

NONCONFORMING TETRAHEDRAL FINITE ELEMENTS FOR FOURTH ORDER ELLIPTIC EQUATIONS

WANG MING AND JINCHAO XU

ABSTRACT. This paper is devoted to the construction of nonconforming finite elements for the discretization of fourth order elliptic partial differential operators in three spatial dimensions. The newly constructed elements include two nonconforming tetrahedral finite elements and one quasi-conforming tetrahedral element. These elements are proved to be convergent for a model biharmonic equation in three dimensions. In particular, the quasi-conforming tetrahedron element is a modified Zienkiewicz element, while the nonmodified Zienkiewicz element (a tetrahedral element of Hermite type) is proved to be divergent on a special grid.

1. INTRODUCTION

The construction of appropriate finite element spaces for fourth order elliptic partial differential equations is an intriguing subject. This problem has been well-studied in two-dimensional spaces, and there have been a lot of interesting constructions of both conforming and nonconforming finite element spaces. In comparison, there has been very little work devoted to three-dimensional problems.

A conforming finite element space for fourth order problems consist of piecewise polynomials that are globally continuously differentiable (C^1). This smoothness requirement can only be met with piecewise polynomials of sufficiently high degree. In two dimensions, it is known [31] that at least a 5th degree polynomial (the well-known Argyris element) is needed on a triangular mesh. Such a high degree polynomial leads to finite element spaces with a very large degree of freedom which is not computationally desirable. As a result, many lower degree nonconforming finite elements have been constructed and used in practice (see [8]).

In three spatial dimensions, even higher degree polynomials are needed to construct a conforming finite element space on, say, a tetrahedral finite element grid. In [30] (see also [17]), a conforming tetrahedral conforming finite element space was first constructed using the 9th degree of polynomials. This element requires C^1 globally, C^2 on all element edges, and C^4 on all element vertices. The degree of freedom for this element is huge, 220 on each element! In order to reduce the degree

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of polynomials, as in two dimensions, there has been some work on the construction of conforming finite element spaces on macro-elements (namely by further partitioning a tetrahedron into sub-tetrahedrons); see [1] and [29] (similar to Clough-Tocher in two dimensions) and [14]. But these elements all still have a very large degree of freedom, and furthermore the macro-elements are often awkward to use in practical applications.

To reduce the degree of polynomials and degree of freedom on each element, one naturally turns to nonconforming elements. Surprisingly, there has been very little work on the construction of nonconforming finite elements for fourth order elliptic boundary value problems in three dimensions. The purpose of this work is to fill in this important gap in the literature for this type of element.

The construction of nonconforming finite elements for fourth order problems in three dimensions is not only important from a mathematical point of view but also potentially important in practical applications. Indeed, two-dimensional biharmonic equations have been much used in modeling linear plates (see [15]), and such practical applications contributed to the importance and interest of studying efficient numerical methods such as nonconforming finite elements to solve this type of equation. We would like to point out that the three-dimensional biharmonic operator also has important applications in practice. One notable example is the Cahn-Hilliard diffusion equation (see [6]) and its modified version (see [13] and the references there). The complex microstructure evolutions for many important material processes, such as the phase separation in binary alloys and the solidifications of metals and alloys (see [5]), can be modeled by the Cahn-Hilliard diffusion equations.

There were many works on the numerical methods for the Cahn-Hilliard equation; see [2, 3, 5], [9]–[12], [19] and their references. In addition to the finite difference method and also the spectral method, the fourth order term in the Cahn-Hilliard equation can also be discretized by the finite element method (see [2, 3], [10]–[12]). The finite element methods of mixed type, namely by writing the biharmonic operator as a product of two Laplacian operators, were discussed in [2, 3, 11]. It is conceivable that the biharmonic operator can also be discretized directly from its original form, as it is often done for biharmonic equations in two dimensions. This kind of finite element method had been applied to Cahn-Hilliard equation in one and two dimensions (see [10, 12]), and there is no work for three dimensions yet. As discussed above, the existing 3-dimensional conforming finite elements are not very practical and the nonconforming finite element methods proposed in this paper can hopefully be used for such applications.

