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A NEW SUPERCONVERGENT COLLOCATION METHOD FOR EIGENVALUE PROBLEMS

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Abstract. Here we propose a new method based on projections for the approximate solution of eigenvalue problems. For an integral operator with a smooth kernel, using an interpolatory projection at Gauss points onto the space of (discontinuous) piecewise polynomials of degree $\leq r-1$, we show that the proposed method exhibits an error of the order of $4r$ for eigenvalue approximation and of the order of $3r$ for spectral subspace approximation. In the case of a simple eigenvalue, we show that by using an iteration technique, an eigenvector approximation of the order $4r$ can be obtained. This improves upon the order $2r$ for eigenvalue approximation in the collocation/iterated collocation method and the orders r and $2r$ for spectral subspace approximation in the collocation method and the iterated collocation method, respectively. We illustrate this improvement in the order of convergence by numerical examples.

1. Introduction

Consider the eigenvalue problem

$$T\phi = \lambda\phi, \tag{1.1}$$

where T is a compact linear operator defined on a complex Banach space. A standard technique for solving (1.1) approximately is to replace T by a finite rank operator. The approximate solution of (1.1) is then obtained by essentially solving a matrix eigenvalue problem. If π_n is a sequence of finite rank projection operators converging to the identity operator pointwise, then in the classical Galerkin method T is replaced by $T_n^G = \pi_n T \pi_n$, and in the iterated Galerkin method proposed by Sloan, T is replaced by $T_n^S = T\pi_n$. When π_n is an interpolatory projection, the choice of $T_n^C = \pi_n T \pi_n$ gives rise to a collocation method, whereas $T_n^S = T\pi_n$ is associated with the iterated collocation method. These methods are extensively studied in literature. (See, for example [1], [3], [4], [5], [6], [7], [10].)

We propose to approximate T by the finite rank operator

$$T_n^M = \pi_n T \pi_n + \pi_n T(I-\pi_n) + (I-\pi_n)T\pi_n.$$

Then $\|T - T_n^M\| = \|(I-\pi_n)T(I-\pi_n)\| \to 0$ as $n \to \infty$.

In this paper an integral operator

$$Tu(s) = \int_0^1 k(s,t)u(t)dt, \quad s \in [0,1],$$

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with a smooth kernel is considered. Note that $T : C[0,1] \to C[0,1]$ is compact.

Consider a partition of $[0,1]$ with n subintervals and norm h. The collocation points are chosen to be the nr Gauss points, obtained by shifting r Gauss points in $[-1,1]$ to each of the subintervals of the partition. Hence the interpolation points do not include the end points of the subintervals, i.e., the partition points. As a consequence, the interpolating piecewise polynomial is discontinuous at the partition points. Let X_n be the space of all (discontinuous) piecewise polynomials of degree $\leq r-1$ with respect to the above partition of $[0,1]$. For $f \in C[0,1]$, let $\pi_n f$ be the unique element of X_n which interpolates f at the nr collocation points. By using a result of [2], π_n can be extended to $L^\infty[0,1]$, and then $\pi_n : L^\infty[0,1] \to X_n$ is a projection.

Let λ be a nonzero eigenvalue of T with algebraic multiplicity m and let P be the associated spectral projection. Let $\hat{\lambda}_n^M$ and $\hat{\lambda}_n^C = \hat{\lambda}_n^S$ denote the arithmetic mean of the m eigenvalues of T_n^M and T_n^C (or T_n^S), respectively, which approximate λ. Let P_n^M, P_n^C and P_n^S denote the spectral projections associated with the group of m eigenvalues of T_n^M, T_n^C and T_n^S, respectively. We prove that

$$|\lambda - \hat{\lambda}_n^M| = O(h^{4r}) \text{ and } \hat{\delta}(R(P), R(P_n^M)) = O(h^{3r}),$$

where $\hat{\delta}$ denotes the gap between the subspaces.

