamma_j + \gamma_{j+2} + \gamma_{j+4})$.

Let F be the cubic subfield of the cyclotomic field $\mathbb{Q}(\zeta_p)$, where $p \equiv 1 \pmod{6}$, $p > 13$. Then $F = \mathbb{Q}(\beta_i)$, where β_i , $i = 1, 2, 3$, are the roots of the polynomial

$$g(X) = X^3 - 3pX - Ap,$$

satisfying $(\beta_1 - \beta_2)(\beta_2 - \beta_3)(\beta_3 - \beta_1) > 0$. The automorphism τ of the field F is defined by $\tau(\beta_i) = \beta_{i+1}$, where the indices are taken modulo 3.

Let E^+ be the maximal real subfield of the field $E = F(\zeta_{13})$. Then E^+ is the compositum of the fields $\mathbb{Q}(\zeta_{13})^+$ and F , and consequently,

$$\text{Gal}(E^+/\mathbb{Q}) = \mathbb{Z}/6 \times \mathbb{Z}/3 = \langle \sigma \rangle \times \langle \tau \rangle,$$

where the automorphisms σ and τ are restricted to E^+ .

For $j = 0, 1, 2$, let E_j be the subfield of E^+ fixed by the automorphism $\sigma^{2j}\tau^{-1}$. Then $E_0 = \mathbb{Q}(\zeta_{13})^+$. It follows that, for $a \in E^+$,

$$\begin{array}{ll} a \in E_1 & \text{iff} \quad \sigma^2(a) = \tau(a), \\ a \in E_2 & \text{iff} \quad \sigma^2(a) = \tau^{-1}(a). \end{array}$$

Define

$$\begin{aligned} \rho_1 &= \gamma_1\beta_1 + \gamma_5\beta_2 + \gamma_3\beta_3, \\ \rho_2 &= \gamma_1\beta_1 + \gamma_3\beta_2 + \gamma_5\beta_3. \end{aligned}$$

Then $\rho_1, \rho_2 \in E^+$ and $\sigma^2(\rho_1) = \tau(\rho_1)$, $\sigma^2(\rho_2) = \tau^{-1}(\rho_2)$. Moreover, $\sigma^k(\rho_j) \neq \rho_j$ for $1 \leq k \leq 5$, $j = 1, 2$. Hence $E_j = \mathbb{Q}(\rho_j)$, for $j = 1, 2$.

Let

$$\begin{aligned} \lambda_1 &= \rho_1 + \sigma\tau(\rho_1) = (\gamma_1 + \gamma_4)\beta_1 + (\gamma_2 + \gamma_5)\beta_2 + (\gamma_3 + \gamma_6)\beta_3, \\ \lambda_2 &= \rho_2 + \sigma\tau^2(\rho_2) = (\gamma_1 + \gamma_4)\beta_1 + (\gamma_3 + \gamma_6)\beta_2 + (\gamma_2 + \gamma_5)\beta_3. \end{aligned}$$

Then $F_j := \mathbb{Q}(\lambda_j)$ is a cubic subfield of E_j , $j = 1, 2$.

There are ten fields F corresponding to primes $p < 5000$ such that $13^2 \parallel \#K_2\mathcal{O}_F$. Namely, for

$$p = 163, 379, 487, 757, 1723, 1867, 3673, 4447, 4999$$

the above divisibility holds, and even in the stronger form $13^2 \parallel \#K_2\mathcal{O}_F$.

The class number of the field $E_1 \cap E_2 = \mathbb{Q}(\sqrt{13})$ is 1, and there are no nontrivial singular primary units with respect to 13. Consequently, for $a \in U'_{E_j}/U_{E_j}^{13}$ we have $N_{E_j/\mathbb{Q}(\sqrt{13})}(a) = 1$.

Similarly, if in the field F_j , $j = 1, 2$, there are no nontrivial singular primary units, $U'_{F_j}/U_{F_j}^{13} = 1$, then for $a \in U'_{E_j}/U_{E_j}^{13}$ we have $N_{E_j/F_j}(a) = 1$.

Since $\text{Gal}(E_j/F_j) = \langle \sigma^3 \rangle$ and $\text{Gal}(E_j/\mathbb{Q}(\sqrt{13})) = \langle \sigma^2 \rangle$, the above equalities for norms can be written as

$$(1 + \sigma^3)a = 1 \quad \text{and} \quad (1 + \sigma^2 + \sigma^4)a = 1 \quad \text{for } a \in U'_{E_j}/U_{E_j}^{13}.$$

The following analogue of Lemma 6.1 holds.

Lemma 7.1. *Let A be an elementary abelian 13-group on which the Galois group $G = \text{Gal}(\mathbb{Q}(\zeta_{13})/\mathbb{Q}) = \langle \sigma \rangle$ acts, and let $a \in A$.*

(i) *If*

$$(7.1) \quad (1 + \sigma^2 + \sigma^4)a = 1 \quad \text{and} \quad (1 + \sigma^3)a = 1,$$

then

$$\varepsilon_j a = 1 \quad \text{for } 0 \leq j < 11, \quad j \neq 2, 10.$$

(ii) *Moreover,*

$$\begin{aligned} \sigma(a) &= a^4 & \text{iff } (\varepsilon_2 a = a \text{ and (7.1) holds}), \\ \sigma(a) &= a^{10} & \text{iff } (\varepsilon_{10} a = a \text{ and (7.1) holds}). \end{aligned}$$