In this paper, we will propose some finite elements for three-dimensional fourth order partial differential equations. We took the natural approach of trying to extend the various nonconforming finite element in two dimensions to three dimensions. In two dimensions, there are well-known nonconforming elements, including the elements named after Morley, Zienkiewicz, Adini, Bogner-Fox-Schmit, etc. (see [4, 8, 16, 18]). There are some other ways of constructing elements, such as the quasi-conforming method [25, 7]. In this paper, we will focus on tetrahedral complete or incomplete cubic elements, and propose and analyze the following three types of elements:

- (1) A cubic tetrahedral element with 20 degrees of freedom and complete cubic polynomial shape function space.

- (2) A incomplete cubic tetrahedral element with 16 degrees of freedom and incomplete cubic polynomial shape function space.
- (3) A quasi-conforming tetrahedral element with 16 degrees of freedom similar to a nine-parameter quasi-conforming element.

The first two are nonconforming elements, and the last one is an element constructed by the quasi-conforming method. For nonconforming elements, the basic mathematical theory has been studied in many works (see [8, 16], [22]–[24], [33]). For quasi-conforming elements, detailed discussions can be found in [32, 33]. Following these theories, we give the convergence analysis of the elements.

The element of Hermite tetrahedron of type (3') in [8], called a three-dimensional Zienkiewicz element in this paper, is also viewed as an element for biharmonic equations just like the two-dimensional Zienkiewicz element. In two-dimensional case, the Zienkiewicz element is not convergent for general meshes. We will also show that the three-dimensional Zienkiewicz element is divergent for some popular grids in three dimensions.

We note that the degree of freedom of each element proposed in this paper is substantially smaller than any known conforming elements. We expect that they can be easily used in practice.

The rest of the paper is organized as follows. Section 2 gives a basic description of the nonconforming element method. Section 3 gives a detailed description of the new finite elements. Section 4 shows the convergence of the new elements and the divergence of the three-dimensional Zienkiewicz element. Some concluding remarks are made at the end of the paper.

2. PRELIMINARIES

In this section, we shall give a brief discussion of a model fourth order elliptic boundary value problem and how it may be discretized by a nonconforming finite element method.

Given a bounded polyhedron domain $\Omega \subset R^3$ with boundary $\partial\Omega$, for a nonnegative integer s , let $H^s(\Omega)$, $\|\cdot\|_{s,\Omega}$, and $|\cdot|_{s,\Omega}$ be the usual Sobolev space, norm, and seminorm, respectively. Let $H_0^s(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$ with respect to the norm $\|\cdot\|_{s,\Omega}$ and (\cdot, \cdot) denote the inner product of $L^2(\Omega)$.

For $f \in L^2(\Omega)$, we consider the following fourth order boundary value problem:

$$(2.1) \quad \begin{cases} \Delta^2 u = f, & \text{in } \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0, \end{cases}$$

where $\nu = (\nu_1, \nu_2, \nu_3)^\top$ is the unit outer normal to $\partial\Omega$ and Δ is the standard Laplacian operator.

For any function $v \in H^1(T)$, set

$$Dv = \left(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_3} \right).$$

When $v \in H^2(\Omega)$, we define

$$(2.2) \quad E(v) = \left(\frac{\partial^2 v}{\partial x_1^2}, \frac{\partial^2 v}{\partial x_2^2}, \frac{\partial^2 v}{\partial x_3^2}, \frac{\partial^2 v}{\partial x_1 \partial x_2}, \frac{\partial^2 v}{\partial x_1 \partial x_3}, \frac{\partial^2 v}{\partial x_2 \partial x_3} \right)^\top.$$

Let $K \in R^{6 \times 6}$ be the matrix given by

$$K = \text{diag}(1, 1, 1, 2, 2, 2).$$

Define

$$(2.3) \quad a(v, w) = \int_{\Omega} E(w)^{\top} K E(v), \quad \forall v, w \in H^2(\Omega).$$

The weak form of problem (2.1) is: find $u \in H_0^2(\Omega)$ such that

$$(2.4) \quad a(u, v) = (f, v), \quad \forall v \in H_0^2(\Omega).$$

For a subset $B \subset R^3$ and a nonnegative integer r , let $P_r(B)$ be the space of all polynomials of degree not greater than r , and let $Q_r(B)$ the space of all polynomials of degree in each coordinate not greater than r .