We further consider the case when λ is a simple eigenvalue of T. Let

$$T_n^M \phi_n^M = \lambda_n^M \phi_n^M, \quad \|\phi_n^M\| = 1,$$

and

$$\psi_n^M = \frac{1}{\lambda_n^M} T \phi_n^M.$$

It is shown that

$$\|P\phi_n^M - \psi_n^M\| = O(h^{4r}).$$

The above estimates should be compared with the following known estimates for the collocation and the iterated collocation methods (see Chatelin [4]):

$$\begin{aligned} &|\lambda - \hat{\lambda}_n^C| = |\lambda - \hat{\lambda}_n^S| = O(h^{2r}), \\ &\hat{\delta}(R(P), R(P_n^C)) = O(h^r), \\ &\hat{\delta}(R(P), R(P_n^S)) = O(h^{2r}). \end{aligned}$$

Note that the range of T_n^M is contained in $X_n \cup T(X_n) = Y_n$, say. Since the dimension of X_n is nr, the dimension of Y_n is $\leq 2nr$. While considering the eigenvalue problem for T_n^M, it is enough to consider the restriction of T_n^M to Y_n. Thus the eigenvalue problem for T_n^M can be reduced to a matrix eigenvalue problem of size $2nr$, whereas the eigenvalue problem for T_n^C or T_n^S is equivalent to a matrix eigenvalue problem of size nr. In the case of an uniform partition of $[0,1]$, the norm of the partition is $h = \frac{1}{n}$. Thus, in the proposed method, an eigenvalue approximation of the order $(\frac{1}{n})^{4r}$ is obtained by solving an eigenvalue problem of size $2nr$, whereas in the collocation method, in order to achieve the same order of convergence, it is necessary to solve an eigenvalue problem of size n^2r. The numerical results given in Section 6 confirm this observation.

It is possible to extend our results in various directions. In [8] we have considered the case of the orthogonal projections. We can also use the new operator T_n^M for approximate solution of an operator equation. As we have introduced a sequence of finite rank operators converging to T in norm, we can define iterative refinement schemes for opertor equations as well as eigenvalue problems, multilevel methods, accelerated spectral approximation and extrapolation using this new operator. It is also possible to choose the interpolation points as Lobatto points. These issues will be studied in future papers.

Here is an outline of the paper. In Section 2 we set the notations and prove some preliminary results. In Section 3 we describe our new method in a general setting and obtain some error estimates for eigenelement approximation. In Section 4 we obtain precise orders of convergence for eigenvalue as well as the spectral subspace approximation of an integral operator with a smooth kernel. The projection operator is chosen to be the interpolatory projection at Gauss points. In Section 5, using an iteration technique, we obtain an eigenvector approximation of the order of $4r$. In Section 6 we illustrate our results by numerical examples.

2. Preliminaries

Let X be a complex Banach space and $\mathrm{BL}(X)$ the space of all bounded linear operators on X along with the operator norm. Let $T : X \to X$ be a compact linear operator and let $\sigma(T)$ and $\rho(T)$ denote the spectrum and the resolvent set of T, respectively. Let λ be a nonzero eigenvalue of T with algebraic multiplicity m. Let ϵ be such that $0 < \epsilon < dist(\lambda, \sigma(T) \setminus \{\lambda\})$ and Γ the positively oriented circle with center λ and radius ϵ. Then $\Gamma \subset \rho(T)$ and

$$\max_{z\in\Gamma} \|(T - zI)^{-1}\| \leq C_1.$$

Note that throughout this paper C_1, C_2, C_3 and C_4 are constants, and C denotes a generic constant, independent of n . Let

$$P = -\frac{1}{2\pi i}\int_\Gamma (T - zI)^{-1} dz,$$

the spectral projection associated with T and λ. Then rank $P = m$.

For nonzero subspaces Y and Z of X, let

$$\delta(Y, Z) = \sup\{ \operatorname{dist}(y, Z) : y \in Y, \ \ \|y\| = 1\}.$$

Then

$$\hat{\delta}(Y, Z) = \max\{\delta(Y, Z), \delta(Z, Y)\}$$

is known as the gap between Y and Z. For $S \in \mathrm{BL}(X)$, we denote by $R(S)$ the range space. Let $\delta = \min\{|z| : z \in \Gamma\} > 0$.