Proof. (i) From the definition of ε_j and (7.1) we get

$$\begin{aligned} \varepsilon_j a &= \frac{1}{12} \sum_{k=0}^{11} 2^{kj} \sigma^{-k} \cdot a \\ &= \frac{1}{12} \sum_{k=0}^2 \left(2^{kj} \sigma^{-k} + 2^{(k+3)j} \sigma^{-k-3} + 2^{(k+6)j} \sigma^{-k-6} + 2^{(k+9)j} \sigma^{-k-9} \right) \cdot a \\ (7.2) \quad &= \frac{1}{12} (1 - 2^{3j} + 2^{6j} - 2^{9j}) \sum_{k=0}^2 2^{kj} \sigma^{-k} \cdot a \\ &= \frac{1}{12} (1 - 2^{3j}) (1 + 2^{6j}) \sum_{k=0}^2 2^{kj} \sigma^{-k} \cdot a. \end{aligned}$$

Since $2^{6j} \equiv (-1)^j \pmod{13}$, it follows that $\varepsilon_j a = 0$ for j odd and for j divisible by 4.

For $j = 6$, by (7.1) we have

$$\sum_{k=0}^2 2^{kj} \sigma^{-k} \cdot a = \sum_{k=0}^2 (-1)^k \sigma^{-k} \cdot a = (1 - \sigma + \sigma^2)a = (1 + \sigma^2 + \sigma^4)a = 0.$$

(ii) Assume (7.1). Then, in particular, (7.2) holds for $j = 2$:

$$\begin{aligned} \varepsilon_2 a &= \frac{1}{12} (1 - 2^6) (1 + 2^{12}) \sum_{k=0}^2 2^{2k} \sigma^{-k} \cdot a \\ &= \frac{1}{3} (1 + 4\sigma^{-1} + 3\sigma^{-2})a = -4(1 - 4\sigma^2 - 3\sigma)a = (2\sigma + 6)a = \sigma(a)^2 \cdot a^6 \end{aligned}$$

in view of (7.1). Hence $\varepsilon_2 a = a$ iff $\sigma(a)^2 \cdot a^5 = 1$, equivalently, $\sigma(a) = a^4$.

Now let $\sigma(a) = a^4$. Then $(1 + \sigma^2 + \sigma^4)a = a \cdot a^{4^2} \cdot a^{4^4} = a^{13} = 1$. Similarly $(1 + \sigma^3)a = a \cdot a^{4^3} = a \cdot a^{-1} = 1$. Consequently, (7.1) holds, and from the above it follows that $\varepsilon_2 a = a$.

The proof of the last equivalence is similar. $\square$

Lemma 7.2. *Assume that the class numbers of the fields E_1 and E_2 are not divisible by 13. Then*

$$13\text{-rank}(K_2\mathcal{O}_F) = \sum_{j=1}^2 13\text{-rank}\left(\varepsilon_2\left(U'_{E_j}/U_{E_j}^{13}\right)\right).$$

Proof. This follows from (5.6). $\square$

The field E_j is totally real of degree 6, so there are five fundamental units in E_j . The automorphism σ acts on the linear space $U_{E_j}/U_{E_j}^{13} = \mathbb{F}_{13}^5$ over $\mathbb{F}_{13}$. Denote by $V_j^{(\lambda)}$ the eigenspace corresponding to an eigenvalue λ . It is known (see [Ma]) that the characteristic polynomial of the action of σ on this linear space is equal to $(X^6 - 1)/(X - 1)$. Since the polynomial has five distinct roots in $\mathbb{F}_{13}$:

$$(X^6 - 1)/(X - 1) = (X + 1)(X - 3)(X - 4)(X - 9)(X - 10),$$

it follows that the eigenspaces $V_j^{(\lambda)}$ corresponding to the eigenvalues $\lambda = -1, 3, 4, 9, 10$ are 1-dimensional. In particular, $\dim V_j^{(4)} = 1$.

Lemma 7.3. *Under the above notation we have*

$$(7.3) \quad V_j^{(4)} \cap \left(U'_{E_j}/U_{E_j}^{13}\right) \subseteq \varepsilon_2\left(U'_{E_j}/U_{E_j}^{13}\right),$$

and if in F_j there are no nontrivial singular primary units, the equality holds in (7.3).

Proof. From Lemma 7.1 (ii) it follows that if $u \in V_j^{(4)}$, i.e., $\sigma(u) = u^4$, then $\varepsilon_2 u = u$. Hence $u \in \varepsilon_2\left(U'_{E_j}/U_{E_j}^{13}\right)$.

Conversely, if $u \in U'_{E_j}/U_{E_j}^{13}$, $\varepsilon_2 u = u$, and there are no nontrivial singular primary units in F_j , then the assumption (7.1) of Lemma 7.1 is satisfied, and by part (ii) of that lemma we get $\sigma(u) = u^4$, i.e., $u \in V_j^{(4)}$. $\square$

It turns out that the 13-primary parts of the class groups of the fields F_j and E_j , $j = 1, 2$, are trivial for all fields F mentioned above. Moreover, in F_j , $j = 1, 2$, there are no nontrivial singular primary units, with one exception: $p = 487$, and the field E_2 . Nevertheless, in this exceptional case, we have $N_{E_2/F_2}(u) = 1$ for every $u \in U'_{E_2}/U_{E_2}^{13}$. Therefore, the equality always holds in (7.3), and this makes it possible to determine the 13-rank of $K_2\mathcal{O}_F$ for every field in Table 1.

First for every field E_j we have determined a unit u generating the 1-dimensional subspace $V_j^{(4)}$, and next we have verified whether u is singular primary.

