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STABILIZED FINITE ELEMENT METHOD BASED ON THE CRANK–NICOLSON EXTRAPOLATION SCHEME FOR THE TIME-DEPENDENT NAVIER–STOKES EQUATIONS

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ABSTRACT. This paper provides an error analysis for the Crank–Nicolson extrapolation scheme of time discretization applied to the spatially discrete stabilized finite element approximation of the two-dimensional time-dependent Navier–Stokes problem, where the finite element space pair (X_h, M_h) for the approximation (u_h^n, p_h^n) of the velocity u and the pressure p is constructed by the low-order finite element: the $Q_1 - P_0$ quadrilateral element or the $P_1 - P_0$ triangle element with mesh size h. Error estimates of the numerical solution (u_h^n, p_h^n) to the exact solution $(u(t_n), p(t_n))$ with $t_n \in (0, T]$ are derived.

1. INTRODUCTION

In a primitive variable formulation for solving the Stokes equations and the Navier–Stokes equations, the importance of ensuring the compatibility of discrete velocity and pressure by satisfying the so-called inf-sup condition is widely understood. In particular, it is well known that the simplest conforming low-order elements, such as the $P_1 - P_0$ (linear velocity, constant pressure) triangular element and the $Q_1 - P_0$ (bilinear velocity, constant pressure) quadrilateral element are not stable. During the last two decades there has been a rapid development in practical stabilization techniques for the $P_1 - P_0$ element and the $Q_1 - P_0$ element for solving the Stokes problem. For this purpose a local "macroelement condition" and some energy methods have been used. The use of such a macroelement condition as a mean of verifying the (Babuška–Brezzi) inf-sup condition is a standard technique (see, for example, Girault and Raviart [18]); the basic idea was first introduced by Boland and Nicolaides [8], and independently by Stenberg [43]. The stabilized mixed finite element approximation under consideration is based on the combination of the standard variational formulation of the Stokes problem and the bilinear form including a jump operator in pressure. The discrete velocity u_h and the pressure p_h are chosen from finite element subspaces X_h and M_h of the Sobolev spaces X and M defined in Section 2, related to conforming low-order elements like the $P_1 - P_0$ triangular element, or the $Q_1 - P_0$ quadrilateral element,

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which do not possess the properties required by the inf-sup condition. Recently, Kechkar and Silvester [33, 42], Kay and Silvester [32], Norburn and Silvester [36] and Silvester and Wathen [41] pursued some interesting work on both mathematical analysis and numerical tests of locally stabilized mixed finite element methods for the Stokes problem. Their work has also been extended to the Navier–Stokes problem and other related problems, (see, e.g., [26] and [21]), and some numerical analysis and tests of the stabilized finite element method for the stationary Navier–Stokes equations were provided by He et al. in [24].

The viscous incompressible Navier–Stokes equations with zero boundary conditions are one of the fundamental systems modelling fluid motion. The mathematical theory of these equations and their numerical solution are an important field of research. Here our aim is to solve the following time-dependent viscous incompressible Navier–Stokes problem:

$$\left\{\begin{array}{l} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f,\ \mathrm{div} u = 0,\ (x,t)\in\Omega\times(0,T]; \\ u(x,0) = u_0(x),\ x\in\Omega;\ u(x,t)|_\Gamma = 0,\ t\in[0,T], \end{array}\right. \tag{1.1}$$

in a bounded two-dimensional domain with some appropriate assumptions stated in Section 2, where $u = u(x,t) = (u_1(x,t), u_2(x,t))$ represents the velocity vector of a viscous incompressible fluid, $p = p(x,t)$ the pressure, $f = f(x,t)$ the prescribed body force, $u_0(x)$ the initial velocity, $\nu > 0$ the viscosity and $T > 0$ a finite time.

For the usual spatial discretization, i.e., time-continuous approximations, the finite element space pair (X_h, M_h) needs to satisfy some appropriate approximate properties and the compatibility properties required by the inf-sup condition (see Heywood and Rannacher [27]), where $0 < h < 1$ is the mesh size. This means that the finite element space pair (X_h, M_h) needs to be established by a more complex element than the $P_1 - P_0$ element or the $Q_1 - P_0$ element [3], [18], [27]. We assume that the data (u_0, f) satisfy the assumption:
(**A1**): $u_0 \in H^2(\Omega)^2 \cap H^1_0(\Omega)^2$ with $\mathrm{div}\, u_0 = 0$ and $f,\ f_t,\ f_{tt} \in L^\infty(0,T;L^2(\Omega)^2)$ with

$$\|u_0\|_2 + \sup_{t\in[0,T]}\{\|f(t)\|_0 + \|f_t(t)\|_0 + \|f_{tt}(t)\|_0\} < \infty.$$

Then the spatial discrete solution $(u_h(t), p_h(t))$ satisfies the error following estimates [27]:

$$\begin{aligned} \|u(t)-u_h(t)\|_0 \quad &+ \quad h\|u(t)-u_h(t)\|_1 + h\sigma^{1/2}(t)\|p(t)-p_h(t)\|_0 \\ &\leq \quad \kappa h^2,\ 0\leq t\leq T, \end{aligned} \tag{1.2}$$

where $\sigma(t) = \min\{1,t\}$ and $\kappa > 0$ is a general constant depending on the data (ν, Ω, u_0, f, T) which may have different values at its different occurrences. For fully discrete approximations, the discrete solution (u_h^n, p_h^n) based on the Crank–Nicolson scheme, in which the viscous term and the nonlinear term are discretized implicitly, satisfies the error estimates [28]

$$\sigma(t_m)\|u_h(t_m) - u_h^m\|_0 \quad\leq\quad \kappa\tau^2, \quad \sigma^{3/2}(t_m)\|p_h(t_m) - p_h^m\|_0 \leq \kappa\tau, \tag{1.3}$$

for all $t_m = m\tau \in (0,T]$, where $\tau \leq \kappa_0$ is the time step size, and κ_0 some fixed value depending on the data (ν, Ω, u_0, f, T).

It is noted that the factors $\sigma^s(t_n)$ with $s = \frac{1}{2},\ 1,\ \frac{3}{2}$ that appeared in (1.2)–(1.3) are due to the nonsmoothness of the time derivatives of the velocity u and pressure p at $t = 0$.

For the usual time discretization, i.e., spatial-continuous approximations, Shen [40] proposed a second-order projection scheme, in which the viscous term and the nonlinear term are treated implicitly and the pressure term explicitly. Guermond and Shen [19] studied a velocity-correction projection scheme for the linearized Navier–Stokes equations. The semi-discrete solution (u^n, p^n) satisfies the error estimates

$$
\begin{aligned}
(\tau \sum_{n=n_0}^{m} \|u(t_n) - u^n\|_0^2)^{1/2} &\leq \kappa\tau^2,\ \|u(t_m) - u^m\|_0 \leq \kappa\tau^{3/2}, \\
\|u(t_m) - u^m\|_1 &+ \|p(t_m) - p^m\|_0 \leq \kappa\tau,
\end{aligned}
\tag{1.4}
$$

for all $t_m = m\tau \in (t_{n_0}, T]$ under the assumption (**A1**), where $0 < t_{n_0} < T$ is a fixed time.

Moreover, a second-order time characteristics and a spatial discretization of the $P_k - P_{k'}$ finite element type for the Navier–Stokes equations in the d-dimensional domain was presented by Boukir, Maday, Métivet and Razafindrakoto [9], and the H^1-error estimate is given by

$$
(\tau \sum_{n=1}^{m} \|p(t_n) - p_h^n\|_0^2)^{1/2} + \|u(t_m) - u_h^m\|_1 \leq \kappa(\tau^2 + h^k + h^{k'+1}),
\tag{1.5}
$$

for all $t_m = m\tau \in (t_{n_0}, T]$ under some stronger regularity assumption of the exact solution $(u(t), p(t))$ and the stability condition $\tau h^{-d/6} \leq \kappa_0$ with $d = 2, 3$.

For simplicity of notation, we confine our attention to the $Q_1 - P_0$ quadrilateral element and the $P_1 - P_0$ triangle element. Let τ_h be a partition (triangles or quadrilaterals) of $\bar{\Omega}$ with mesh size h, assumed to be uniformly regular in the usual sense. Here the finite element space pair (X_h, M_h) does not posses the compatibility properties required by the inf-sup condition. An earlier paper [26] dealt mainly with spatial discretization (time continuous approximations) and a later paper [21] studied a fully discrete stabilized finite element approximation, in which time is discretized by the backward Euler semi-implicit scheme with the time step $0 < \tau < 1$.

Assume that the initial velocity $u_0 \in H_0^1(\Omega)^2$ with $\operatorname{div} u_0 = 0$ and $f,\ f_t \in L^\infty(0, T; L^2(\Omega)^2)$, He et al. [25] have proved that the spatial discrete solution $(u_h(t), p_h(t))$ satisfies the error estimates [26]

$$
\begin{aligned}
\sigma^{1/2}(t)\|u(t) - u_h(t)\|_0 &+ h\sigma^{1/2}(t)\|u(t) - u_h(t)\|_1 + h\sigma(t)\|p(t) - p_h(t)\|_0 \\
&\leq \kappa h^2,\ 0 \leq t \leq T,
\end{aligned}
\tag{1.6}
$$

while the fully discrete solution (u_h^n, p_h^n) based on the backward Euler semi-implicit scheme satisfies the error estimates

$$
\begin{aligned}
\sigma^{1/2}(t_m)\|u(t_m) - u_h^m\|_0 &\leq \kappa(h^2 + \tau), \\
\sigma^{1/2}(t_m)\|u(t_m) - u_h^m\|_1 + \sigma(t_m)\|p_h(t_m) - p_h^m\|_0 &\leq \kappa(h + \tau^{1/2}),
\end{aligned}
\tag{1.7}
$$

(see He [21]) for all $t_m = m\tau \in (0, T]$ and $\tau|\log h|^{1/2} \leq \kappa_0$.