Let T_n be a sequence in BL converging to T in collectively compact fashion. We quote the following results from Osborn [9].

For all large $n, \Gamma \subset \rho(T_n)$ and $\max\{\|(T_n - zI)^{-1}\| : z \in \Gamma\} \leq C_2$.
As the spectral projection

$$P_n = -\frac{1}{2\pi i}\int_\Gamma (T_n - zI)^{-1} dz$$

is of rank m, the spectrum of T_n inside Γ consists of m eigenvalues $\lambda_{n,1}, \ldots, \lambda_{n,m}$, counted according to their algebraic multiplicities. Let

$$\hat{\lambda}_n = \frac{\lambda_{n,1} + \cdots + \lambda_{n,m}}{m}$$

denote their arithmetic mean.

Theorem 2.1 (Osborn [9])**.** *For all large n,*

$$\hat{\delta}(R(P), R(P_n)) \leq C\|(T - T_n)T\|.$$

Below we prove a modified version of Theorem 2 of Osborn [9].

Theorem 2.2. *For all large n,*

$$|\lambda - \hat{\lambda}_n| \leq C\|T_n(T - T_n)T\|.$$

Proof. Let $A_n = P_{n|R(P)} : R(P) \to R(P_n)$. The argument given in Theorem 2 of [9] shows that A_n is bijective and $\|A_n^{-1}\| \leq 2$ for all large n. Define $\hat{T} = T_{|R(P)}$ and $\hat{T}_n = A_n^{-1} T_n A_n$. Then

$$\begin{aligned} |\lambda - \hat{\lambda}_n| &= \frac{1}{m} |\mathrm{trace}(\hat{T} - \hat{T}_n)| \leq \|\hat{T} - \hat{T}_n\| \\ &= \sup\{ \|A_n^{-1} P_n (T - T_n) x\| : \ x \in R(P), \|x\| = 1 \ \} \\ &\leq 2\|P_n(T - T_n)P\|. \end{aligned}$$

Using the integral representations of P and P_n, it can be seen that

$$\|P_n(T - T_n)P\| \leq (\frac{\epsilon}{\delta})^2 C_1 C_2 \|T_n(T - T_n)T\|,$$

which proves the result. □

3. A new projection method

Let $\pi_n : X \to X$ be a sequence of bounded projections such that for each $x \in X$, $\pi_n x \to x$ as $n \to \infty$ and $X_n = R(\pi_n)$ is finite dimensional.

We propose to approximate

$$T\phi = \lambda\phi$$

by

$$(3.1) \qquad (\ \pi_n T \pi_n + \pi_n T(I - \pi_n) + (I - \pi_n) T \pi_n\ )\ \phi_n^M = \lambda_n^M \phi_n^M;$$

that is,

$$T_n^M \phi_n^M = \lambda_n^M \phi_n^M.$$

Since T_n^M converges to T in the norm, for all large n, T_n^M has m eigenvalues $\lambda_{n,1}^M, \ldots, \lambda_{n,m}^M$ inside Γ. Let $\hat{\lambda}_n^M$ denote the arithmetic mean of these m eigenvalues and let P_n^M denote the associated spectral projection. Then the following result follows from Theorems 2.1 and 2.2.

Theorem 3.1. *For all large n*

$$(3.2) \qquad \hat{\delta}(R(P), R(P_n^M)) \leq C\|(I - \pi_n)T(I - \pi_n)T\|,$$

$$(3.3) \qquad |\lambda - \hat{\lambda}_n^M| \leq C\|T(I - \pi_n)T(I - \pi_n)T\|.$$

Let $\hat{\lambda}_n^C = \hat{\lambda}_n^S$ denote the arithmetic mean of m eigenvalues of $T_n^C = \pi_n T \pi_n$ in the Galerkin method or $T_n^S = T\pi_n$ in the Sloan method, approximating eigenvalue λ of T. Let P_n^C and P_n^S denote the spectral projections associated with T_n^C and T_n^S, respectively. Then the following result follows from Theorems 2.1 and 2.2.