Similarly to the case $q = 7$ the minimal polynomial for ρ_j is an Eisenstein polynomial with respect to $q = 13$, and every unit $u \in U_{E_j}$ can be written in the form

$$u = a_0 + a_1x + \cdots + a_5x^5,$$

where $x = \rho_j$, $a_0, \dots, a_5 \in \mathbb{Q}$, and $v_{13}(a_0) = 0$. We can assume that $a_0 \equiv 1 \pmod{13}$, replacing u by u^{12} .

Then u is a singular primary unit iff $a_0 \equiv 1 \pmod{13^2}$ and $a_i \equiv 0 \pmod{13}$ for $i = 1, \dots, 5$. Table 3 (at the end of the paper) gives the results concerning the 13-rank of $K_2\mathcal{O}_F$. We use the following notation: $v_{13} := v_{13}(\#K_2\mathcal{O}_F)$, $d_j := \dim(V_j^{(4)} \cap (U'_{E_j}/U_{E_j}^{13}))$, and $(K_2\mathcal{O}_F)_{13}$ denotes the 13-primary part of $K_2\mathcal{O}_F$.

8. DESCRIPTION OF TABLE 1

The first column of the table lists all primes $p \equiv 1 \pmod{6}$, $p < 5000$, and the second the corresponding values of A . The third and fourth columns give the orders of $K_2\mathcal{O}_F$; moreover, the fourth column provides information about the structure of the group $K_2\mathcal{O}_F$. To save space we use the following convention. If the order of a group is written in the form $(n_1)^{k_1}(n_2)^{k_2}\dots$, where $(n_j)^{k_j}$ is in roman font, it means that the group is isomorphic to the product of k_1 copies of $\mathbb{Z}/n_1$, k_2 copies of $\mathbb{Z}/n_2$, etc. However, if a factor $(n_j)^{k_j}$ is written in **bold** type, it means that there is a direct summand of order $n_j^{k_j}$, but its structure is unknown. Thus $(2)^2$ means $\mathbb{Z}/2 \times \mathbb{Z}/2$, and (2^2) means $\mathbb{Z}/4$, but **(31^2)** means a group of order 31^2 , i.e., one of the groups $\mathbb{Z}/31 \times \mathbb{Z}/31$ and $\mathbb{Z}/31^2$.

The fifth column of the table gives the class group of F , provided it is nontrivial. We use the same convention here, e.g. $(2)^2$ means $\mathbb{Z}/2 \times \mathbb{Z}/2$.

The structure of the 2-primary part of $K_2\mathcal{O}_F$ can be determined as follows. The ideal (2) splits in $\mathcal{O}_F$ into nonprincipal ideals iff p belongs to the following list: 277, 397, 1399, 1459, 1699, 1777, 1789, 1831, 2689, 3271, 4339, 4567. It turns out that in all these cases the 2-primary part of the group $Cl(\mathcal{O}_{F,2})$ is trivial, even for $p = 1777$, when $Cl(\mathcal{O}_F) = \mathbb{Z}/4 \times \mathbb{Z}/4$.

For other fields in the table, 2-rank $Cl(\mathcal{O}_{F,2}) = 2$ -rank $Cl(\mathcal{O}_F)$. Consequently, by Lemma 3.2, 2-rank $(K_2\mathcal{O}_F) = 3$ or 5 for every $p < 5000$.

Now we can apply the second inequality of Corollary 3.6. Namely, it turns out that the fundamental unit ε_1 is totally positive iff $p = 1009$ or 4567 (for $p < 5000$). Fortunately, for these two fields the corresponding elements γ_j are not totally positive. Then by Corollary 3.6, 4-rank $(K_2\mathcal{O}_F) \leq 2$ -rank $(K_2\mathcal{O}_F) - 3$. Hence 4-rank $(K_2\mathcal{O}_F) = 0$ or 2, and if 2-rank $(K_2\mathcal{O}_F) = 3$, then 4-rank $(K_2\mathcal{O}_F) = 0$, for $p < 5000$. It follows that if $e_2 := 2$ -rank $(K_2\mathcal{O}_F)$ and $v_2(\#K_2\mathcal{O}_F) = r$, then the 2-primary part of $K_2\mathcal{O}_F$ is isomorphic to

$$(\mathbb{Z}/2)^{e_2-2} \times (\mathbb{Z}/2^s)^2, \quad \text{where } s = (r - e_2)/2 + 1.$$

To determine the structure of the 3-primary part of $K_2\mathcal{O}_F$ we use Theorem 4.7: 3-rank $(K_2\mathcal{O}_F) = 3$ -rank $(Cl(\mathcal{O}_E))$. It can be verified that for all $p < 5000$ we have 3-rank $(Cl(\mathcal{O}_E)) \leq 2$. Then the structure of $K_2\mathcal{O}_F$ is determined uniquely, unless 3-rank $(Cl(\mathcal{O}_E)) = 2$ and $v_3(\#K_2\mathcal{O}_F) > 3$. The latter is the case for $p = 1747$, 2593, 3061, 4789, where the 9-rank of $K_2\mathcal{O}_F$ is unknown.

The following facts have been verified for $p < 5000$, but we do not know if they are true in general.