This paper continues our analysis of the stabilized mixed finite element method based on the $Q_1 - P_0$ quadrilateral element and the $P_1 - P_0$ triangle element [26, 21] for solving the two-dimensional time-dependent Navier–Stokes equations with respect to the data (u_0, f) satisfying the assumption (**A1**). We consider the second order fully discrete scheme based on the Crank–Nicolson extrapolation scheme in which we use an implicit scheme for the viscous and pressure terms and a semi-implicit scheme for the nonlinear term. The discrete solution (u_h^n, p_h^n) satisfies the error estimates

$$(1.8) \quad \|u(t_m) - u_h^m\|_0 \leq \kappa(h^2 + \tau^{3/2}),$$

$$(1.9) \quad \|u(t_m) - u_h^m\|_1 \leq (h + \tau^{3/4}),\ \sigma^{1/2}(t_m)\|\bar{p}(t_m) - \bar{p}_h^m\|_0 \leq \kappa(h + \tau^{3/4}),$$

for all $t_m \in (0, T]$, where $I_\tau = \{t_m\}_0^N$ is a given set in the interval $[0, T]$ with a time step size $\tau = \max_{1\leq m\leq N}(t_m - t_{m-1})$, and

$$\bar{p}_h^m = \frac{1}{2}(p_h^m + p_h^{m-1}),\ \bar{p}(t_m) = \frac{1}{2}(p(t_m) + p(t_{m-1})).$$

The contents of this paper are divided into sections as follows. In Section 2, the abstract functional setting of the Navier–Stokes problem is given with some basic statements. Stabilized finite element approximations are recalled in Section 3. Some key technical lemmas and known results are provided in Section 4. The fully discrete stabilized finite element method with the Crank–Nicolson extrapolation scheme and the corresponding fully discrete duality problem are considered in section Section 5. The L^2- and H^1-error estimates for the discrete velocity and L^2-error estimate for the discrete pressure are derived in Section 6.

2. Functional setting of the Navier–Stokes problem

For the mathematical setting of problem (1.1), first we introduce the Hilbert spaces

$$X = H_0^1(\Omega)^2, \quad Y = L^2(\Omega)^2, \quad M = L_0^2(\Omega) = \{q \in L^2(\Omega); \int_\Omega q dx = 0\}.$$

The spaces $L^2(\Omega)^m, m = 1, 2, 4$, are endowed with the L^2-scalar product and L^2-norm denoted by $(\cdot,\cdot)$ and $\|\cdot\|_0$. The space X is equipped with its equivalent scalar product $(\nabla u, \nabla v)$ and norm $|u|_1 = \|\nabla u\|_0$. Next, let the closed subset V of X be given by

$$V = \{v \in X; \text{div } v = 0\},$$

and denote by H the closed subset of Y; i.e.,

$$H = \{v \in Y; \text{div } v = 0, v \cdot n|_{\partial\Omega} = 0\}.$$

We refer the readers to [1, 2, 6, 7, 18, 27, 45] for more detail on these spaces. We also denote the Laplace operator by $A = -\Delta$ and denote the Stokes operator by $\bar{A} = -P\Delta$, where P is the L^2-orthogonal projection of Y onto H.

As mentioned above, we need a further assumption on Ω:

(A2) Assume that Ω is smooth so that the unique solution $(v, q) \in (X, M)$ of the steady Stokes problem

$$-\Delta v + \nabla q = g, \text{div} v = 0 \quad \text{in } \Omega,\ v|_{\partial\Omega} = 0,$$

for prescribed $g \in Y$ exists and satisfies

$$\|v\|_2 + \|q\|_1 \leq c\|g\|_0,$$

where $c > 0$ is a generic constant depending on Ω and ν which may stand for a different value at its different occurrences, and $\|\cdot\|_i$ denotes the usual norm of Sobolev spaces $H^i(\Omega)^m;\ i = 0, 1, 2,\ m = 1, 2, 4.$

We remark that the validity of assumption **(A2)** is known (see [18, 27, 31, 30, 45]) if $\partial\Omega$ is of C^2, or if Ω is a two-dimensional convex polygon. From assumption **(A2)**, it is known [1, 27, 34] that

$$\|Av\|_0^2 \quad \le \quad \|v\|_2^2 \le c\|\bar{A}v\|_0^2,\ v \in H^2(\Omega)^2 \cap V, \tag{2.1}$$

$$\|v\|_0 \quad \le \quad \gamma_0|v|_1,\ v \in X,\ |v|_1 \le \gamma_0\|v\|_2 \le c\|Av\|_0,\ v \in H^2(\Omega)^2 \cap X, \tag{2.2}$$

where γ_0 is positive constant depending only on Ω.

We define the continuous bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $X \times X$ and $X \times M$, respectively, by

$$a(u,v) = \nu(\nabla u, \nabla v),\ u,v \in X,$$
$$d(v,q) = -(v,\nabla q) = (q, \mathrm{div} v),\ v \in X, q \in M,$$

and a trilinear form on $X \times X \times X$ by

$$\begin{aligned} b(u,v,w) &= ((u\cdot\nabla)v,w) + \frac{1}{2}((\mathrm{div}u)v,w) \\ &= \frac{1}{2}((u\cdot\nabla)v,w) - \frac{1}{2}((u\cdot\nabla)w,v),\ \forall u,v,w \in X. \end{aligned}$$

With the above notations, the variational formulation of problem (1.1) reads as follows. Find $(u,p) \in (X,M),\ t \in (0,T]$ such that for all $(v,q) \in (X,M)$,

$$(u_t,v) + a(u,v) - d(v,p) + d(u,q) + b(u,u,v) \quad = \quad (f,v), \tag{2.3}$$

$$u(0) \quad = \quad u_0. \tag{2.4}$$

A simple modification to the regularity argument given in [29, 27] allows us to obtain the following regularity results.

Theorem 2.1. *Assume that assumptions* (**A1**) *and* (**A2**) *are valid. Then the problem* (2.3)–(2.4) *admits a unique solution* (u,p) *satisfying the regularity results*

$$\|u(t)\|_2^2 + \|p(t)\|_1^2 + \|u_t(t)\|_0^2 + \int_0^t \|u_t\|_1^2 ds \le \kappa, \tag{2.5}$$

$$\sigma(t)\|u_t(t)\|_1^2 + \int_0^t \sigma(s)(\|u_{tt}\|_0^2 + \|u_t\|_2^2 + \|p_t\|_1^2) ds \le \kappa, \tag{2.6}$$

$$\sigma^2(t)(\|u_t(t)\|_2^2 + \|p_t(t)\|_1^2 + \|u_{tt}(t)\|_0^2) + \int_0^t \sigma^2(s)\|u_{tt}\|_1^2 ds \le \kappa, \tag{2.7}$$

$$\sigma^3(t)\|u_{tt}(t)\|_1^2 + \int_0^t \sigma^3(s)(\|u_{ttt}\|_0^2 + \|u_{tt}\|_2^2 + \|p_{tt}\|_1^2) ds \le \kappa, \tag{2.8}$$

for all $t \in [0,T]$.

3. Stabilized finite element approximation

Let $h > 0$ be a real positive parameter. The finite element subspace (X_h, M_h) of (X,M) is characterized by $\tau_h = \tau_h(\Omega)$, a partitioning of $\bar{\Omega}$ into triangles K or quadrilaterals K, assumed to be uniformly regular as $h \to 0$. For further details, the reader can refer to Ciarlet [13] and Girault and Raviart [18]. The mesh parameter h is given by $h = \max\{h_K\}$, and the set of all interelement boundaries will be denoted by Γ_h.

Finite element subspaces of interest in this paper are defined by setting

$$R_1(K) = \begin{cases} P_1(K) \text{ if } K \text{ is triangular,} \\ Q_1(K) \text{ if } K \text{ is quadrilateral,} \end{cases} \tag{3.1}$$

giving the continuous piecewise (bi)linear velocity subspace

$$X_h = \{v \in C^0(\bar{\Omega})^2 \cap X;\ v_i|_K \in R_1(K),\ i = 1, 2,\ K \in \tau_h\},$$

and the piecewise constant pressure subspace

$$M_h = \{q \in M;\ q|_K \in P_0(K), K \in \tau_h\}.$$

Note that neither of these methods are stable in the standard Babuška–Brezzi sense; the $P_1 - P_0$ triangle "locks" on regular grids (since there are more discrete incompressibility constraints than velocity degrees of freedom), and the $Q_1 - P_0$ quadrilateral is one example of unstable mixed methods, as elucidated by Sani et al. in [38].

In order to define a locally stabilized formulation of the time-dependent Navier–Stokes problem, we introduce *a macroelement partitioning* Λ_h as follows. Given any subdivision τ_h, a macroelement partitioning Λ_h may be defined such that each macroelement $\mathcal{K}$ is a connected set of adjoining elements from τ_h. Every element K must lie in exactly one macroelement, which implies that macroelements do not overlap. For each $\mathcal{K}$, the set of interelement edges which are strictly in the interior of $\mathcal{K}$ will be denoted by $\Gamma_{\mathcal{K}}$. The length of edge $e \in \Gamma_{\mathcal{K}}$ is denoted by h_e.

With these additional definitions a locally stabilized discrete formulation of the problem (2.3)–(2.4) can be stated as follows.

Definition 3.1. Locally Stabilized Formulation. Find $(u_h, p_h) \in (X_h, M_h)$ such that for all $t \in (0, T]$ and $(v_h, q_h) \in (X_h, M_h)$,

$$(u_{ht}, v_h) + \mathcal{B}_h((u_h, p_h); (v_h, q_h)) + b(u_h, u_h, v_h) = (f, v_h), \tag{3.2}$$

$$u_h(0) = u_{0h}, \tag{3.3}$$

where $u_{0h} \in X_h$ is an approximation of u_0 and

$$\mathcal{B}_h((u_h, p_h); (v_h, q_h)) = \mathcal{B}((u_h, p_h); (v_h, q_h)) + \beta \mathcal{C}_h(p_h, q_h),$$

for all $(u_h, p_h), (v_h, q_h) \in (X_h, M_h)$, here

$$\mathcal{B}((u, p); (v, q)) = a(u, v) - d(v, p) + d(u, q)\ \forall (u, p),\ (v, q) \in (X, M),$$

$$\mathcal{C}_h(p, q) = \sum_{\mathcal{K} \in \Lambda_h} \sum_{e \in \Gamma_{\mathcal{K}}} h_e \int_e [p]_e [q]_e ds,$$

for all $p,\ q$ in the algebraic sum $H^1(\Omega) + M_h$, $[\cdot]_e$ is the jump operator across $e \in \Gamma_{\mathcal{K}}$ and $\beta > 0$ is the local stabilization parameter.