Theorem 3.2. *For all large n*

$$\hat{\delta}(R(P), R(P_n^C)) \leq C\|(T - \pi_n T \pi_n)T\|, \tag{3.4}$$

$$\hat{\delta}(R(P), R(P_n^S)) \leq C\|T(I - \pi_n)T\|, \tag{3.5}$$

$$|\lambda - \hat{\lambda}_n^C| = |\lambda - \hat{\lambda}_n^S| \leq C\|T(I - \pi_n)T\|. \tag{3.6}$$

A comparison of bounds (3.2)–(3.3) and (3.4)–(3.6) suggests that the eigenelement approximation using T_n^M may be better than Galerkin or Sloan approximation. In the next section it is shown that, in fact, the order of convergence is doubled by using the new operator.

4. Orders of convergence

Let $X = C[0,1]$ with the supremum norm. Choose $r \geq 1$ and assume that $k(.,.) \in C^{2r}([0,1]\times[0,1])$.

Consider the integral operator

$$(Tu)(s) = \int_0^1 k(s,t)u(t)dt, \quad s \in [0,1].$$

Then $T : C[0,1] \to C[0,1]$ is a compact linear operator. In fact, $R(T) \subset C^{2r}[0,1]$. For $u \in C^{2r}[0,1]$, $u^{(2r)}$ denotes the $2r$-th derivative of u. We set

$$D^{i,j}k(s,t) = \frac{\partial^{i+j}}{\partial s^i \partial t^j} k(s,t), \quad s,t \in [0,1],$$

$$\|k\|_{2r,\infty} = \sum_{i=0}^{2r} \sum_{j=0}^{2r} \|D^{i,j}k\|_\infty$$

and

$$\|u\|_{2r,\infty} = \sum_{i=0}^{2r} \|u^{(i)}\|_\infty.$$

Consider a partition

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

of $[0,1]$ and for $j = 1, \ldots, n$, set $h_j = t_j - t_{j-1}$, $\;h = \max\{h_j : j = 1, \ldots, n\}$. We assume that $h \to 0$ as $n \to \infty$. Let X_n be the space of all piecewise polynomials of order r (i.e., of degree $\leq r-1$) with breakpoints at $t_1, \ldots, t_{n-1}$. We impose no continuity conditions at the breakpoints.

Let $B_r = \{\tau_1, \ldots, \tau_r\}$ denote the set of r Gauss points, i.e., the zeros of the (Legendre) polynomial $\dfrac{d^r}{ds^r}(s^2-1)^r$ in the interval $[-1,1]$.

Define $f_j : [-1,1] \to [t_{j-1}, t_j]$ as

$$f_j(t) = \frac{1-t}{2} t_{j-1} + \frac{1+t}{2} t_j, \quad t \in [-1,1].$$

Let $A = \bigcup_{j=1}^n f_j(B_r)$, the set of nr Gauss points.

The map $\pi_n : C[0,1] \to X_n$ is defined by

$$\pi_n u \in X_n, \; (\pi_n u)(t) = u(t), \quad t \in A.$$

Then $\pi_n u \to u$ as $n \to \infty$ for each $u \in C[0,1]$ and the results of Section 3 are applicable. Note that $\pi_n u$ is, in general, discontinuous at the breakpoints.

In what follows we use crucially the following two estimates.