- (i) If $9 \mid \#K_2\mathcal{O}_F$, then 3-rank $(K_2\mathcal{O}_F) = 2$.
- (ii) $3 \parallel \#K_2\mathcal{O}_F$ iff $3 \parallel \#Cl(\mathcal{O}_E)$.
- (iii) $9 \parallel \#K_2\mathcal{O}_F$ iff $\text{Syl}_3 Cl(\mathcal{O}_E) = \mathbb{Z}/3 \times \mathbb{Z}/3$.
- (iv) $27 \parallel \#K_2\mathcal{O}_F$ iff $\mathbb{Z}/3 \times \mathbb{Z}/9 \subseteq Cl(\mathcal{O}_E)$.

Thus the 3-primary part of the group $Cl(\mathcal{O}_E)$ is never cyclic provided $9 \mid \#K_2\mathcal{O}_F$.

TABLE 1.

p	A	$\#K_2\mathcal{O}_F$	$K_2\mathcal{O}_F$	$Cl(\mathcal{O}_F)$
7	1	8	$(2)^3$	$(2)^2$
13	-5	8	$(2)^3$	
19	7	24	$(2)^3(3)$	
31	4	224	$(2)^5(7)$	
37	-11	168	$(2)^3(3)(7)$	
43	-8	608	$(2)^5(19)$	
61	1	1064	$(2)^3(7)(19)$	
67	-5	1544	$(2)^3(193)$	
73	7	1896	$(2)^3(3)(79)$	
79	-17	1592	$(2)^3(199)$	
97	19	2936	$(2)^3(367)$	
103	13	5096	$(2)^3(7^2)(13)$	
109	-2	9888	$(2)^5(3)(103)$	
127	-20	17376	$(2)^5(3)(181)$	
139	-23	8600	$(2)^3(5)^2(43)$	
151	19	15848	$(2)^3(7)(283)$	
157	-14	30944	$(2)^5(967)$	
163	25	16224	$(2)^5(3)(13^2)$	
181	7	23064	$(2)^3(3)(\mathbf{31}^2)$	
193	-23	34328	$(2)^3(7)(613)$	
199	-11	29448	$(2)^3(3)^2(409)$	
211	13	35528	$(2)^3(4441)$	
223	28	90272	$(2)^5(7)(13)(31)$	
229	22	94304	$(2)^5(7)(421)$	
241	-17	51464	$(2)^3(7)(919)$	
271	-29	93816	$(2)^3(3)^2(1303)$	$(2)^2$
277	-26	162944	$(2)^3(2^2)^2(19)(67)$	
283	-32	171104	$(2)^5(5347)$	
307	16	311424	$(2)^3(2^2)^2(3)(811)$	(7)
313	-35	119336	$(2)^3(7)(2131)$	
331	1	122888	$(2)^3(15361)$	
337	-5	152264	$(2)^3(7)(2719)$	$(2)^2$
349	37	141728	$(2)^5(43)(103)$	
367	-35	270392	$(2)^3(73)(463)$	
373	13	184232	$(2)^3(23029)$	$(2)^2$
379	-29	198744	$(2)^3(3)(7^2)(13)^2$	
397	34	472704	$(2)^3(2^2)^2(3)(1231)$	
409	31	260648	$(2)^3(31)(1051)$	
421	19	261704	$(2)^3(32713)$	
433	-2	606336	$(2)^3(2^2)^2(3)(1579)$	
439	28	952448	$(2)^3(2^2)^2(7)(1063)$	
457	10	832544	$(2)^5(26017)$	

TABLE 1. (continued)

p	A	$\#K_2\mathcal{O}_F$	$K_2\mathcal{O}_F$	$Cl(\mathcal{O}_F)$
463	-23	338600	$(2)^3(5)^2(1693)$	$(2)^2$
487	25	450216	$(2)^3(3)^2(13)^2(37)$	
499	-32	1344032	$(2)^5(97)(433)$	
523	43	674856	$(2)^3(3)^2(7)(13)(103)$	
541	-29	543528	$(2)^3(3)^2(7549)$	
547	1	746144	$(2)^5(7)(3331)$	
571	31	635768	$(2)^3(7)(11353)$	
577	-11	908088	$(2)^3(3)(157)(241)$	
601	-26	1649504	$(2)^5(19)(2713)$	
607	49	759776	$(2)^5(23743)$	$(2)^2$
613	-47	1067928	$(2)^3(3)(44497)$	
619	-17	1115912	$(2)^3(7)(19927)$	
631	43	905064	$(2)^3(3)(43)(877)$	
643	40	3180512	$(2)^5(99391)$	
661	49	1434104	$(2)^3(7)(25609)$	
673	37	1043672	$(2)^3(7)(18637)$	
691	-8	2807168	$(2)^3(2^2)^2(7)(13)(241)$	
709	-53	1159328	$(2)^5(36229)$	
727	-44	4195808	$(2)^5(19)(67)(103)$	$(2)^2$
733	-50	3434144	$(2)^5(7)(15331)$	
739	16	3466752	$(2)^3(2^3)^2(3)(37)(61)$	
751	-41	1439672	$(2)^3(13)(109)(127)$	
757	-29	1975272	$(2)^3(3)(13^2)(487)$	
769	49	1763528	$(2)^3(13)(31)(547)$	
787	31	2218136	$(2)^3(19)(14593)$	
811	-56	4291872	$(2)^5(3)(13)(19)(181)$	
823	-5	2084600	$(2)^3(5)^2(7)(1489)$	
829	7	1997304	$(2)^3(3)(83221)$	$(2)^2$
853	-35	3450272	$(2)^5(7)(73)(211)$	
859	13	2109992	$(2)^3(73)(3613)$	
877	-59	2241848	$(2)^3(7)(7^2)(19)(43)$	
883	-47	2377608	$(2)^3(3)(157)(631)$	
907	19	2474312	$(2)^3(309289)$	
919	52	8472672	$(2)^5(3)^2(13)(31)(73)$	
937	61	2641632	$(2)^5(3)(7)(3931)$	
967	-41	4288664	$(2)^3(31)(17293)$	
991	61	4490088	$(2)^3(3)(61)(3067)$	$(2)^2$
997	10	11895008	$(2)^5(371719)$	
1009	43	4821504	$(2)^3(2^3)^2(3)(43)(73)$	
1021	-14	12182144	$(2)^3(2^2)^2(13)(7321)$	
1033	-53	3748376	$(2)^3(103)(4549)$	
1039	-59	4077704	$(2)^3(19)(139)(193)$	