Let (T, P_T, Φ_T) be a finite element where T is the geometric shape, P_T the shape function space, and Φ_T the vector of degrees of freedom, and let Φ_T be P_T -unisolvent (see [8]). Let \mathcal{T}_h be a triangulation of Ω with mesh size h . For each element $T \in \mathcal{T}_h$, let h_T be the diameter of the smallest ball containing T and let ρ_T be the diameter of the largest ball contained in T .

Let $\{\mathcal{T}_h\}$ be a family of triangulations with $h \rightarrow 0$. Throughout the paper, we assume that $\{\mathcal{T}_h\}$ is quasi-uniform, namely, it satisfies that $h_T \leq h \leq \eta \rho_T$, $\forall T \in \mathcal{T}_h$ for a positive constant η independent of h .

For each \mathcal{T}_h , let V_{h0} be the corresponding finite element space associated with (T, P_T, Φ_T) for the discretization of $H_0^2(\Omega)$. This defines a family of finite element spaces $\{V_{h0}\}$. In the case of a nonconforming element, $V_{h0} \not\subset H_0^2(\Omega)$.

For $v, w \in H^2(\Omega) + V_{h0}$, we define

$$(2.5) \quad a_h(v, w) = \sum_{T \in \mathcal{T}_h} \int_T E(w)^{\top} K E(v).$$

The finite element method for problem (2.4) corresponding to element (T, P_T, Φ_T) is: find $u_h \in V_{h0}$ such that

$$(2.6) \quad a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_{h0}.$$

We introduce the following mesh-dependent norm $\|\cdot\|_{m,h}$ and seminorm $|\cdot|_{m,h}$:

$$\|v\|_{m,h} = \left(\sum_{T \in \mathcal{T}_h} \|v\|_{m,T}^2 \right)^{1/2}, \quad |v|_{m,h} = \left(\sum_{T \in \mathcal{T}_h} |v|_{m,T}^2 \right)^{1/2}$$

for all functions $v \in L^2(\Omega)$ with $v^T \in H^m(T)$, $\forall T \in \mathcal{T}_h$.

For each element $T \in \mathcal{T}_h$, let Π_T denote the canonical interpolation operator of (T, P_T, Φ_T) , and define Π_h by $(\Pi_h v)|_T = \Pi_T(v|_T)$, where $T \in \mathcal{T}_h$ and v is piecewise smooth.

3. TETRAHEDRAL ELEMENTS

Given a tetrahedron T with vertices $a_i = (x_{i1}, x_{i2}, x_{i3})^{\top}$, $0 \leq i \leq 3$, denote by F_i the facet opposite a_i , by b_i the barycenter of F_i , $0 \leq i \leq 3$, and by $\lambda_0, \dots, \lambda_3$ the barycentric coordinates of T .

Let \hat{T} be the reference tetrahedron with vertices \hat{a}_i given by

$$\hat{a}_0 = (0, 0, 0)^{\top}, \quad \hat{a}_1 = (1, 0, 0)^{\top}, \quad \hat{a}_2 = (0, 1, 0)^{\top}, \quad \hat{a}_3 = (0, 0, 1)^{\top}.$$