For $u \in C^r[0,1]$ (see Chatelin [4]),

$$\|(I-\pi_n)u\|_\infty \le C_3 \|u^{(r)}\|_\infty h^r. \tag{4.1}$$

Let $f \in C^r[0,1]$ and $g \in C^{2r}[0,1]$. Then (see de-Boor–Swartz [7])

$$\left| \int_0^1 f(t)(I-\pi_n)g(t)dt \right| \le C_4 \|f\|_{r,\infty} \|g\|_{2r,\infty} h^{2r}. \tag{4.2}$$

Theorem 4.1. *If $\pi_n : C[0,1] \to X_n$ is the interpolatory projection defined above and T is an integral operator with kernel $k(\cdot,\cdot) \in C^{2r}([0,1])\times[0,1])$, then*

$$\|(I-\pi_n)T\| = O(h^r), \tag{4.3}$$

$$\|T(I-\pi_n)T\| = O(h^{2r}), \tag{4.4}$$

$$\|(I-\pi_n)T(I-\pi_n)T\| = O(h^{3r}), \tag{4.5}$$

$$\|T(I-\pi_n)T(I-\pi_n)T\| = O(h^{4r}). \tag{4.6}$$

Proof. Let $u \in C[0,1]$. Since for $s \in [0,1]$ and $i = 0, 1 \ldots, 2r$,

$$(Tu)^{(i)}(s) = \int_0^1 \frac{\partial^i}{\partial s^i} k(s,t)u(t)dt,$$

it follows that

$$\|(Tu)^{(i)}\|_\infty \le \|D^{i,0}k\|_\infty \|u\|_\infty$$

and

$$\|Tu\|_{2r,\infty} \le \|k\|_{2r,\infty} \|u\|_\infty. \tag{4.7}$$

Then by (4.1)

$$\|(I-\pi_n)Tu\|_\infty \le C_3 \|(Tu)^{(r)}\|_\infty h^r \le C_3 \|D^{r,0}k\|_\infty \|u\|_\infty h^r,$$

which proves (4.3).

Since

$$(T(I-\pi_n)Tu)(s) = \int_0^1 k(s,t)(I-\pi_n)(Tu)(t)dt,$$

by (4.2) and (4.7),

$$\begin{aligned} \|T(I-\pi_n)Tu\|_\infty &\le C_4 \|k\|_{r,\infty} \|Tu\|_{2r,\infty} h^{2r} \\ &\le C_4 \|k\|_{r,\infty} \|k\|_{2r,\infty} \|u\|_\infty h^{2r}, \end{aligned} \tag{4.8}$$

which proves (4.4).

For $i = 0, 1 \ldots, 2r$,

$$(T(I-\pi_n)Tu)^{(i)}(s) = \int_0^1 \frac{\partial^i}{\partial s^i} k(s,t)(I-\pi_n)(Tu)(t)dt.$$

Hence by (4.2),

$$|(T(I-\pi_n)Tu)^{(i)}(s)| \le C_4 (\sum_{j=0}^{r} \|D^{i,j}k\|_\infty) \|Tu\|_{2r,\infty} h^{2r}.$$

As a consequence, using (4.7), we get

$$\|(T(I-\pi_n)Tu)^{(r)}\|_\infty \leq C_4\|k\|_{r,\infty}\|k\|_{2r,\infty}\|u\|_\infty h^{2r} \tag{4.9}$$

and

$$\|T(I-\pi_n)Tu\|_{2r,\infty} \leq C_4(\|k\|_{2r,\infty})^2\|u\|_\infty h^{2r}. \tag{4.10}$$

Next, by (4.1) and (4.9),

$$\begin{aligned}\|(I-\pi_n)T(I-\pi_n)Tu\|_\infty &\leq C_3\|(T(I-\pi_n)Tu)^{(r)}\|_\infty h^r\\ &\leq C_3C_4\|k\|_{r,\infty}\|k\|_{2r,\infty}\|u\|_\infty h^{3r},\end{aligned}$$

which proves (4.5).

Lastly, by (4.8) and (4.10),

$$\begin{aligned}\|T(I-\pi_n)T(I-\pi_n)Tu\|_\infty &\leq C_4\|k\|_{r,\infty}\|T(I-\pi_n)Tu\|_{2r,\infty}h^{2r}\\ &\leq (C_4)^2\|k\|_{r,\infty}(\|k\|_{2r,\infty})^2\|u\|_\infty h^{4r},\end{aligned}$$

which completes the proof. □

Combining the results of Theorems 3.1 and 4.1 we obtain the following orders of convergence for eigenvalue and spectral subspace approximation using the new method.