TABLE 1. (continued)

p	A	$\#K_2\mathcal{O}_F$	$K_2\mathcal{O}_F$	$Cl(\mathcal{O}_F)$
1051	64	8639072	$(2)^5(13)(19)(1093)$	(13)
1063	-65	4472664	$(2)^3(3)(7)(79)(337)$	
1069	-62	9159584	$(2)^5(7)(103)(397)$	
1087	55	5176088	$(2)^3(647011)$	
1093	22	14307488	$(2)^5(13)(163)(211)$	(7)
1117	-65	7521864	$(2)^3(3)(7)(44773)$	
1123	-35	5642744	$(2)^3(31)(61)(373)$	
1129	67	4681832	$(2)^3(37)(15817)$	
1153	7	5324856	$(2)^3(3)(127)(1747)$	
1171	-11	5391816	$(2)^3(3)(271)(829)$	
1201	-59	6078488	$(2)^3(13)(211)(277)$	
1213	-17	5861816	$(2)^3(853)(859)$	
1231	19	6344408	$(2)^3(7)(277)(409)$	
1237	-41	6324152	$(2)^3(790519)$	
1249	-53	8869736	$(2)^3(1108717)$	
1279	43	6901272	$(2)^3(3)^2(7)(13693)$	
1291	67	7776776	$(2)^3(7)(19)(7309)$	
1297	25	8220744	$(2)^3(3)(3^2)(7)(5437)$	
1303	55	11756264	$(2)^3(13)(113041)$	
1321	-71	10453448	$(2)^3(31)(61)(691)$	
1327	4	18672032	$(2)^5(583501)$	$(2)^2$
1381	31	8621288	$(2)^3(13)(19)(4363)$	
1399	-68	30029696	$(2)^3(2^2)^2(283)(829)$	
1423	-20	26058336	$(2)^5(3)(521)^2$	
1429	-71	10536248	$(2)^3(1317031)$	
1447	-35	12013736	$(2)^3(7)(214531)$	
1453	67	10408088	$(2)^3(1301011)$	
1459	-56	28097664	$(2)^3(2^2)^2(3)(7)(10453)$	
1471	76	24495200	$(2)^5(5)^2(67)(457)$	(13)
1483	37	10679384	$(2)^3(37)(109)(331)$	
1489	-77	11908904	$(2)^3(7)(109)(1951)$	
1531	7	19844424	$(2)^3(3)^2(367)(751)$	
1543	-77	18481832	$(2)^3(19)(121591)$	(19)
1549	-11	20078664	$(2)^3(3)(836611)$	
1567	79	12582024	$(2)^3(3)(7)^2(13)(823)$	
1579	-32	32194208	$(2)^5(19)(52951)$	
1597	-50	48989024	$(2)^5(7^2)(157)(199)$	
1609	19	21766568	$(2)^3(2720821)$	
1621	79	19523448	$(2)^3(3)^2(7)(38737)$	
1627	-80	38016992	$(2)^5(13)(91387)$	
1657	70	41556192	$(2)^5(3)(19)(22783)$	
1663	73	15879128	$(2)^3(1984891)$	
				(7)

TABLE 1. (continued)

p	A	$\#K_2\mathcal{O}_F$	$K_2\mathcal{O}_F$	$Cl(\mathcal{O}_F)$
1669	67	21204728	$(2)^3(139)(19069)$	$(2)^2$
1693	-47	16202232	$(2)^3(3)^2(307)(733)$	
1699	64	42115712	$(2)^3(2^2)^2(151)(2179)$	
1723	40	46730528	$(2)^5(13)^2(8641)$	
1741	49	18534152	$(2)^3(7^2)(13)(3637)$	
1747	61	17597736	$(2)^3(\mathbf{3^4})(13)(2089)$	$(2)^2$
1753	10	46850528	$(2)^5(1464079)$	
1759	31	28347032	$(2)^3(7)(37)(13681)$	
1777	-14	45192704	$(2)^3(2^3)^2(61)(1447)$	
1783	-83	26028744	$(2)^3(3)(7)(154933)$	
1789	82	44748416	$(2)^3(2^2)^2(73)(4789)$	(7)
1801	-74	44090592	$(2)^5(3)(7)(7^2)(13)(103)$	
1831	-68	53752736	$(2)^5(1679773)$	
1861	37	33785624	$(2)^3(853)(4951)$	
1867	85	34006856	$(2)^3(13^2)(25153)$	
1873	-65	25249128	$(2)^3(3)(31)(33937)$	$(2)^2$
1879	73	30150368	$(2)^5(942199)$	
1933	-62	79342496	$(2)^5(2479453)$	
1951	1	24210656	$(2)^5(61)(79)(157)$	
1987	-89	25099304	$(2)^3(19)(61)(2707)$	
1993	13	26971448	$(2)^3(7)(481633)$	$(2)^2$
1999	52	64523808	$(2)^5(3)^2(224041)$	
2011	-59	26608088	$(2)^3(13)(255847)$	
2017	34	62057088	$(2)^3(2^2)^2(3)^2(103)(523)$	
2029	-77	41567288	$(2)^3(7^2)(19)(5581)$	
2053	-83	30386616	$(2)^3(3)(13)(17)^2(337)$	$(2)^2$
2083	-23	29384504	$(2)^3(733)(5011)$	
2089	-38	69126432	$(2)^5(3)(223)(3229)$	
2113	82	70391072	$(2)^5(61)(36061)$	
2131	91	48170528	$(2)^5(7)^2(31)(991)$	
2137	85	38567048	$(2)^3(13)(370837)$	$(2)^2$
2143	-92	73613664	$(2)^5(3)(3^2)(85201)$	
2161	-29	32856264	$(2)^3(3)^2(7^2)(67)(139)$	
2179	88	113959392	$(2)^5(3)(157)(7561)$	
2203	-8	116452064	$(2)^5(19)(191533)$	
2221	-53	56434376	$(2)^3(7054297)$	$(2)^2$
2239	91	44756552	$(2)^3(19)(277)(1063)$	
2251	16	125175264	$(2)^5(3)(127)(10267)$	
2269	-83	53021592	$(2)^3(3)^2(13)(37)(1531)$	
2281	-86	90707552	$(2)^5(13)(218047)$	