Set

$$B_T = \begin{pmatrix} x_{11} - x_{01} & x_{21} - x_{01} & x_{31} - x_{01} \\ x_{12} - x_{02} & x_{22} - x_{02} & x_{32} - x_{02} \\ x_{13} - x_{03} & x_{23} - x_{03} & x_{33} - x_{03} \end{pmatrix} = (a_1 - a_0, a_2 - a_0, a_3 - a_0),$$

and $F_T \hat{x} = B_T \hat{x} + a_0$, $\hat{x} \in R^3$; then

$$T = F_T \hat{T}, \quad a_i = F_T \hat{a}_i, \quad 0 \leq i \leq 3.$$

Set $B_T^{-1} = (\xi_{ij})_{3 \times 3}$. Let B_1, B_2, B_3 be the row vectors of B_T^{-1} and

$$B_0 = -(B_1 + B_2 + B_3);$$

then

$$(3.1) \quad D\lambda_i = B_i, \quad 0 \leq i \leq 3.$$

3.1. The cubic tetrahedral element. For the first nonconforming element, called the cubic tetrahedral element, (T, P_T, Φ_T) is defined by (see Figure 1)

- 1) T is a tetrahedron,
- 2) $P_T = P_3(T)$,
- 3) Φ_T is the degree of freedom vector with components

$$v(a_j), \frac{\partial v}{\partial \nu}(b_j), \quad 0 \leq j \leq 3, \quad Dv(a_i)(a_j - a_i), \quad 0 \leq i \neq j \leq 3, \quad \forall v \in C^1(T).$$

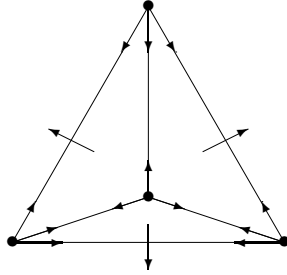


FIGURE 1

We claim that Φ_T is P_T -unisolvant since, with respect to Φ_T , we can obtain the following basis functions of P_T :

$$(3.2) \quad \begin{cases} q_i = \frac{9}{4\|B_i\|} \left(\sum_{\substack{0 \leq j < k < l \leq 3 \\ j \neq i, k \neq i, l \neq i}} \lambda_j \lambda_k \lambda_l - \lambda_i \sum_{\substack{0 \leq j < k \leq 3 \\ j \neq i, k \neq i}} \lambda_j \lambda_k \right), & 0 \leq i \leq 3, \\ p_i = 3\lambda_i^2 - 2\lambda_i^3 + \sum_{\substack{0 \leq k \leq 3 \\ k \neq i}} \frac{4B_i B_k^\top}{3\|B_k\|} q_k, & 0 \leq i \leq 3, \\ p_{ij} = \lambda_i^2 \lambda_j + \frac{\|B_j\|}{9} q_j + \sum_{\substack{0 \leq k \leq 3 \\ k \neq i, k \neq j}} \frac{(2B_i + B_j) B_k^\top}{9\|B_k\|} q_k, & 0 \leq i \neq j \leq 3. \end{cases}$$

In fact, by a direct calculation (see below), we have that

$$(3.3) \quad \begin{cases} q_i(a_k) = 0, & Dq_i(a_k) = 0, & \frac{\partial q_i}{\partial \nu}(b_k) = \delta_{ik}, \\ p_i(a_k) = \delta_{ik}, & Dp_i(a_k) = 0, & \frac{\partial p_i}{\partial \nu}(b_k) = 0, \\ p_{ij}(a_k) = 0, & Dp_{ij}(a_k)(a_l - a_k) = \delta_{ik}\delta_{jl}, & \frac{\partial p_{ij}}{\partial \nu}(b_k) = 0, \end{cases}$$

when $0 \leq i \neq j \leq 3$ and $0 \leq k \neq l \leq 3$, and where δ_{ik} and δ_{jl} are Kronecker deltas.

The corresponding interpolation operator Π_T can be written by, $\forall v \in C^1(T)$

$$(3.4) \quad \Pi_T v = \sum_{0 \leq i \leq 3} p_i v(a_i) + \sum_{0 \leq i \leq 3} q_i \frac{\partial v}{\partial \nu}(b_i) + \sum_{0 \leq i \neq j \leq 3} p_{ij} Dv(a_i)(a_j - a_i).$$