Theorem 4.2. *For all large n,*

$$\hat{\delta}(R(P), R(P_n^M)) = O(h^{3r}), \tag{4.11}$$

$$|\lambda - \hat{\lambda}_n^M| = O(h^{4r}). \tag{4.12}$$

Also, from Theorems 3.2 and 4.1 we obtain the following orders of convergence for the collocation and the iterated collocation methods. These orders of convergence are well known (see, for example, Chatelin [4]):

$$\hat{\delta}(R(P), R(P_n^C)) = O(h^r), \tag{4.13}$$

$$\hat{\delta}(R(P), R(P_n^S)) = O(h^{2r}), \tag{4.14}$$

$$|\lambda - \hat{\lambda}_n^C| = |\lambda - \hat{\lambda}_n^S| = O(h^{2r}). \tag{4.15}$$

A comparison of (4.11)–(4.12) and (4.13)–(4.15) show that the order of convergence h^{2r} for eigenvalue approximation in the collocation/iterated collocation method is improved to h^{4r} in the new method. For spectral subspaces the improvement is from h^r in the collocation method and from h^{2r} in the iterated collocation method to h^{3r} in the new method. In Section 6 we illustrate the above results by numerical examples. In the next section we show that in the case of a simple eigenvalue, the order of convergence for eigenvector approximation can be further improved to h^{4r} by using an iteration technique.

5. Improvement by iteration

In this section we restrict ourselves to the case when λ is a simple eigenvalue. Let

$$T_n^M\phi_n^M = \lambda_n^M\phi_n^M, \quad \|\phi_n^M\| = 1.$$

We define

$$\psi_n^M = \frac{1}{\lambda_n^M}T\phi_n^M.$$

Let

$$\phi = P\phi_n^M.$$

Theorem 5.1. *For all large n*

$$\|\phi - \psi_n^M\| = O(h^{4r}). \tag{5.1}$$

Proof. Recall that Γ is a circle with center λ and radius ϵ. Choose n big enough so that $|\lambda - \lambda_n^M| \leq \frac{\epsilon}{2}$. Then since $|\lambda| > \epsilon$, we have $|\lambda_n^M| > \frac{\epsilon}{2}$ and for $z \in \Gamma$, $|z - \lambda_n^M| > \frac{\epsilon}{2}$. Consider

$$\begin{aligned} \phi - \psi_n^M &= \frac{1}{\lambda}T\phi - \frac{1}{\lambda_n^M}T\phi_n^M \\ &= (\frac{1}{\lambda} - \frac{1}{\lambda_n^M})T\phi + \frac{1}{\lambda_n^M}T(\phi - \phi_n^M). \end{aligned}$$

Now

$$\|(\frac{1}{\lambda} - \frac{1}{\lambda_n^M})T\phi\| \leq \frac{|\lambda - \lambda_n^M|\|P\|}{|\lambda_n^M|} \leq \frac{2}{\epsilon}\|P\||\lambda - \lambda_n^M| \leq Ch^{4r}, \tag{5.2}$$

by (4.12).

Next

$$\begin{aligned} T(\phi - \phi_n^M) &= T(P - P_n^M)\phi_n^M \\ &= -\frac{1}{2\pi i}\int_\Gamma T((T - zI)^{-1} - (T_n^M - zI)^{-1})\phi_n^M dz \\ &= \frac{1}{2\pi i}\int_\Gamma T(T - zI)^{-1}(T - T_n^M)(T_n^M - zI)^{-1}\phi_n^M dz \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{(T - zI)^{-1}T(I - \pi_n)T(I - \pi_n)\phi_n^M}{\lambda_n^M - z}dz \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{(T - zI)^{-1}T(I - \pi_n)T(I - \pi_n)T\pi_n\phi_n^M}{\lambda_n^M(\lambda_n^M - z)}dz. \end{aligned}$$