A general framework for analyzing the locally stabilized formulation (3.2)–(3.3) can be developed using the notion of an equivalent class of macroelements. As in Stenberg [43], each equivalence class, denoted by $\mathcal{E}_{\hat{\mathcal{K}}}$, contains macroelements which are topologically equivalent to a reference macroelement $\hat{\mathcal{K}}$. To illustrate the idea, two practical examples of locally stabilized mixed approximations are given below.

Example 3.1. The first example is the standard $Q_1 - P_0$ approximation pair. A locally stabilized formulation (3.2)–(3.3) can be constructed in this case, if τ_h is such that the elements K can be grouped into 2×2 macroelements

$$\mathcal{K} = \{K_1,\ K_2,\ K_3,\ K_4\},$$

with the reference macroelement

$$\hat{\mathcal{K}} = \{\hat{K}_1,\ \hat{K}_2,\ \hat{K}_3,\ \hat{K}_4\}.$$

An obvious way of constructing such a partitioning in practice is to form the grid τ_h by uniformly refining a coarse grid Λ_h, such as by joining the midedge points.

Example 3.2. The triangular $P_1 - P_0$ approximation pair can be stabilized similarly if the partitioning τ_h is constructed such that the elements can be grouped into disjoint macroelements, all consisting of four elements.

The following properties are classical (see [7, 13, 47, 48]):

$$|v_h|_1 \leq ch^{-1}\|v_h\|_0, \; v_h \in X_h. \tag{3.4}$$

The following stability results of this mixed method for the macroelement partitions defined above were formally established by Kay and Silvester [32] and Kechkar and Silvester [33].

Theorem 3.2. *Given a stabilization parameter $\beta \geq \beta_0 > 0$, suppose that every macroelement $\mathcal{K} \in \Lambda_h$ belongs to one of the equivalence classes $\mathcal{E}_{\hat{\mathcal{K}}}$, and that the following macroelement connectivity condition is valid: for any two neighboring macroelements $\mathcal{K}_1$ and $\mathcal{K}_2$ with $\int_{\mathcal{K}_1 \cap \mathcal{K}_2} ds \neq 0$ there exists $v \in X_h$ such that*

$$\text{supp } v \subset \mathcal{K}_1 \cup \mathcal{K}_2 \quad \text{and} \quad \int_{\mathcal{K}_1 \cap \mathcal{K}_2} v \cdot n ds \neq 0. \tag{3.5}$$

Then,

$$|\mathcal{C}_h(p,q)| \leq c \sum_{K \in \tau_h} (\int_K (\|p\|_0^2 + h^2\|\nabla p\|_0^2) dx)^{1/2} (\int_K (\|q\|_0^2 + h^2\|\nabla q\|_0^2) dx)^{1/2}, \tag{3.6}$$

for all $p, \; q \in H^1(\Omega) + M_h$, and

$$\alpha(|u_h|_1 \quad + \quad \|p_h\|_0) \leq \sup_{(v_h,q_h) \in (X_h, M_h)} \frac{\mathcal{B}_h((u_h,p_h);(v_h,q_h))}{\|v_h\| + |q_h|}, \tag{3.7}$$

for all $(u_h, p_h) \in (X_h, M_h)$, and

$$\mathcal{C}_h(p, q_h) \quad = \quad 0, \; \mathcal{C}_h(p_h, q) = 0, \; \mathcal{C}_h(p,q) = 0, \; \forall p, \; q \in H^1(\Omega), \; p_h, \; q_h \in M_h, \tag{3.8}$$

where $\alpha > 0$ is a constant independent of h and β, and β_0 is some fixed positive constant and n is the unit outward normal vector.

4. Technical preliminaries

This section considers preliminary estimates which will be very useful in error estimates of the finite element solution (u_h, p_h).

With the statements in Section 3, a discrete analogue $A_h = -\Delta_h$ of the Laplacian operator $A = -\Delta$ is defined through the condition that $(-\Delta_h u_h, v_h) = ((u_h, v_h))$ for all $u_h, v_h \in X_h$. The operator $A_h : \; X_h \rightarrow X_h$ is invertible, with its inverse denoted A_h^{-1}. Since A_h^{-1} is self-adjoint and positive definite, we may define "discrete" Sobolev norms on X_h, of any order $r \in R$, by setting

$$|v_h|_r = \|A_h^{r/2} v_h\|_0, \quad \forall v_h \in X_h.$$

These norms will be assumed to have various properties similar to their continuous counterparts, an assumption that implicitly imposes conditions on the structure of the spaces X_h and M_h. In particular, it holds that

$$|v_h|_0 = \|v_h\|_0, \qquad |v_h|_1 = \|\nabla v_h\|_0, \qquad |v_h|_2 = \|A_h v_h\|_0, \quad \forall v_h \in X_h.$$

By the way, we derive from (2.2) that

$$\|v_h\|_0 \leq \gamma_0 \|\nabla v_h\|_0, \qquad \|\nabla v_h\|_0 \leq \gamma_0 \|A_h v_h\|_0, \quad \forall v_h \in V_h, \tag{4.1}$$

where $\gamma_0 > 0$ is a constant depending only on Ω.

Moreover, we define the discrete gradient operator ∇_h for $q_h \in M_h$ as

$$(v_h, \nabla_h q_h) = -d(v_h, q_h), \ \forall v_h \in X_h.$$

Now, by using a slightly modified argument on the estimates of the trilinear form b provided in [21, 22, 23, 27, 28, 29], we can obtain the following results on b.

Lemma 4.1. *The trilinear form b satisfies the estimates*

$$b(u_h, v_h, w_h) = -b(u_h, v_h, w_h), \tag{4.2}$$

$$\begin{aligned} &|b(u_h, v_h, w_h)| + |b(v_h, u_h, w_h)| + |b(w_h, u_h, v_h)| \\ &\quad \leq \frac{c_0}{2}\|u_h\|_0^{1/2}|u_h|_1^{1/2}|v_h|_1\|w_h\|_0^{1/2}|w_h|_1^{1/2} \\ &\quad + \frac{c_0}{2}|u_h|_1\|v_h\|_0^{1/2}|v_h|_1^{1/2}\|w_h\|_0^{1/2}|w_h|_1^{1/2}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} &|b(u_h, v_h, w_h)| + |b(v_h, u_h, w_h)| + |b(w_h, u_h, v_h)| \\ &\quad \leq \frac{c_0}{2}\|A_h v_h\|_0^{1/2}\|v_h\|_1^{1/2}\|u_h\|_0^{1/2}\|u_h\|_1^{1/2}\|w_h\|_0 \\ &\quad + \frac{c_0}{2}\|A_h v_h\|_0^{1/2}\|v_h\|_0^{1/2}\|u_h\|_1\|w_h\|_0, \end{aligned} \tag{4.4}$$

for all $u_h, v_h, w_h \in X_h$, where $c_0 > 0$ is a constant depending only on Ω.

In order to derive the error estimates of the finite element solution (u_h, p_h), we also define the Galerkin projection $(R_h, Q_h) : (X, Y) \rightarrow (X_h, M_h)$ by requiring

$$\begin{aligned} &\mathcal{B}_h((R_h(u,p), Q_h(u,p)); (v_h, q_h)) = \mathcal{B}((u,p); (v_h, q_h)), \\ &\qquad \forall (u,p) \in (X, M), \ (v_h, q_h) \in (X_h, M_h). \end{aligned} \tag{4.5}$$

Note that, due to Theorem 3.2, (R_h, Q_h) is well defined and satisfies the approximate properties [24]

$$\|R_h(u,p) - u\|_0 + h(|R_h(u,p) - u|_1 + \|Q_h(u,p) - p\|_0) \leq ch^2(\|u\|_2 + \|p\|_1), \tag{4.6}$$

for all $(u,p) \in (H^2(\Omega)^2 \cap X, \ H^1(\Omega) \cap M)$.

Moreover, we need to introduce the following L^2-orthogonal projection P_h : $L^2(\Omega)^2 \rightarrow X_h$ defined by

$$(P_h v, v_h) = (v, v_h), \ v \in L^2(\Omega)^2, \ v_h \in X_h.$$

Using some slight modifications of the literature [26, 27], we can obtain the following error estimates.

Theorem 4.2. *Assume the assumptions of Theorems* 2.1 *and* 3.2 *are valid and set* $(u_{0h}, p_{0h}) = (R_h(u_0, p_0), Q_h(u_0, p_0))$. *Then* (u_h, p_h) *satisfies*

$$\|u(t) - u_h(t)\|_0 + h(\|u(t) - u_h(t)\|_1 + \sigma^{1/2}(t)\|p(t) - p_h(t)\|_0) \leq \kappa h^2, \tag{4.7}$$

$$\left(\int_0^t \sigma(s)|u_t - u_{ht}|_1^2 ds\right)^{1/2} \leq \kappa h, \tag{4.8}$$

for all $t \in (0, T]$, where $p_0 = \lim_{t\rightarrow 0} p(t) \in H^1(\Omega) \cap M$.

Since our error analysis for the time discretization depends heavily on some regularity estimates of the semi-discrete solution (u_h, p_h), we will provide the following regularity results.

Theorem 4.3. *Under the assumptions of Theorems* 4.2*, the finite element solution* (u_h, p_h) *satisfies the regularities*

$$\int_0^t (\|A_h^{-1}(u_{htt} + P_h\nabla_h p_{ht})\|_0^2 + \|u_{ht}\|_0^2 + \|A_h u_h\|_0^2)ds \\ + \nu|u_h(t)|_1^2 + \beta\mathcal{C}_h(p_h(t)a, p_h(t)) \le \kappa, \tag{4.9}$$

$$\int_0^t (\|A_h^{-1/2}(u_{htt} + P_h\nabla_h p_{ht})\|_0^2 + \nu|u_{ht}|_1^2 + \beta\mathcal{C}_h(p_{ht}, p_{ht}))ds \\ + (\|u_{ht}(t)\|_0^2 + \|\nabla_h p_h(t)\|_0^2 + \|A_h u_h(t)\|_0^2) \le \kappa, \tag{4.10}$$

$$\int_0^t \sigma(s)(\|u_{htt}\|_0^2 + \|P_h\nabla_h p_{ht}\|_0^2 + \|A_h u_{ht}\|_0^2)ds \le \kappa, \tag{4.11}$$

$$\int_0^t \sigma(s)\|A_h^{-1}(u_{httt} + P_h\nabla_h p_{htt})\|_0^2 ds \\ + \sigma(t)(\nu|u_{ht}(t)|_1^2 + \beta\mathcal{C}_h(p_{ht}(t), p_{ht}(t))) \le \kappa, \tag{4.12}$$

$$\int_0^t \sigma^2(s)(\|A_h^{-1/2}(u_{httt} + P_h\nabla_h p_{htt})\|_0^2 + \nu|u_{htt}|_1^2 + \beta\mathcal{C}_h(p_{htt}, p_{htt}))ds \\ + \sigma^2(t)\|u_{htt}(t)\|_0^2 \le \kappa, \tag{4.13}$$

$$\int_0^t \sigma^3(s)(\|u_{httt}\|_0^2 + \|P_h\nabla_h p_{htt}\|_0^2 + \|A_h u_{htt}\|_0^2)ds \le \kappa, \tag{4.14}$$

for all $t \in [0, T]$.