Now we verify (3.3). For the function q_0 , we have

$$\begin{aligned} q_0 &= \frac{9}{4\|B_0\|} \left(\lambda_1 \lambda_2 \lambda_3 - \lambda_0 (\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1) \right), \\ Dq_0 &= \frac{9}{4\|B_0\|} \left(\lambda_2 \lambda_3 D\lambda_1 + \lambda_1 \lambda_3 D\lambda_2 + \lambda_1 \lambda_2 D\lambda_3 \right. \\ &\quad \left. - (\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1) D\lambda_0 - \lambda_0 \sum_{1 \leq i \neq j \leq 3} \lambda_i D\lambda_j \right). \end{aligned}$$

Since $\lambda_i(a_j) = \delta_{ij}$ we obviously have $q_0(a_j) = 0$ and $Dq_0(a_j) = 0$ ($0 \leq j \leq 3$). Furthermore,

$$\begin{aligned} Dq_0(b_0) &= \frac{1}{4\|B_0\|} (D\lambda_1 + D\lambda_2 + D\lambda_3 - 3D\lambda_0) = -\frac{D\lambda_0}{\|B_0\|} = -\frac{B_0}{\|B_0\|}, \\ Dq_0(b_j) &= \frac{1}{4\|B_0\|} \left(D\lambda_j - D\lambda_0 - 2D\lambda_j - \sum_{\substack{1 \leq i \leq 3 \\ i \neq j}} D\lambda_i \right) = 0, \quad 1 \leq j \leq 3. \end{aligned}$$

Thanks to (3.1) the outer normal of each F_i is just $-B_i$. Hence $\frac{\partial}{\partial \nu} q_0(b_j) = \delta_{0j}$, $0 \leq j \leq 3$. For other q_i we can use a completely similar argument. Thus the first line of (3.3) is verified.

Let $0 \leq i, k \leq 3$. Obviously $p_i(a_k) = \delta_{ik}$ and

$$Dp_i = 6(1 - \lambda_i) \lambda_i D\lambda_i + \sum_{\substack{0 \leq j \leq 3 \\ j \neq i}} \frac{4B_i B_j^\top}{3\|B_j\|} Dq_j.$$

This directly leads to the fact that $Dp_i(a_k) = 0$. By the properties of q_j , we have $Dp_i(b_i) = 0$ and

$$Dp_i(b_k) B_k^\top = \frac{4}{3} B_i B_k^\top + \frac{4B_i B_k^\top}{3\|B_k\|} Dq_k(b_k) B_k^\top = 0$$

when $k \neq i$. Thus the second line of (3.3) is also verified.

Finally, we consider p_{ij} . Let $0 \leq i \neq j \leq 3$ and $0 \leq k \leq 3$. Then $p_{ij}(a_k) = 0$ by definition, and

$$Dp_{ij} = 2\lambda_i \lambda_j D\lambda_i + \lambda_i^2 D\lambda_j + \frac{\|B_j\|}{9} Dq_j + \sum_{\substack{0 \leq m \leq 3 \\ m \neq i, m \neq j}} \frac{(2B_i + B_j) B_m^\top}{9\|B_m\|} Dq_m.$$

Now, for $l \neq k$, since $D\lambda_j(a_l - a_k) = \delta_{jl} - \lambda_j(a_k)$ (see [8, p. 65]), we have

$$Dp_{ij}(a_k)(a_l - a_k) = \lambda_i^2(a_k)D\lambda_j(a_l - a_k) = \delta_{ik}(\delta_{jl} - \delta_{jk}) = \delta_{ik}\delta_{jl}.$$

Using the properties of q_m , $0 \leq m \leq 3$, we deduce that

$$\begin{aligned} Dp_{ij}(b_i)B_i^\top &= 0, \quad Dp_{ij}(b_j)B_j^\top = \frac{\|B_j\|}{9}(\|B_j\| + Dq_j(b_j)B_j^\top) = 0, \\ Dp_{ij}(b_k)B_k^\top &= \frac{2}{9}B_iB_k^\top + \frac{1}{9}B_jB_k^\top + \frac{(2B_i + B_j)B_k^\top}{9\|B_k\|}Dq_k(b_k)B_k^\top = 0, \quad k \neq i, j. \end{aligned}$$

Hence the last line of (3.3) is verified.