TABLE 1. (continued)

p	A	$\#K_2\mathcal{O}_F$	$K_2\mathcal{O}_F$	$Cl(\mathcal{O}_F)$
2287	-20	144755808	$(2)^5(3)(79)(19087)$	$(2)^2$
2293	37	39299096	$(2)^3(991)(4957)$	
2311	-89	42889952	$(2)^5(7)(191473)$	
2341	-74	139225632	$(2)^5(3)(7)(13)(15937)$	
2347	64	105761888	$(2)^5(1039)(3181)$	
2371	-41	43322888	$(2)^3(7)(\mathbf{19}^2)(2143)$	
2377	79	43970328	$(2)^3(3)^2(610699)$	
2383	28	157511072	$(2)^5(37)(133033)$	
2389	-59	70897688	$(2)^3(97)(211)(433)$	
2437	1	48333848	$(2)^3(6041731)$	(7)
2467	-11	50148936	$(2)^3(3)(3^2)(232171)$	
2473	73	50147624	$(2)^3(67)(93559)$	
2503	-47	50764200	$(2)^3(3)(5)^2(19)(61)(73)$	
2521	97	59782056	$(2)^3(3)(19)(131101)$	
2539	-83	53568600	$(2)^3(3)(5)^2(19)(37)(127)$	
2551	49	57461816	$(2)^3(7182727)$	
2557	-101	53301624	$(2)^3(3)(439)(5059)$	
2593	25	63726264	$(2)^3(\mathbf{3}^6)(7^2)(223)$	
2617	91	90367832	$(2)^3(11295979)$	
2647	-29	60219096	$(2)^3(3)(7)(358447)$	(19)
2659	103	63408968	$(2)^3(7)(73)(15511)$	
2671	-44	210615392	$(2)^5(13)(283)(1789)$	
2677	31	62355176	$(2)^3(13)(61)(9829)$	
2683	97	68405928	$(2)^3(3)(19)(67)(2239)$	
2689	-62	148030976	$(2)^3(2^3)^2(19)(15217)$	
2707	55	74568200	$(2)^3(5)^2(7^3)(1087)$	
2713	103	91568648	$(2)^3(103)(111127)$	
2719	-101	74672952	$(2)^3(3)(127)(24499)$	
2731	-104	154184576	$(2)^3(2^2)^2(7^2)(13)(31)(61)$	
2749	-14	191244800	$(2)^3(2^3)^2(5)^2(67)(223)$	$(2)^2$
2767	76	176114144	$(2)^5(5503567)$	
2791	-92	188965248	$(2)^3(2^2)^2(3)(163)(3019)$	
2797	-89	73500128	$(2)^5(13^2)(13591)$	
2803	-95	117740000	$(2)^5(\mathbf{5}^4)(7)(29)^2$	
2833	-98	187258496	$(2)^3(2^2)^2(1462957)$	
2851	73	119950136	$(2)^3(673)(22279)$	
2857	-41	75490952	$(2)^3(19)(31)(37)(433)$	
2887	91	86493368	$(2)^3(13)(349)(2383)$	
2917	106	186688224	$(2)^5(3)^2(19)(109)(313)$	
2953	70	231006048	$(2)^5(3)(7)(13)(31)(853)$	$(2)^2$
2971	-56	298295808	$(2)^3(2^3)^2(3)(194203)$	