Proof. The proof is by a fairly standard energy argument. Hereafter, we will make frequent use of (4.1)–(4.4) without explicit mention.

First, differentiating (3.2) with respect to t results in the equations

$$(u_{htt} + P_h\nabla_h p_{ht}, v_h) + a(u_{ht}, v_h) + d(u_{ht}, q_h) + \beta\mathcal{C}_h(p_{ht}, q_h) \\ + b_t(u_h, u_h, v_h) = (f_t, v_h), \tag{4.15}$$

$$(u_{httt} + P_h\nabla_h p_{htt}, v_h) + a(u_{htt}, v_h) + d(u_{htt}, q_h) + \beta\mathcal{C}_h(p_{htt}, q_h) \\ + b_{tt}(u_h, u_h, v_h) = (f_{tt}, v_h). \tag{4.16}$$

If we take $q_h = 0$ in (4.15)–(4.16), a simple calculation yields

$$\int_0^t \|A_h^{-i/2}(u_{htt} + P_h\nabla_h p_{ht})\|_0^2 ds \le c\int_0^t \|A_h^{1-i/2}u_{ht}\|_0^2(1 + |u_h|_1^2) + \|f_t\|_0^2)ds, \tag{4.17}$$

$$\int_0^t \sigma^{3-i}(s)\|A_h^{-i/2}(u_{httt} + P_h\nabla_h p_{htt})\|_0^2 ds \le c\int_0^t \sigma^{3-i}(s)\|A_h^{1-i/2}u_{ht}\|_0^2|u_{ht}|_1^2 ds \\ + c\int_0^t (\sigma^{3-i}(s)\|A_h^{1-i/2}u_{htt}\|_0^2(1 + |u_h|_1^2) + \|f_{tt}\|_0^2)ds, \tag{4.18}$$

for $i = 2, 1, 0$.

Next, using (3.4), we obtain

$$\|A_h u_h\|_0 = \sup_{v_h \in X_h} \frac{|(A_h u_h, v_h)|}{\|v_h\|_0} \quad \le \quad (ch^{-1}|u - u_h|_1 + \|Au\|_0), \tag{4.19}$$

for all $u_h \in X_h$ and $u \in X$. Hence, we derive from Theorems 2.1 and 4.2 that

$$\|A_h u_h(t)\|_0^2 \leq c(h^{-2}|u(t)-u_h(t)|_1^2 + \|Au(t)\|_0^2) \leq \kappa, \ \forall 0 \leq t \leq T. \tag{4.20}$$

Now, by recalling [26, 21], we have

$$\nu|u_h(t)|_1^2 + \beta\mathcal{C}_h(p_h(t),p_h(t)) + \int_0^t \|u_{ht}\|_0^2 ds \leq \kappa, \ \forall 0 \leq t \leq T. \tag{4.21}$$

Taking $(v_h,q_h) = (u_{ht},p_{ht})$ in (4.15) and using (2.3) and (4.1)–(4.3), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_{ht}\|_0^2 + \nu|u_{ht}|_1^2 + \beta\mathcal{C}_h(p_{ht},p_{ht}) + b(u_{ht},u_h,u_{ht}) = (f_t,u_{ht}), \tag{4.22}$$

$$|b(u_{ht},u_h,u_{ht})| \leq c\|u_{ht}\|_0|u_{ht}|_1\|A_h u_h\|_0 \leq \frac{\nu}{4}|u_{ht}|_1^2 + c\|A_h u_h\|_0^2\|u_{ht}\|_0^2, \tag{4.23}$$

$$(f_t,u_{ht}) \leq \frac{\nu}{4}|u_{ht}|_1^2 + c\|f_t\|_0^2. \tag{4.24}$$

From the definition of $(u_h(0),p_h(0)) = (R_h(u_0,p_0),Q_h(u_0,p_0))$, it holds that

$$(u_{ht}(0),v_h) - \nu(\Delta u_0,v_h) + (v_h,\nabla p_0) + b(u_h(0),u_h(0),v_h) = (f(0),v_h), \tag{4.25}$$

which with (4.1) and (4.2) yield

$$\|u_{ht}(0)\|_0 \leq c\|u_0\|_2 + \|p_0\|_1 + c\|A_h u_h(0)\|_0|u_h(0)|_1 + \|f(0)\|_0. \tag{4.26}$$

Now we integrate (4.22) and use (4.23)–(4.24), (4.26) and (4.20)–(4.21) to obtain

$$\|u_{ht}(t)\|_0^2 + \int_0^t (\nu|u_{ht}|_1^2 + \beta\mathcal{C}_h(p_{ht},p_{ht}))ds \leq \kappa, \ 0 \leq t \leq T. \tag{4.27}$$

Combining (4.17) with (4.20)–(4.21) and (4.27) and using (3.2) yields (4.9)–(4.10).
Furthermore, we derive from (4.19), (4.8), (4.2) and (2.6) that

$$\int_0^t \sigma(s)\|A_h u_{ht}\|_0^2 ds \leq c\int_0^t \sigma(s)(h^{-2}|u_t-u_{ht}|_1^2 + \|Au_t\|_0^2)ds \leq \kappa, \tag{4.28}$$

for all $0 \leq t \leq T$, which (4.9) and (4.17) with $i=0$ yield

$$\int_0^t \sigma(s)\|u_{htt} + \nabla_h p_{ht}\|_0^2 ds \leq \kappa, \ \forall 0 \leq t \leq T. \tag{4.29}$$

Moreover, we derive from (4.15) that

$$\begin{aligned}(u_{htt},v_h) &+ a(u_{ht},v_h) - d(v_h,p_{ht}) + d(u_{htt},q_h) + \beta\mathcal{C}_h(p_{htt},q_h)\\ &+ b(u_{ht},u_h,v_h) + b(u_h,u_{ht},v_h) = (f_t,v_h).\end{aligned} \tag{4.30}$$

By taking $(v_h,q_h) = \sigma(t)(u_{htt},p_{ht})$ in (4.30), we obtain

$$\begin{aligned}\sigma(t)\|u_{htt}\|_0^2 &+ \frac{1}{2}\frac{d}{dt}(\sigma(t)\nu|u_{ht}|_1^2 + \sigma(t)\beta\mathcal{C}_h(p_{ht},p_{ht}))\\ &+ \sigma(t)b(u_{ht},u_h,u_{htt}) + \sigma(t)b(u_h,u_{ht},u_{htt})\\ &= \frac{1}{2}\frac{d}{dt}\sigma(t)(\nu|u_{ht}|_1^2 + \beta\mathcal{C}_h(p_{ht},p_{ht})) + \sigma(t)(f_t,u_{htt}).\end{aligned} \tag{4.31}$$

Using (4.1)–(4.3), we have

$$\begin{aligned}\sigma(t)|b(u_{ht},u_h,u_{htt})| &+ \sigma(t)|b(u_h,u_{ht},u_{htt})|\\ &\le c\sigma(t)|u_{ht}|_1\|A_hu_h\|_0\|u_{htt}\|_0^2\\ &\le \frac{1}{4}\sigma(t)\|u_{htt}\|_0^2+c\sigma(t)|u_{ht}|_1^2\|A_hu_h\|_0^2,\\ \sigma(t)|(f_t,u_{htt})| &\le \frac{1}{4}\sigma(t)\|u_{htt}\|_0^2+c\|f_t\|_0^2.\end{aligned}$$

Combining these inequalities with (4.31) yields

$$\begin{aligned}\sigma(t)\|u_{htt}\|_0^2 &+ \frac{d}{dt}(\sigma(t)\nu|u_{ht}|_1^2+\sigma(t)\beta\mathcal{C}_h(p_{ht},p_{ht}))\\ &\le \nu|u_{ht}|_1^2+\beta\mathcal{C}_h(p_{ht},p_{ht})+c\sigma(t)|u_{ht}|_1^2\|A_hu_h\|_0^2+\|f_t\|_0^2.\end{aligned}\tag{4.32}$$

In view of (4.10), there exists a sequence $\epsilon_n\to 0$ such that

$$\sigma(\epsilon_n)\nu|u_{ht}(\epsilon_n)|_1^2+\sigma(\epsilon_n)\beta\mathcal{C}_h(p_{ht}(\epsilon_n),p_{ht}(\epsilon_n))\to 0.$$

Therefore, integrating (4.32) from ϵ_n to t and letting $\epsilon_n\to 0$, one finds

$$\begin{aligned}&\int_0^t\sigma(s)\|u_{htt}\|_0^2ds+\sigma(t)\nu|u_{ht}|_1^2+\sigma(t)\beta\mathcal{C}_h(p_{ht}(t),p_{ht}(t))\\ &\quad\le \int_0^t(\nu|u_{ht}|_1^2+\beta\mathcal{C}_h(p_{ht},p_{ht}))ds\\ &\qquad+ c\int_0^t\sigma(s)|u_{ht}|_1^2\|A_hu_h\|_0^2ds+c\int_0^t\|f_t\|_0^2ds.\end{aligned}\tag{4.33}$$

Applying the Gronwall lemma to (4.33) and using (4.10), we obtain

$$\int_0^t\sigma(s)\|u_{htt}\|_0^2ds+\sigma(t)\nu|u_{ht}(t)|_1^2+\sigma(t)\beta\mathcal{C}_h(p_{ht}(t),p_{ht}(t))\le k,\ 0\le t\le T.\tag{4.34}$$

Combining (4.34) with (4.28)–(4.29) and using (4.18) with $i=2$ yields (4.11)–(4.12).