Hence

$$\|T(\phi - \phi_n^M)\| \leq \frac{4}{\epsilon}C_1\|\pi_n\|\|T(I - \pi_n)T(I - \pi_n)T\|,$$

and by (4.6)

$$\|T(\phi - \phi_n^M)\| = O(h^{4r}).$$

The result now follows by combining (5.2) and the above estimate. □

6. Numerical examples

We consider the integral operator T given by

$$(Tu)(s) = \int_0^1 \exp(st)u(t)dt, \quad s \in [0, 1].$$

In actual computation T is replaced by its approximation $\tilde{T}$ given by

$$\tilde{T}u(s) = \sum_{j=1}^m w_j^{(m)} \exp(st_j^{(m)})u(t_j^{(m)}), \quad u \in C[0, 1], \quad s \in [0, 1],$$

where m is very large.

Here the nodes $t_1^{(m)}, \ldots, t_m^{(m)}$ in $[0,1]$ and the weights $w_1^{(m)}, \ldots, w_m^{(m)}$ in $\mathbb{C}$ give a convergent quadrature formula

$$Qu = \sum_{j=1}^{m} w_j^{(m)} u(t_j^{(m)}), \quad x \in C[0,1].$$

Collocation at Gauss Points. We choose X_n to be the space of piecewise constant functions ($r = 1$) or the space of piecewise linear functions ($r = 2$) with respect to the equidistant partition

$$0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n}{n} = 1.$$

The collocation points are either midpoints

$$t_j^{(n)} = \frac{2j-1}{n}, \quad j = 1, \ldots, n,$$

or Gauss 2 points

$$t_j^{(n)} = \begin{cases} \dfrac{j - \frac{1}{\sqrt{3}}}{n}, & \text{if } j \text{ is odd}, \\ \dfrac{j - 1 + \frac{1}{\sqrt{3}}}{n}, & \text{if } j \text{ is even}, \end{cases}$$

$j = 1, \ldots, n$.

The projection $\pi_n : C[0,1] \to X_n$ is the interpolatory projection.

We choose

$$w_j^{(m)} = \frac{1}{m}, \quad t_j^{(m)} = \frac{2j-1}{m}, \quad j = 1, \ldots, m,$$

when $r = 1$ and

$$w_j^{(m)} = \frac{1}{m}, \quad t_j^{(m)} = \begin{cases} \dfrac{j - \frac{1}{\sqrt{3}}}{m}, & \text{if } j \text{ is odd}, \\ \dfrac{j - 1 + \frac{1}{\sqrt{3}}}{m}, & \text{if } j \text{ is even}, \end{cases}$$

when $r = 2$.

We fix $m = 512$. For $r = 1$, we choose $n = 4, 8, 16, 32, 64$ and for $r = 2$, we choose $n = 2, 4, 8, 16$.

Let $\tilde{\lambda}$ be the largest eigenvalue of $\tilde{T}$, in modulus, and let λ_n^M, λ_n^C be the eigenvalues obtained by using the new method and the collocation method, respectively. Let ϕ_n^M and ϕ_n^C be the associated eigenvectors.

We write

$$|\tilde{\lambda} - \lambda_n^C| \simeq K_1 h^\alpha, \quad |\tilde{\lambda} - \lambda_n^M| \simeq K_2 h^\beta,$$
$$\|\tilde{P}\phi_n^C - \phi_n^C\| \leq K_3 h^\gamma, \quad \|\tilde{P}\phi_n^M - \phi_n^M\| \simeq K_4 h^\eta.$$

Since $h = \frac{1}{n}$, we use two successive values of n to determine α, β, γ and η.

In Table 6.1 we give the error in the eigenvalue approximation and the computed values of α and β in the collocation/iterated collocation method at midpoints and the new method. Note that the theoretically predicted values are $\alpha = 2, \beta = 4$.