TABLE 1. (continued)

p	A	$\#K_2\mathcal{O}_F$	$K_2\mathcal{O}_F$	$Cl(\mathcal{O}_F)$
3001	-77	151503416	$(2)^3(19)(409)(2437)$	$(2)^2$
3019	13	136579688	$(2)^3(7)(907)(2689)$	
3037	49	96292448	$(2)^5(7^3)(31)(283)$	
3049	-17	139208552	$(2)^3(7)(2485867)$	
3061	-38	245335392	$(2)^5(\mathbf{3}^4)(94651)$	
3067	19	143062472	$(2)^3(7)(367)(6961)$	
3079	79	152143368	$(2)^3(3)(13)(337)(1447)$	
3109	-23	146303288	$(2)^3(109)(167779)$	
3121	-89	100077224	$(2)^3(13)(379)(2539)$	
3163	112	252655616	$(2)^3(2^4)^2(19)(43)(151)$	
3169	97	105161208	$(2)^3(3)(907)(4831)$	
3181	94	341303936	$(2)^3(2^2)^2(139)(19183)$	
3187	-29	162777192	$(2)^3(3)(619)(10957)$	
3217	31	164205608	$(2)^3(7)(79)(37117)$	(7)
3229	46	291821984	$(2)^5(67)(136111)$	
3253	-113	156813992	$(2)^3(19)(37)(27883)$	$(2)^2$
3259	88	285120384	$(2)^3(2^2)^2(3)(19)(39079)$	
3271	4	272604032	$(2)^3(2^2)^2(2129719)$	
3301	109	122600504	$(2)^3(13)(1178851)$	
3307	-59	118606136	$(2)^3(11)^2(122527)$	
3313	115	132209448	$(2)^3(3)(7^2)(19)(61)(97)$	
3319	37	183639176	$(2)^3(7)(23)^2(6199)$	
3331	16	282038784	$(2)^3(2^3)^2(3)(139)(1321)$	
3343	-68	400784864	$(2)^5(31)(404017)$	
3361	-113	138729752	$(2)^3(7)(2477317)$	
3373	-98	436401824	$(2)^5(379)(35983)$	
3391	-116	293541152	$(2)^5(9173161)$	
3433	-77	143234072	$(2)^3(1483)(12073)$	
3457	-110	362770944	$(2)^3(2^3)^2(3)(103)(2293)$	
3463	28	344540192	$(2)^5(10766881)$	(19)
3469	103	138196856	$(2)^3(7)^2(352543)$	
3499	-89	223275464	$(2)^3(37)(73)(10333)$	
3511	79	143220168	$(2)^3(3^3)(2)(7)(13)(21859)$	
3517	109	201534944	$(2)^5(13)(484459)$	
3529	-47	217436856	$(2)^3(3)(7)(13)(99559)$	
3541	58	395526368	$(2)^5(12360199)$	
3547	-119	153726792	$(2)^3(3)(6405283)$	
3559	67	152045816	$(2)^3(13)(1461979)$	
3571	1	147892568	$(2)^3(31)(596341)$	
3583	7	161986392	$(2)^3(\mathbf{3}^4)(457)(547)$	
3607	-116	504481952	$(2)^5(13)(1212697)$	

TABLE 1. (continued)

p	A	$\#K_2\mathcal{O}_F$	$K_2\mathcal{O}_F$	$Cl(\mathcal{O}_F)$
3613	13	154051112	$(2)^3(43)(447823)$	$(2)^2$
3631	76	518241248	$(2)^5(7)^2(67)(4933)$	
3637	115	192358872	$(2)^3(3)^2(2671651)$	
3643	-17	158957528	$(2)^3(19869691)$	
3673	-83	159534648	$(2)^3(3)^2(7)(13^2)(1873)$	
3691	-101	163217832	$(2)^3(3)(6800743)$	
3697	-71	167071592	$(2)^3(409)(51061)$	
3709	-119	203851464	$(2)^3(3)(8493811)$	
3727	25	188158752	$(2)^5(3)^2(103)(6343)$	
3733	55	290980136	$(2)^3(19)(31)(37)(1669)$	
3739	-113	247660184	$(2)^3(30957523)$	
3769	73	172890536	$(2)^3(7)(13)(237487)$	
3793	103	178884776	$(2)^3(7)(3194371)$	
3823	100	513848672	$(2)^5(109)(147319)$	
3847	-59	283234088	$(2)^3(1621)(21841)$	
3853	115	299061672	$(2)^3(3)(7)(13)(19)(7207)$	
3877	-35	228407816	$(2)^3(7)(7^2)(13)(19)(337)$	
3889	-2	639625056	$(2)^5(3)(7)(103)(9241)$	
3907	61	288686664	$(2)^3(3)^2(7)(572791)$	
3919	-77	213262376	$(2)^3(37)(720481)$	
3931	-89	198399272	$(2)^3(24799909)$	
3943	52	477244032	$(2)^3(2^2)^2(3)(1242823)$	$(2)^2$
3967	-125	325012856	$(2)^3(7)(97)(59833)$	
4003	-107	215594744	$(2)^3(67)(241)(1669)$	
4021	91	226861496	$(2)^3(28357687)$	
4027	-104	527257472	$(2)^3(2^2)^2(7)(193)(3049)$	
4051	-56	551322528	$(2)^5(3)(1933)(2971)$	
4057	-26	729226976	$(2)^5(22788343)$	
4093	127	320646728	$(2)^3(40080841)$	
4099	67	332880224	$(2)^5(193)(53899)$	
4111	109	226897784	$(2)^3(28362223)$	
4129	-98	557796224	$(2)^3(2^2)^2(19)(211)(1087)$	$(2)^2(7)$
4153	-122	558884768	$(2)^5(13)(193)(6961)$	
4159	-83	234335448	$(2)^3(3)^2(31)(67)(1567)$	
4177	34	793463904	$(2)^5(3)(3^2)(918361)$	
4201	127	267278408	$(2)^3(103)(277)(1171)$	
4219	1	288301664	$(2)^5(7)(1287061)$	
4231	7	294942552	$(2)^3(3)(211)(58243)$	
4243	85	286981544	$(2)^3(35872693)$	
4261	13	287072096	$(2)^5(79)(113557)$	
4273	-53	251825912	$(2)^3(13)(2421403)$	