Similarly, we can prove (4.13)–(4.14) by using (3.2), (3.4), (4.1)–(4.4) and (4.9)–(4.12). □

Also, we will introduce some discrete versions of the Gronwall lemma by a slightly improved argument used in [16, 39].

Lemma 4.4. *Let C, τ and a_n,b_n,d_n, for integers $n\ge n_1$, be nonnegative numbers such that*

$$a_m+\tau\sum_{n=n_1}^{m}b_n\le\tau\sum_{n=n_1}^{m-1}a_nd_n+C,\ \forall m\ge n_1.\tag{4.35}$$

Then,

$$a_m+\tau\sum_{n=n_1}^{m}b_n\le\exp(\tau\sum_{n=n_1}^{m-1}d_n)C,\ \forall m\ge n_1.\tag{4.36}$$

5. Fully discrete stabilized finite element method

In this section we consider the time discretization of the stabilized finite element approximation. Let $t_n = n\tau$, where $\tau = \frac{T}{N}$ and N is an integer. The Crank–Nicolson extrapolation scheme applied to the stabilized finite element approximation is to determine the series $\{u_h^n\}_{n=1}^N \subset X_h$, $\{p_h^n\}_{n=1}^N \subset M_h$ as the solution of the recursive linear equation

$$(d_t u_h^n, v_h) + \mathcal{B}_h((\bar{u}_h^n, \bar{p}_h^n); (v_h, q_h)) + b(\phi(u_h^n), \bar{u}_h^n, v_h) = (\bar{f}(t_n), v_h), \tag{5.1}$$

for all $(v_h, q_h) \in (X_h, M_h)$ for the initial value $(u_h^0, p_h^0) = (u_{0h}, p_{0h})$. Here and after, we define a small time step size $\tau_0 = T^{-1}\tau^{3/2}$ on time interval $(0, \tau^{1/2}]$ and a large time step size $\tau = \frac{T}{n_0}$ on $[\tau^{1/2}, T]$ for some fixed integer n_0, where

$$\tau_0 n_0 = \tau^{1/2},\ N = n_0 + \frac{T - \tau^{1/2}}{\tau} = \frac{2T}{\tau} - \tau^{-1/2},$$

$$d_t u_h^n = \frac{1}{k}(u_h^n - u_h^{n-1}), k = \tau_0 \text{ as } 1 \le n \le n_0,\ k = \tau \text{ as } n_0 + 1 \le n \le N,$$

$$\phi(u_h^0) = \phi(u_h^1) = u_h^0,\ \phi(u_h^n) = \frac{3}{2}u_h^{n-1} - \frac{1}{2}u_h^{n-2},\ 2 \le n \le N,$$

$$\bar{u}_h^n = \frac{1}{2}(u_h^n + u_h^{n-1}),\ \bar{u}_h(t_n) = \frac{1}{2}(u_h(t_n) + u_h(t_{n-1})).$$

From the definition of ϕ, we find that the Crank–Nicolson extrapolation scheme is an implicit scheme for the viscous and pressure terms and a semi-implicit scheme for the nonlinear term.

In order to analyze the discretization errors $(e_h^n, \bar{\mu}^n) = (u_h(t_n) - u_h^n, \bar{p}_h(t_n) - \bar{p}_h^n)$ with $(e_h^0, \mu_h^0) = (0, 0)$, we deduce from (3.2) that

$$\begin{aligned} &\frac{1}{k}(u_h(t_n) - u_h(t_{n-1}), v_h) + \frac{1}{k}\int_{t_{n-1}}^{t_n} \mathcal{B}_h((u_h(t), p_h(t)); (v_h, q_h))dt \\ &+ \frac{1}{k}\int_{t_{n-1}}^{t_n} b(u_h(t), u_h(t), v_h)dt = \frac{1}{k}\int_{t_{n-1}}^{t_n} (f(t), v_h)dt \end{aligned} \tag{5.2}$$

for all $(v_h, q_h) \in (X_h, M_h)$. Subtracting (5.1) from (5.2) and using the following formulas

$$d(\bar{u}_h(t_n), q_h) + \beta\mathcal{C}_h(\bar{p}_h(t_n), q_h) = 0,$$

$$\frac{1}{k}\int_{t_{n-1}}^{t_n} (d(u_h(t), q_h) + \beta\mathcal{C}_h(p_h(t), q_h))dt = 0,$$

$$\bar{\phi}(t_n) - \frac{1}{k}\int_{t_{n-1}}^{t_n} \phi(t)dt = \frac{1}{2k}\int_{t_{n-1}}^{t_n} (t_n - t)(t - t_{n-1})\phi_{tt}(t)dt,$$

for all $\phi \in H^2(t_{n-1}, t_n; F)$ for some Hilbert space F, then using (3.2), results in

$$\begin{aligned} &(d_t e_h^n, v_h) + \mathcal{B}_h((\bar{e}_h^n, \tilde{\mu}_h^n); (v_h, q_h)) + b(\phi(e_h^n), \bar{u}_h(t_n), v_h) + \beta(\phi(u_h^n), \bar{e}_h^n, v_h) \\ &\quad = (\tilde{E}_n, v_h) - \frac{\beta}{2k}\int_{t_{n-1}}^{t_n} (t_n - t)(t - t_{n-1})\mathcal{C}_h(p_{htt}, q_h)dt, \end{aligned} \tag{5.3}$$

for all $(v_h, q_h) \in (X_h, M_h)$, and

$$\begin{aligned} &(d_t e_h^n, v_h) + \mathcal{B}_h((\bar{e}_h^n, \bar{\mu}_h^n); (v_h, q_h)) + b(\phi(e_h^n), \bar{u}_h(t_n), v_h) + b(\phi(u_h^n), \bar{e}_h^n, v_h) \\ &\quad = (E_n, v_h), \end{aligned} \tag{5.4}$$

for all $(v_h, q_h) \in (X_h, M_h)$, where $\tilde{\mu}_h^n = \frac{1}{k}\int_{t_{n-1}}^{t_n} p_h(t)dt - \bar{p}_h^n$ and

$$(5.5)\quad (\tilde{E}_1, v_h) = -\frac{1}{2k}\int_{t_0}^{t_1}(t_1-t)(t-t_0)(u_{httt} + P_h\nabla_h p_{htt}, v_h)dt - \frac{1}{2}b(\int_{t_0}^{t_1} u_{ht}dt, u_h(t_1), v_h),$$

$$(5.6)\quad \begin{aligned}(\tilde{E}_n, v_h) = &-\frac{1}{2k}\int_{t_{n-1}}^{t_n}(t_n-t)(t-t_{n-1})(u_{httt} + P_h\nabla_h p_{htt}, v_h)dt \\ &- \frac{1}{2}b(\int_{t_{n-1}}^{t_n}(t_n-t)u_{htt}dt - \int_{t_{n-2}}^{t_{n-1}}(t-t_{n-2})u_{htt}dt, \bar{u}_h(t_n), v_h) \\ &- \frac{1}{4}b(\int_{t_{n-1}}^{t_n} u_{ht}dt, \int_{t_{n-1}}^{t_n} u_{ht}dt, v_h),\end{aligned}$$

for all $2 \le n \le N$, and

$$(5.7)\quad (E_1, v_h) = -\frac{1}{2k}\int_{t_0}^{t_1}(t_1-t)(t-t_0)(u_{httt}, v_h)dt - \frac{1}{2}b(\int_{t_0}^{t_1} u_{ht}dt, u_h(t_1), v_h),$$

$$(5.8)\quad \begin{aligned}(E_n, v_h) = &-\frac{1}{2k}\int_{t_{n-1}}^{t_n}(t_n-t)(t-t_{n-1})(u_{httt}, v_h)dt \\ &- \frac{1}{2}b(\int_{t_{n-1}}^{t_n}(t_n-t)u_{htt}dt - \int_{t_{n-2}}^{t_{n-1}}(t-t_{n-2})u_{htt}dt, \bar{u}_h(t_n), v_h) \\ &- \frac{1}{4}b(\int_{t_{n-1}}^{t_n} u_{ht}dt, \int_{t_{n-1}}^{t_n} u_{ht}dt, v_h),\end{aligned}$$

for all $2 \le n \le N$.

In order to provide the bound of the error $(e_h^n, \bar{\mu}_h^n)$, we need to provide the bound of $\tilde{E}_n$ and E_n.

Lemma 5.1. *Under the assumptions of Theorem* 4.2*, it holds that*

$$(5.9)\quad \tau_0\sum_{n=1}^{n_0}\sigma^i(t_n)\|A_h^{-1/2}P_h\tilde{E}_n\|_0^2 \le \kappa\tau_0^{i+2}, \quad i=0,1,$$

$$(5.10)\quad \tau\sum_{n=n_0+1}^{N}\|A_h^{-1/2}P_h\tilde{E}_n\|_0^2 \le \kappa\tau^3,$$

$$(5.11)\quad \tau_0\sum_{n=1}^{n_0}\sigma^i(t_n)\|P_hE_n\|_0^2 \le \kappa\tau_0^{i+1}, \quad i=0,1,$$

$$(5.12)\quad \tau\sum_{n=n_0+1}^{N}\sigma^i(t_n)\|P_hE_n\|_0^2 \le \kappa\tau^{5/2+i/2}, \quad i=0,1.$$

Proof. In this proof for brevity we will make frequent use of (4.1)–(4.4) and the facts

$$\begin{aligned}\sigma(t_n) &\le \sigma(t_{n-1}) + \tau_0, \quad 1 \le n \le n_0,\\ \sigma(t_{n-1}) &\le 2\sigma(t_{n-2}), \quad n_0 + 1 \le n \le N,\end{aligned} \tag{5.13}$$

without explicit mention in this section.