For $0 \leq i \leq 3$, vectors $a_j - a_i$, $j \neq i$, form a basis of R^3 . Thus the degrees of freedom of the cubic tetrahedral element can be replaced by

$$v(a_j), \quad Dv(a_j), \quad \frac{\partial v}{\partial \nu}(b_j), \quad 0 \leq j \leq 3, \quad \forall v \in C^1(T).$$

For the cubic tetrahedral element, we can define the corresponding finite element space V_{h0} as follows: $v \in V_{h0}$ if and only if (1) $v|_T \in P_3(T)$, $\forall T \in \mathcal{T}_h$, (2) v and Dv are continuous at all vertices of elements in \mathcal{T}_h and vanish at all vertices belonging to $\partial\Omega$, and (3) $\frac{\partial v}{\partial \nu}$ is continuous at the barycenters of all facets of elements in \mathcal{T}_h and vanishes at barycenters of all facets on $\partial\Omega$.

Unlike the Zienkiewicz element, this complete cubic finite element space is not always contained in $C^0(\bar{\Omega})$. To see this, let us choose two different elements $T, T' \in \mathcal{T}_h$ such that they have a common facet F and $T \cup T' \subset \Omega$. Denote by b the barycenter of F and by ν the unit outer normal to F with respect to T . Let v_h be the function in V_{h0} satisfying: $v_h \equiv 0$ outside $T \cup T'$ and $\frac{\partial v_h}{\partial \nu}(b) = 1$. Then $v_h|_T$ is just one of the basis functions q_i given by (3.2). From (3.2) $v_h|_T$ does not vanish at all relative inner points of each facet of T . Hence v_h is not continuous through the facets of T different from F .

Lemma 3.1. *Let V_{h0} be the finite element space of the cubic tetrahedral element. If $T, T' \in \mathcal{T}_h$ have a common facet F , then*

$$(3.5) \quad \int_F D(v_h|_T) = \int_F D(v_h|_{T'}), \quad v_h \in V_{h0}.$$

If a facet F of $T \in \mathcal{T}_h$ is on $\partial\Omega$, then

$$(3.6) \quad \int_F D(v_h|_T) = 0, \quad v_h \in V_{h0}.$$

Proof. Let $v_h \in V_{h0}$ and let F be the common facet of $T, T' \in \mathcal{T}_h$. Denote the unit normal of F relative to T by ν , and choose $\nu, \tau^{(1)}, \tau^{(2)}$ as an orthogonal unit basis of R^3 . Let $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3$ be vertices of F and let \tilde{b}_0 be barycenter of F . Denote by $\tilde{\lambda}_i$ the area coordinate of F corresponding to vertex \tilde{a}_i . Then

$$\begin{aligned} \int_F \tilde{\lambda}_j^2 &= \frac{|F|}{6} = \frac{|F|}{12} \left(\sum_{i=1}^3 \tilde{\lambda}_j^2(\tilde{a}_i) + 9\tilde{\lambda}_j^2(\tilde{b}_0) \right), \quad 1 \leq j \leq 3, \\ \int_F \tilde{\lambda}_j \tilde{\lambda}_k &= \frac{|F|}{12} = \frac{|F|}{12} \left(\sum_{i=1}^3 \tilde{\lambda}_j \tilde{\lambda}_k(\tilde{a}_i) + 9\tilde{\lambda}_j \tilde{\lambda}_k(\tilde{b}_0) \right), \quad 1 \leq j \neq k \leq 3. \end{aligned}$$

From the fact that $P_2(F) = \text{span}\{\tilde{\lambda}_1^2, \tilde{\lambda}_2^2, \tilde{\lambda}_3^2, \tilde{\lambda}_1\tilde{\lambda}_2, \tilde{\lambda}_2\tilde{\lambda}_3, \tilde{\lambda}_3\tilde{\lambda}_1\}$, we obtain

$$(3.7) \quad \int_F p = \frac{|F|}{12} \left(\sum_{i=1}^3 p(\tilde{a}_i) + 9p(\tilde{b}_0) \right), \quad \forall p \in P_2(F).$$