In Table 6.2 the errors in the eigenvector approximation and the computed orders of convergence γ and η in the collocation at midpoints and the new method are given. Note that the theoretically predicted values are $\gamma = 1, \eta = 3$.

TABLE 6.1. **Collocation at midpoint** ($r = 1$). Theoretically predicted values: $\alpha = 2, \beta = 4$

n	$\lvert\lambda - \lambda_n^C\rvert$	$\lvert\lambda - \lambda_n^M\rvert$	α	β
4	4.00×10^{-3}	3.81×10^{-5}		
8	1.01×10^{-3}	2.42×10^{-6}	1.99	3.98
16	2.52×10^{-4}	1.52×10^{-7}	2.00	4.00
32	6.28×10^{-5}	9.42×10^{-9}	2.01	4.01
64	1.54×10^{-5}	5.75×10^{-10}	2.02	4.03

TABLE 6.2. **Collocation at midpoint** ($r = 1$). Theoretically predicted values: $\gamma = 1, \eta = 3$

n	$\lVert P\phi_n^C - \phi_n^C\rVert$	$\lVert P\phi_n^M - \phi_n^M\rVert$	γ	η
4	4.75×10^{-2}	3.44×10^{-4}		
8	1.73×10^{-2}	3.19×10^{-5}	1.46	3.43
16	6.13×10^{-3}	2.84×10^{-6}	1.56	3.49
32	2.11×10^{-3}	2.45×10^{-7}	1.53	3.53
64	7.01×10^{-4}	2.01×10^{-8}	1.59	3.61

TABLE 6.3. **Collocation at Gauss 2 points** ($r = 2$). Theoretically predicted values: $\alpha = 4, \beta = 8$

n	$\lvert\lambda - \lambda_n^C\rvert$	$\lvert\lambda - \lambda_n^M\rvert$	α	β
2	4.30×10^{-5}	2.35×10^{-8}		
4	2.73×10^{-6}	9.93×10^{-11}	3.98	7.88
8	1.71×10^{-7}	3.98×10^{-13}	3.99	7.96
16	1.07×10^{-8}	4.44×10^{-15}	4.00	6.49

TABLE 6.4. **Collocation at Gauss 2 points** ($r = 2$). Theoretically predicted values: $\gamma = 2, \eta = 6$

n	$\lVert P\phi_n^C - \phi_n^C\rVert$	$\lVert P\phi_n^M - \phi_n^M\rVert$	γ	η
2	5.56×10^{-3}	1.90×10^{-6}		
4	1.04×10^{-3}	2.40×10^{-8}	2.42	6.30
8	1.86×10^{-4}	2.78×10^{-10}	2.48	6.43
16	3.20×10^{-5}	3.04×10^{-12}	2.54	6.52

The corresponding results for collocation at Gauss 2 points are listed in Tables 6.3 and 6.4. In this case the theoretically predicted values are $\alpha = 4, \beta = 8, \gamma = 2, \eta = 6$.

Note from Table 6.1 that both the errors $|\lambda - \lambda_4^M|$ and $|\lambda - \lambda_{64}^C|$ are of the order of 10^{-5}. The computation of λ_4^M needs the solution of a matrix eigenvalue problem of size 8 whereas λ_{64}^C is obtained by solving a matrix eigenvalue problem of size 64. Also λ_{32}^M, which is obtained by solving a matrix eigenvalue problem of size 64, has error of the order of 10^{-8}. Similar observations can be made from Tables 6.2–6.4.

In Tables 6.1 and 6.3 the observed values of α and β match well with the theoretically predicted values. In the case of the collocation at midpoints, the expected values of γ and η are 1 and 3, respectively, whereas their observed values are about 1.5 and 3.5. Similarly, in the case of the collocation at Gauss 2 points, the expected values of γ and η are 2 and 6, respectively, whereas their observed values are about 2.5 and 6.5. The theoretically predicted values for γ and η are obtained by using an upper bound for the error in the eigenvector approximation, and the error seems to converge faster than the upper bound.

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