First, we derive from (5.6) and the Schwarz inequality that

$$\begin{aligned}\|A_h^{-1/2} P_h \tilde{E}_n\|_0 &= \sup_{v_h \in X_h} \frac{|(\tilde{E}_n, v_h)|}{\|A_h^{1/2} v_h\|_0}\\ &\le \frac{1}{k}\int_{t_{n-1}}^{t_n} (t_n - t)(t - t_{n-1})\|A_h^{-1/2}(u_{httt} + P_h \nabla_h p_{htt})\|_0 dt\\ &\quad + \kappa(\int_{t_{n-1}}^{t_n} (t_n - t)|u_{htt}|_1 dt + \int_{t_{n-2}}^{t_{n-1}} (t - t_{n-2})|u_{htt}|_1 dt)\\ &\quad + \kappa(\int_{t_{n-1}}^{t_n} |u_{ht}|_1 dt)^2\\ &\le \kappa k^{-1/2}(\int_{t_{n-1}}^{t_n} (t_n - t)^2 k^2\\ &\qquad \times (\|A_h^{-1/2}(u_{httt} + P_h \nabla_h p_{htt})\|_0^2 + |u_{htt}|_1^2) dt)^{1/2}\\ &\quad + \kappa k^{-1/2}(\int_{t_{n-2}}^{t_{n-1}} (t - t_{n-2})^2 k^2 |u_{htt}|_1^2 dt)^{1/2} + \kappa k \int_{t_{n-1}}^{t_n} |u_{ht}|_1^2 dt,\end{aligned} \tag{5.14}$$

with $k = \tau_0$ for $1 \le n \le n_0$ and $k = \tau$ for $n_0 + 1 \le n \le N$. Similarly, we deduce from (5.5) that

$$\begin{aligned}\|A_h^{-1/2} P_h \tilde{E}_1\|_0 &\le c_0\gamma_0 \|A_h u_h(t_1)\|_0 \sup_{t_0 \le t \le t_1} \|u_{ht}(t)\|_0 \tau_0\\ &\quad + \kappa\tau_0^{1/2}(\int_{t_0}^{t_1} \sigma^2(t)\|A_h^{-1/2}(u_{httt} + P_h \nabla_h p_{htt})\|_0^2 dt)^{1/2}.\end{aligned} \tag{5.15}$$

Thus, by using (5.13)–(5.14) and Theorem 4.3, we obtain

$$\begin{aligned}&\sigma^i(t_n)\|A_h^{-1/2}\tilde{E}_n\|_0^2 \tau_0\\ &\quad \le \kappa\tau_0^{2+i}\int_{t_{n-1}}^{t_n} (\sigma^2(t)\|A_h^{-1/2}(u_{httt} + P_h \nabla_h p_{htt})\|_0^2 + |u_{ht}|_1^2) dt\\ &\qquad + \kappa\tau_0^{2+i}\int_{t_{n-2}}^{t_n} \sigma^2(t)|u_{htt}|_1^2 dt,\ 2 \le n \le n_0,\ i = 0, 1,\end{aligned} \tag{5.16}$$

$$\begin{aligned}&\|A_h^{-1/2}\tilde{E}_n\|_0^2 \tau\\ &\quad \le \kappa\tau^4\sigma^{-2}(t_{n-1})\int_{t_{n-1}}^{t_n} \sigma^2(t)(\|A_h^{-1/2}(u_{httt} + P_h \nabla_h p_{htt})\|_0^2\\ &\qquad + |u_{ht}|_1^2) dt + \kappa\tau^4\sigma^{-2}(t_{n-1})\int_{t_{n-2}}^{t_n} \sigma^2(t)|u_{htt}|_1^2 dt,\end{aligned} \tag{5.17}$$

for all $n_0 + 1 \le n \le N$. Summing (5.16) from $n = 2$ to $n = n_0$ and (5.17) from $n = n_0 + 1$ to $n = N$, respectively, using (5.15) and Theorem 4.3 and noting $\sigma^{-2}(t_{n-1}) \le t_{n_0}^{-2} = \tau^{-1}$ for all $n_0 + 1 \le n \le N$, we have obtained (5.9) and (5.10).

Furthermore, we derive from (5.7)–(5.8) that

$$
\begin{aligned}
\|P_hE_1\|_0 &= \sup_{v_h\in X_h}\frac{|(E_1,v_h)|}{\|v_h\|_0}\\
&\le c_0\gamma_0\tau_0^{1/2}\|A_hu_h(t_1)\|_0(\int_{t_0}^{t_1}|u_{ht}(t)|_1^2dt)^{1/2}\\
&\quad+2\sup_{t_0\le t\le t_1}\|u_{ht}(t)\|_0,
\end{aligned}\tag{5.18}
$$

$$
\begin{aligned}
\|P_hE_n\|_0 &= \sup_{v_h\in X_h}\frac{|(E_n,v_h)|}{\|v_h\|_0}\\
&\le k^{-1/2}(\int_{t_{n-1}}^{t_n}(t_n-t)^2(t-t_{n-1})^2\|u_{httt}\|_0dt)^{1/2}\|A_h\bar{u}_h(t_n)\|_0\\
&\quad+ck^{-1/2}(\int_{t_{n-1}}^{t_n}(t_n-t)^2k^2|u_{htt}|_1^2dt)^{1/2}\\
&\quad+ck^{-1/2}(\int_{t_{n-2}}^{t_{n-1}}(t-t_{n-2})^2k^2|u_{htt}|_1^2dt)^{1/2}\|A_h\bar{u}_h(t_n)\|_0\\
&\quad+c\kappa^{1/2}(\int_{t_{n-1}}^{t_n}k\|A_hu_{ht}\|_0^2dt)^{1/2}(\int_{t_{n-1}}^{t_n}|u_{ht}|_1^2dt)^{1/2},
\end{aligned}\tag{5.19}
$$

for all $2\le n\le N$. Hence, we derive from (5.19), (5.13), and Theorem 4.3 that

$$
\begin{aligned}
\sigma^i(t_n)\|P_hE_n\|_0^2\tau_0 &\le \kappa\tau_0^{i+1}\int_{t_{n-1}}^{t_n}(\sigma^3(t)\|u_{httt}\|_0^2+|u_{ht}|_1^2)dt\\
&\quad+\kappa\tau_0^{i+1}\int_{t_{n-2}}^{t_n}\sigma^2(t)|u_{htt}|_1^2dt,\quad 2\le n\le n_0,
\end{aligned}\tag{5.20}
$$

$$
\begin{aligned}
\sigma^i(t_n)\|P_hE_n\|_0^2\tau &\le \kappa\tau^4\sigma^{-3+i}(t_{n-1})\int_{t_{n-1}}^{t_n}(\sigma^3(t)\|u_{httt}\|_0^2+|u_{ht}|_1^2)dt\\
&\quad+\kappa\tau^4\sigma^{-3+i}(t_{n-1})\int_{t_{n-2}}^{t_n}\sigma^3(t)|u_{htt}|_1^2dt,\quad n_0+1\le n\le N.
\end{aligned}\tag{5.21}
$$

Summing (5.20) from $n=2$ to $n=n_0$ and (5.21) from $n=n_0+1$ to $n=N$, respectively, using Theorem 4.3 and (5.18) and noting $\sigma^{-1}(t_{n-1})\le t_{n_0}^{-1}\le\tau^{-1/2}$ for all $n_0+1\le n\le N$, we arrive at (5.11) and (5.12). □

6. Error analysis

In this section we will analyze the error $(e_h^n,\bar{\mu}_h^n)$. With the aid of Lemma 5.1, we shall obtain the following lower-order error estimates.

Lemma 6.1. *Under the assumptions of Theorem 4.2, it holds that*

$$
\|e_h^m\|_0^2+\tau_0\sum_{n=1}^m(\nu|\bar{e}_h^n|_1^2+\beta\mathcal{C}_h(\tilde{\mu}_h^n,\tilde{\mu}_h^n))\le\kappa\tau_0^2,\quad 1\le m\le n_0,\tag{6.1}
$$

$$
\|e_h^m\|_0^2+\tau\sum_{n=1}^m(\nu|\bar{e}_h^n|_1^2+\beta\mathcal{C}_h(\tilde{\mu}_h^n,\tilde{\mu}_h^n))\le\kappa\tau_0^2+\kappa\tau^3,\quad n_0+1\le m\le N.\tag{6.2}
$$

Proof. First, we take $(v_h, q_h) = 2(\bar{e}_h^n, \tilde{\mu}_h^n)k$ in (5.3) and use (4.1)–(4.2) and (4.4), obtaining

$$
\begin{aligned}
&\|e_h^n\|_0^2 - \|e_h^{n-1}\|_0^2 + 2\nu|\bar{e}_h^n|_1^2 k + 2\beta\mathcal{C}_h(\tilde{\mu}_h^n, \tilde{\mu}_h^n)k \\
&\quad = -2b(\phi(e_h^n), \bar{u}_h(t_n), \bar{e}_h^n)k + 2(\tilde{E}_n, \bar{e}_h^n)k \\
&\quad\quad - \beta\int_{t_{n-1}}^{t_n}(t-t_n)(t-t_{n-1})\mathcal{C}_h(p_{htt}, \tilde{\mu}_h^n)dt \\
&\quad \le \nu|\bar{e}_h^n|_1^2 k + \beta\mathcal{C}_h(\tilde{\mu}_h^n, \tilde{\mu}_h^n)k + 4\nu^{-1}\|A_h^{-1/2}P_h\tilde{E}_n\|_0^2 k \\
&\quad\quad + 4\nu^{-1}c_0^2\gamma_0^2\|A_h\bar{u}_h(t_n)\|_0^2\|\phi(e_h^n)\|_0^2 k \\
&\quad\quad + \beta\int_{t_{n-1}}^{t_n}(t-t_n)^2(t-t_{n-1})^2\mathcal{C}_h(p_{htt}, p_{htt})dt,
\end{aligned} \tag{6.3}
$$

for all $1 \le n \le N$ with $k = \tau_0$ for $1 \le n \le n_0$ and $k = \tau$ for $n_0 + 1 \le n \le N$. Summing (6.3) from $n = 1$ to $n = m$ for $m \le n_0$ and $n = n_0 + 1$ to $n = m$ for $n_0 + 1 \le m \le N$, respectively, noting $e_h^0 = 0$, $\sigma^{-2}(t_{n-1}) \le \tau_{n_0}^{-2} = \tau^{-1}$, $n_0 + 1 \le n \le N$ and using Lemma 5.1 and Theorem 4.3, we obtain

$$
\begin{aligned}
&\|e^m\|_0^2 + \tau_0\sum_{n=1}^{m}(\nu|\bar{e}_h^n|_1^2 + \beta\mathcal{C}_h(\tilde{\mu}_h^n, \tilde{\mu}_h^n)) \\
&\quad \le \kappa\tau_0^2 + \kappa\tau_0\sum_{n=1}^{m-1}\|e_h^n\|_0^2, \quad 1 \le m \le n_0,
\end{aligned} \tag{6.4}
$$

$$
\begin{aligned}
&\|e^m\|_0^2 + \tau\sum_{n=1}^{m}(\nu|\bar{e}_h^n|_1^2 + \beta\mathcal{C}_h(\tilde{\mu}_h^n, \tilde{\mu}_h^n)) \\
&\quad \le \|e_h^{n_0}\|_0^2 + \kappa\tau^3 + \kappa\tau\sum_{n=1}^{m-1}\|e_h^n\|_0^2, \quad n_0 + 1 \le m \le N.
\end{aligned} \tag{6.5}
$$

Applying Lemma 4.4 to (6.4) and (6.5), respectively, we arrive at (6.1) and (6.2). □

Lemma 6.2. *Under the assumptions of Theorem* 4.2*, the following estimates hold:*

$$
\nu|e_h^m|_1^2 + \beta\mathcal{C}_h(\mu_h^m, \mu_h^m) + \tau_0\sum_{n=1}^{m}\|d_t e_h^n\|_0^2 \le \kappa\tau_0, \ 1 \le m \le n_0, \tag{6.6}
$$

$$
\nu|e_h^m|_1^2 + \beta\mathcal{C}_h(\mu_h^m, \mu_h^m) + \tau\sum_{n=n_0+1}^{m}\|d_t e_h^n\|_0^2 \le \kappa\tau_0 + \kappa\tau^2, \ n_0 + 1 \le m \le N, \tag{6.7}
$$

where $\mu^0 = 0,\ \mu^n = 2\bar{\mu}^n - \mu^{n-1}$.