TABLE 1. (continued)

p	A	$\#K_2\mathcal{O}_F$	$K_2\mathcal{O}_F$	$Cl(\mathcal{O}_F)$
4297	-131	258540416	$(2)^3(2^2)^2(2019847)$	$(2^2)^2$
4327	55	308430968	$(2)^3(38553871)$	
4339	-128	872818656	$(2)^5(3)(19)(151)(3169)$	(7)
4357	-119	300064128	$(2)^3(2^2)^2(3)(7)(13)(31)(277)$	$(2)^2$
4363	127	293035496	$(2)^3(13)(2817649)$	
4423	-68	678133376	$(2)^3(2^2)^2(5297917)$	
4441	-59	291978248	$(2)^3(37)(241)(4093)$	
4447	115	327955992	$(2)^3(3)(7)(13^2)(11551)$	
4483	133	458846808	$(2)^3(3)(7)(19)(43)(3343)$	
4507	91	323606792	$(2)^3(1693)(23893)$	
4513	-50	1127785568	$(2)^5(7)^2(13)(61)(907)$	
4519	124	789244128	$(2)^5(3)(3^2)(31)(79)(373)$	
4549	-86	810085472	$(2)^5(7)(3616453)$	
4561	37	346779776	$(2)^3(2^2)^2(7)(387031)$	$(2)^2$
4567	4	735532544	$(2)^3(2^3)^2(43)(33409)$	$(2)^2$
4591	133	375756264	$(2)^3(3^2)(13)(157)(2557)$	
4597	-134	747504224	$(2)^5(2551)(9157)$	
4603	103	318625496	$(2)^3(7)(5689741)$	
4621	-131	342124616	$(2)^3(42765577)$	
4639	-41	368370176	$(2)^3(2^3)^2(\mathbf{19}^2)(1993)$	$(2)^2$
4651	136	759974816	$(2)^5(307)(77359)$	
4657	130	871034528	$(2)^5(7)(13)(31)(9649)$	
4663	-20	867009408	$(2)^3(2^2)^2(3)(2257837)$	
4723	-125	391395992	$(2)^3(13)(3763423)$	
4729	58	1160693888	$(2)^3(2^2)^2(19)(477259)$	
4759	28	893525024	$(2)^5(7)(229)(17419)$	
4783	85	608581928	$(2)^3(76072741)$	(7)
4789	97	361591776	$(2)^5(\mathbf{3}^5)(7^2)(13)(73)$	$(2)^2$
4801	121	374007776	$(2)^5(109)(107227)$	$(2)^2$
4813	-107	360943256	$(2)^3(45117907)$	
4831	-71	366576728	$(2)^3(7)(\mathbf{19}^2)(18133)$	
4861	-137	417409800	$(2)^3(3)(5)^2(757)(919)$	
4903	73	389371256	$(2)^3(19)(2561653)$	
4909	94	1006854752	$(2)^5(193)(163027)$	
4933	7	586541592	$(2)^3(3)^2(7)(13)(89521)$	
4951	-11	566435016	$(2)^3(3)^2(7)(1123879)$	
4957	-89	594452264	$(2)^3(7)(10615219)$	
4969	133	603100536	$(2)^3(3)(\mathbf{31}^2)(79)(331)$	
4987	-101	408631704	$(2)^3(3)(13)(1309717)$	
4993	-17	569508056	$(2)^3(13)(5476039)$	
4999	139	450022664	$(2)^3(7^2)(13^2)(6793)$	

TABLE 2.

p	v_7	d_1	d_2	h_1	h_2	7-rank $K_2\mathcal{O}_F$	$(K_2\mathcal{O}_F)_7$
103	2	1	0	0	0	1	(7^2)
379	2	0	1	0	0	1	(7^2)
877	3	1	1	0	0	2	$(7)(7^2)$
1567	2	1	1	0	0	2	$(7)^2$
1597	2	1	0	0	0	1	(7^2)
1741	2	1	0	0	0	1	(7^2)
1801	3	1	1	0	0	2	$(7)(7^2)$
2029	2	0	1	0	0	1	(7^2)
2131	2	1	1	0	0	2	$(7)^2$
2161	2	1	0	0	0	1	(7^2)
2593	2	0	1	0	0	1	(7^2)
2707	3	1	0	0	0	1	(7^3)
2731	2	0	1	0	0	1	(7^2)
3037	3	0	1	0	0	1	(7^3)
3313	2	0	0	0	1	1	(7^2)
3469	2	1	1	0	0	2	$(7)^2$
3631	2	1	1	0	0	2	$(7)^2$
3877	3	1	1	0	0	2	$(7)(7^2)$
4513	2	1	1	≤ 1	0	2	$(7)^2$
4789	2	1	0	0	0	1	(7^2)
4999	2	0	1	0	0	1	(7^2)

TABLE 3.

p	v_{13}	d_1	d_2	13-rank $K_2\mathcal{O}_F$	$(K_2\mathcal{O}_F)_{13}$
163	2	1	0	1	(13^2)
379	2	1	1	2	$(13)^2$
487	2	1	1	2	$(13)^2$
757	2	1	0	1	(13^2)
1723	2	1	1	2	$(13)^2$
1867	2	0	1	1	(13^2)
2797	2	0	1	1	(13^2)
3673	2	1	0	1	(13^2)
4447	2	1	0	1	(13^2)
4999	2	1	0	1	(13^2)

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