Proof. In view of the fact

$$d(e_h^0, q_h) + \beta\mathcal{C}_h(\mu_h^0, q_h) = 0, \quad \forall q_h \in M_h,$$

we derive from (5.4) that

$$
\begin{aligned}
&(d_t e_h^n, v_h) + a(\bar{e}_h^n, v_h) - d(v_h, \bar{\mu}_h^n) + d(d_t e_h^n, q_h) + \beta\mathcal{C}_h(d_t\mu_h^n, q_h) \\
&\quad + b(\phi(u_h^n), \bar{e}_h^n, v_h) + b(\phi(e_h^n), \bar{u}_h(t_n), v_h) \\
&\quad = (E_n, v_h), \quad \forall(v_h, q_h) \in (X_h, M_h).
\end{aligned} \tag{6.8}
$$

By setting $(v_h, q_h) = 2(d_t e_h^n, \bar{\mu}_h^n)k$ in (6.8), we obtain

$$\begin{aligned}&2\|d_t e_h^n\|_0^2 k + \nu(|e_h^n|_1^2 - |e_h^{n-1}|_1^2) + \beta(\mathcal{C}_h(\mu_h^n,\mu_h^n) - \mathcal{C}_h(\mu_h^{n-1},\mu_h^{n-1}))\\ &\quad + 2b(\phi(e_h^n), \bar{u}_h(t_n), d_t e_h^n)k + 2b(\phi(u_h(t_n)), \bar{e}_h^n, d_t e_h^n)k\\ &\quad - 2b(\phi(e_h^n), \bar{e}_h^n, d_t e_h^n)k = 2(E_n, d_t e_h^n)k.\end{aligned} \tag{6.9}$$

Thus, it follows from (4.1)–(4.4), Theorem 4.3 and Lemma 6.1 that

$$\begin{aligned}&2|b(\phi(e_h^n), \bar{u}_h(t_n), d_t e_h^n)|k + 2|b(\phi(u_h(t_n)), \bar{e}_h^n, d_t e_h^n)|k\\ &\quad\le 2c_0\gamma_0(\|A_h\phi(u_h(t_n))\|_0|\bar{e}_h^n|_1 + \|A_h\bar{u}_h(t_n)\|_0|\phi(e_h^n)|_1)\|d_t e^n\|_0 k\\ &\quad\le \frac{1}{2}\|d_t e^n\|_0^2 k + \kappa(|e_h^{n-1}|_1^2 + |e_h^{n-2}|_1^2 + |\bar{e}_h^n|_1^2)k,\end{aligned}$$

$$2|(E_n, d_t e_h^n)|k \le \frac{1}{4}\|d_t e^n\|_0^2 k + 4\|P_h E_n\|_0^2 k,$$

$$\begin{aligned}2|b(\phi(e_h^n), \bar{e}_h^n, d_t e_h^n)|k &\le c_0\|\phi(e_h^n)\|_0^{1/2}|\phi(e_h^n)|_1^{1/2}|\bar{e}_h^n|_1\|d_t e_h^n\|_0^{1/2}|d_t e_h^n|_1^{1/2}k\\ &\quad + c_0\|\phi(e_h^n)\|_0^{1/2}|\phi(e_h^n)|_1^{1/2}\ |\bar{e}_h^n\|_0^{1/2}|\bar{e}_h^n|_1^{1/2}|d_t e_h^n|_1 k\\ &\le \kappa|\phi(e_h^n)|_1^{1/2}(|\bar{e}_h^n|_1|e_h^n - e_h^{n-1}|_1^{1/2} + |\bar{e}_h^n|_1^{1/2}|e_h^n - e_h^{n-1}|_1)k\\ &\le \kappa(|\bar{e}^n|_1^2 + |e_h^{n-1}|_1^2 + |e_h^{n-2}|_1^2)k.\end{aligned}$$

Combining these inequalities with (6.9) yields

$$\begin{aligned}&\nu(|e_h^n|_1^2 - |e_h^{n-1}|_1^2) + \beta(\mathcal{C}_h(\mu_h^n,\mu_h^n) - \mathcal{C}_h(\mu_h^{n-1},\mu_h^{n-1})) + \|d_t e_h^n\|_0^2\tau_0\\ &\quad\le \kappa(|e_h^{n-1}|_1^2 + |e_h^{n-2}|_1^2)\tau_0 + \kappa(|\bar{e}_h^n|_1^2 + \|P_h E_n\|_0^2)\tau_0,\end{aligned} \tag{6.10}$$

$$\begin{aligned}&\nu(|e_h^n|_1^2 - |e_h^{n-1}|_1^2) + \beta(\mathcal{C}_h(\mu_h^n,\mu_h^n) - \mathcal{C}_h(\mu_h^{n-1},\mu_h^{n-1})) + \|d_t e_h^n\|_0^2\tau\\ &\quad\le \kappa(|e_h^{n-1}|_1^2 + |e_h^{n-2}|_1^2)\tau + \kappa(|\bar{e}_h^n|_1^2 + \|P_h E_n\|_0^2)\tau,\end{aligned} \tag{6.11}$$

Summing (6.10) from $n = 1$ to $n = m$ for $1 \le m \le n_0$ and summing (6.11) from $n = n_0 + 1$ to $n = m$ for $n_0 + 1 \le m \le N$, respectively, then using Lemmas 5.1 and 6.1 and Theorem 4.3, we obtain

$$\begin{aligned}&\nu|e_h^m|_1^2 + \beta\mathcal{C}_h(\mu_h^m,\mu_h^m) + \tau_0\sum_{n=1}^{m}\|d_t e_h^n\|_0^2\\ &\quad\le \kappa\tau_0 + \kappa\tau_0\sum_{n=1}^{m-1}|e_h^n|_1^2,\ 1 \le m \le n_0,\end{aligned} \tag{6.12}$$

$$\begin{aligned}&\nu|e_h^m|_1^2 + \beta\mathcal{C}_h(\mu_h^m,\mu_h^m) + \tau_0\sum_{n=1}^{m}\|d_t e_h^n\|_0^2\\ &\quad\le \nu|e_h^{n_0}|_1^2 + \beta\mathcal{C}_h(\mu_h^{n_0},\mu_h^{n_0}) + \kappa\tau^2 + \kappa\tau\sum_{n=n_0+1}^{m-1}|e_h^n|_1^2,\ n_0+1 \le n \le N.\end{aligned} \tag{6.13}$$

Applying Lemma 4.4 to (6.12) and 6.13), respectively, we obtain (6.6)–(6.7). □

Lemma 6.3. *Under the assumptions of Theorem* 4.2*, we have*

$$\sigma(t_m)\|d_t e_h^m\|_0^2 \le \kappa\tau_0, \quad 1 \le n \le n_0, \tag{6.14}$$

$$\sigma(t_m)\|d_t e_h^m\|_0^2 \le \kappa\tau_0 + \kappa\tau^{3/2}, \quad n_0 + 1 \le n \le N. \tag{6.15}$$

Proof. From (5.3) we derive

$$
\begin{aligned}
&\frac{1}{k}(d_t e_h^n - d_t e_h^{n-1}, v_h) + \mathcal{B}_h((d_t\bar{e}_h^n, d_t\tilde{\mu}_h^n);(v_h,q_h))\\
&\quad + b(\phi(d_t u_h^n), \bar{e}_h^n, v_h) + b(\phi(d_t u_h^{n-1}), d_t\bar{e}_h^n, v_h)\\
&\quad + b(\phi(d_t e_h^n), \bar{u}_h(t_n), v_h) + b(\phi(e_h^{n-1}), d_t\bar{u}_h(t_n), v_h)\\
&\quad = (d_t\tilde{E}_n, v_h) - \frac{\beta}{2k^2}\int_{t_{n-1}}^{t_n}(t_n - t)(t-t_{n-1})\mathcal{C}_h(p_{htt}, q_h)dt\\
&\quad - \frac{\beta}{2k^2}\int_{t_{n-2}}^{t_{n-1}}(t_{n-1}-t)(t-t_{n-2})\mathcal{C}_h(p_{htt}, q_h)dt,\\
&\qquad\qquad \forall (v_h,q_h)\in(X_h,M_h),
\end{aligned}
\tag{6.16}
$$

where we use the notation of $t_{-2} = t_{-1} = t_0$.

By setting $(v_h, q_h) = 2(d_t\bar{e}_h^n, d_t\bar{\mu}_h^n)k$ in (6.16) and using (4.2), we obtain

$$
\begin{aligned}
&\|d_t e_h^n\|_0^2 - \|d_t e_h^{n-1}\|_0^2 + 2(\nu|d_t\bar{e}_h^n|_1^2 + \beta\mathcal{C}_h(d_t\tilde{\mu}_h^n, d_t\tilde{\mu}_h^n))k\\
&\quad + \ -2b(\phi(d_t e_h^n), \bar{e}_h^n, d_t\bar{e}_h^n)k + 2b(\phi(d_t e_h^n), \bar{u}_h(t_n), d_t\bar{e}_h^n)\\
&\quad + 2b(\phi(d_t u_h(t_n)), \bar{e}_h^n, d_t\bar{e}_h^n)k + 2b(\phi(e_h^{n-1}), d_t\bar{u}_h(t_n), d_t\bar{e}_h^n)k\\
&= 2(d_t\tilde{E}_n, d_t\bar{e}_h^n)k - \frac{\beta}{2k}\int_{t_{n-1}}^{t_n}(t_n-t)(t-t_{n-1})\mathcal{C}_h(p_{htt}, d_t\tilde{\mu}_h^n)dt\\
&\quad - \frac{\beta}{2k}\int_{t_{n-2}}^{t_{n-1}}(t_{n-1}-t)(t-t_{n-2})\mathcal{C}_h(p_{htt}, d_t\tilde{\mu}_h^n)dt.
\end{aligned}
\tag{6.17}
$$

Then, a simple calculation, using (4.1)–(4.4), Theorem 4.3 and Lemmas 6.1 and 6.2, yields

$$
\begin{aligned}
2|b(\phi(d_t e_h^n), \bar{e}_h^n, d_t\bar{e}_h^n)|k &\le 2c_0\gamma_0|\phi(d_t e_h^n)|_1|\bar{e}_h^n|_1|d_t\bar{e}_h^n|_1 k\\
&\le \frac{\nu}{4}|d_t\bar{e}_h^n|_1^2 k + 4\nu^{-1}c_0^2\gamma_0^2|\phi(e_h^n - e_h^{n-1})|_1^2|\bar{e}_h^n|_1^2 k^{-1},\\
2|b(\phi(e_h^{n-1}), d_t\bar{u}_h(t_n), d_t\bar{e}_h^n)|k &\le 2c_0\gamma_0|\phi(e_h^{n-1})|_1|d_t\bar{u}_h(t_n)|_1|d_t\bar{e}_h^n|_1 k\\
&\le \frac{\nu}{4}|d_t\bar{e}_h^n|_1^2 k + 4\nu^{-1}c_0^2\gamma_0^2\int_{t_{n-2}}^{t_n}|u_{ht}|_1^2 dt|\phi(e_h^{n-1})|_1^2,\\
2|b(\phi(d_t u_h(t_n)), \bar{e}_h^n, d_t\bar{e}_h^n)|k &\le 2c_0\gamma_0|\phi(d_t u_h(t_n))|_1|\bar{e}_h^n|_1|d_t\bar{e}_h^n|_1 k\\
&\le \frac{\nu}{4}|d_t\bar{e}_h^n|_1^2 k + 5\nu^{-1}c_0^2\gamma_0^2\int_{t_{n-3}}^{t_{n-1}}|u_{ht}|_1^2 dt|\bar{e}_h^n|_1^2,\\
2|b(\phi(d_t e_h^n), \bar{u}_h(t_n), d_t\bar{e}_h^n)|k &\le 2c_0\gamma_0\|\phi(d_t e_h^n)\|_0\|A_h\bar{u}_h(t_n)\|_0|d_t\bar{e}_h^n|_1 k\\
&\le \frac{\nu}{4}|d_t\bar{e}_h^n|_1^2 k + 4\nu^{-1}c_0^2\gamma_0^2\|\phi(d_t e_h^n)\|_0^2\|A_h\bar{u}_h(t_n)\|_0^2 k,
\end{aligned}
$$

$$2|(d_tE_n, d_t\bar{e}_h^n)|k \le \frac{\nu}{4}|d_t\bar{e}_h^n|_1^2k + 8\nu^{-1}(\|A_h^{-1/2}P_hE_n\|_0^2 + \|A_h^{-1/2}P_hE_{n-1}\|_0^2)k^{-1},$$

$$\frac{\beta}{2k}|\int_{t_{n-1}}^{t_n}(t_n-t)(t-t_{n-1})\mathcal{C}_h(p_{htt}, d_t\tilde{\mu}_h^n)dt| \le \frac{\beta}{2}\mathcal{C}_h(d_t\tilde{\mu}_h^n, d_t\tilde{\mu}_h^n)k + \frac{\beta}{8k^2}\int_{t_{n-1}}^{t_n}(t_n-t)^2(t-t_{n-1})^2\mathcal{C}_h(p_{htt}, p_{htt})dt,$$

$$\frac{\beta}{2k}|\int_{t_{n-2}}^{t_{n-1}}(t_{n-1}-t)(t-t_{n-2})\mathcal{C}_h(p_{htt}, d_t\tilde{\mu}_h^n)dt| \le \frac{\beta}{2}\mathcal{C}_h(d_t\tilde{\mu}_h^n, d_t\tilde{\mu}_h^n)k + \frac{\beta}{8k^2}\int_{t_{n-2}}^{t_{n-1}}(t_{n-1}-t)^2(t-t_{n-2})^2\mathcal{C}_h(p_{htt}, p_{htt})dt,$$

for all $2 \le n \le N$.

Combining these inequalities with (6.17) and using (5.13) yields

$$\begin{aligned}
&\sigma(t_n)\|d_te_h^n\|_0^2 - \sigma(t_{n-1})\|d_te_h^{n-1}\|_0^2 \le \|d_te_h^{n-1}\|_0^2k \\
&+ 8\nu^{-1}(\sigma(t_n)\|A_h^{-1/2}P_hE_n\|_0^2 + 2\sigma(t_{n-1})\|A_h^{-1/2}P_hE_{n-1}\|_0^2)k^{-1} \\
&+ 4\nu^{-1}c_0^2\gamma_0^2|\phi(e_h^n - e_h^{n-1})|_1^2|\bar{e}_h^n|_1^2k^{-1} + 4\nu^{-1}c_0^2\gamma_0^2\int_{t_{n-2}}^{t_n}|u_{ht}|_1^2dt|\phi(e_h^{n-1})|_1^2 \\
&+ 5\nu^{-1}c_0^2\gamma_0^2\int_{t_{n-3}}^{t_{n-1}}|u_{ht}|_1^2dt|\bar{e}_h^n|_1^2 + 4\nu^{-1}c_0^2\gamma_0^2\|\phi(d_te_h^n)\|_0^2\|A_h\bar{u}_h(t_n)\|_0^2k \\
&+ \frac{\beta}{4}k\int_{t_{n-1}}^{t_n}(t-t_{n-1})\sigma(t)\mathcal{C}_h(p_{htt}, p_{htt})dt \\
&+ \frac{\beta}{4}k\int_{t_{n-2}}^{t_{n-1}}(t-t_{n-2})\sigma(t)\mathcal{C}_h(p_{htt}, p_{htt})dt.
\end{aligned} \tag{6.18}$$

Summing (6.18) from $n = 1$ to $n = m$ for $1 \le m \le n_0$ and (6.18) from $n = n_0 + 1$ to $n = m$ for $n_0 + 1 \le m \le N$, then using Lemmas 5.1, 6.1, and 6.2, and Theorem 4.3 and noting $\sigma^{-1}(t_{n-1}) \le t_{n_0}^{-1} = \tau^{-1/2}$ for all $n_0 + 1 \le n \le N$, we obtain (6.14) and (6.15). □

Lemma 6.4. *Under the assumptions of Theorem 4.2, the error $\bar{\mu}^m = \bar{p}(t_m) - \bar{p}_h^m$ satisfies the bound*

$$\sigma^{1/2}(t_m)\|\bar{p}_h(t_m) - \bar{p}_h^m\|_0 \le \kappa\tau_0^{1/2} + \kappa\tau^{3/4},\ 1 \le m \le N. \tag{6.19}$$

Proof. First, we derive from (5.4) and Theorem 3.2 that

$$\begin{aligned}
\sigma^{1/2}(t_m)\|\bar{\mu}_h^m\|_0 \ &\le\ c|\bar{e}_h^m|_1 + c\|d_te_h^m\|_0 + c|\phi(e_h^m)|_1|u_h(t_m)|_1 + c|\phi(u_h(t_m))|_1|\bar{e}_h^m|_1 \\
&+\ c|\phi(e_h^m)|_1|\bar{e}_h^m|_1 + c\sigma^{1/2}(t_m)\|E_m\|_0,\ 1 \le m \le N.
\end{aligned}$$

Then, by using Theorem 4.3 and Lemmas 6.2, 6.3, and 5.1 in the above estimate, we obtain

$$\begin{aligned}
\sigma^{1/2}(t_m)\|\bar{\mu}_h^m\|_0 \ &\le\ \kappa(|\bar{e}_h^m|_1 + |\phi(e_h^m)|_1 + \sigma^{1/2}(t_m)\|d_te_h^m\|_0) + c\sigma^{1/2}(t_m)\|E_m\|_0 \\
&\le\ \kappa\tau_0^{1/2} + \kappa\tau^{3/4},\ 1 \le m \le N,
\end{aligned} \tag{6.20}$$

which is (6.19). □

Theorem 6.5. *Under the assumptions of Theorem 4.2, the error* $(e^m, \bar{\mu}^m) = (u(t_m) - u_h^m, \bar{p}(t_m) - \bar{p}_h^m)$ *satisfies the bound*

$$\|u(t_m) - u_h^m\|_0 \le \kappa(h^2 + \tau^{3/2}), \tag{6.21}$$

$$\|u(t_m) - u_h^m\|_1 \le (h + \tau^{3/4}),\ \sigma^{1/2}(t_m)\|\bar{p}(t_m) - \bar{p}_h^m\|_0 \le \kappa(h + \tau^{3/4}), \tag{6.22}$$

for all $t_m \in (0, T]$.

Proof. By combining Lemmas 6.1, 6.2, and 6.4 with Theorem 4.2 and noting that $\tau_0 = \frac{1}{T}\tau^{3/2}$, completes the proof of Theorem 6.5. □

Remark. We have presented the convergence analysis of a stabilized finite element method with the Crank–Nicolson extrapolation scheme in time direction for the two-dimensional time-dependent Navier–Stokes problem. We have proved that the scheme is unconditionally stable. However, the error estimate obtained in this paper is not optimal in time direction, which is mainly for lack of discrete Stokes operator on the finite element space pair (X_h, M_h) by the $Q_1 - P_0$ quadrilateral element or the $P_1 - P_0$ triangle element. In this case, the estimate

$$\tau \sum_{n=1}^{N} \|\bar{u}_h(t_n) - \bar{u}_h^n\|_{-1}^2 \le \kappa\tau^4,$$

used in [28] is no longer available.

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