By the definition of V_{h0} , $\frac{\partial}{\partial \nu}(v_h|_T)$ and $\frac{\partial}{\partial \nu}(v_h|_{T'})$ are quadratic polynomials on F , and they equal $\frac{\partial}{\partial \nu}v_h$ at \tilde{b}_0 and \tilde{a}_i , $1 \leq i \leq 3$. Hence

$$(3.8) \quad \int_F \frac{\partial(v_h|_T)}{\partial \nu} = \frac{|F|}{12} \left(\sum_{i=1}^3 \frac{\partial v_h}{\partial \nu}(\tilde{a}_i) + 9 \frac{\partial v_h}{\partial \nu}(\tilde{b}_0) \right) = \int_F \frac{\partial(v_h|_{T'})}{\partial \nu}.$$

Denote all sides of F by S_1, S_2, S_3 , and the unit out normal of S_i by $n^{(i)}$, viewed as the boundary of a triangle in the two-dimensional space spanned by directions $\tau^{(1)}$ and $\tau^{(2)}$. Then for $i \in \{1, 2\}$ Green's formula gives

$$\int_F \frac{\partial(v_h|_T)}{\partial \tau^{(i)}} = \sum_{j=1}^3 n_i^{(j)} \int_{S_j} v_h|_T, \quad \int_F \frac{\partial(v_h|_{T'})}{\partial \tau^{(i)}} = \sum_{j=1}^3 n_i^{(j)} \int_{S_j} v_h|_{T'}.$$

By the definition of V_{h0} , $v_h|_T = v_h|_{T'}$ on S_j . Therefore

$$(3.9) \quad \int_F \frac{\partial(v_h|_T)}{\partial \tau^{(i)}} = \int_F \frac{\partial(v_h|_{T'})}{\partial \tau^{(i)}}, \quad i = 1, 2.$$

Equality (3.5) follows from (3.8) and (3.9). Similarly, we can show (3.6). \square

3.2. The incomplete cubic tetrahedral element. We shall construct a new element by removing the degrees of freedom $\frac{\partial v}{\partial \nu}(b_j)$ from the cubic tetrahedral element.

For $0 \leq i < j < k \leq 3$, let $a_{ijk} = (a_i + a_j + a_k)/3$ and let ν_{ijk} be the unit out normal of the facet with a_i, a_j, a_k as vertices. For $v \in C^1(T)$, define $\tilde{\psi}_{ijk}(v) \in R$ by

$$\tilde{\psi}_{ijk}(v) = \frac{\partial v}{\partial \nu_{ijk}}(a_{ijk}) - \frac{1}{3} \sum_{l=i,j,k} \frac{\partial v}{\partial \nu_{ijk}}(a_l).$$

Define

$$P_3''(T) = \{p \in P_3(T) \mid \tilde{\psi}_{ijk}(p) = 0, \ 0 \leq i < j < k \leq 3\}.$$

For a linear polynomial q , we have

$$q(a_{ijk}) - \frac{1}{3} \sum_{l=i,j,k} q(a_l) = 0.$$

Then $\tilde{\psi}_{ijk}(p) = 0$ when $p \in P_2(T)$, that is, $P_2(T) \subset P_3''(T)$. For incomplete cubic tetrahedral element, (T, P_T, Φ_T) is defined by (see Figure 2)

- 1) T is a tetrahedron,
- 2) $P_T = P_3''(T)$,
- 3) Φ_T is the vector with its component the following degrees of freedom,

$$v(a_j), \ 0 \leq j \leq 3, \quad Dv(a_i)(a_j - a_i), \ 0 \leq i \neq j \leq 3, \ \forall v \in C^1(T).$$

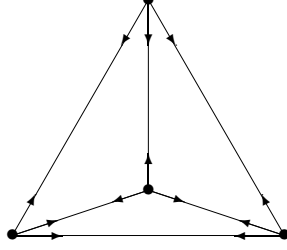


FIGURE 2

The basis functions of the incomplete cubic tetrahedral element can be derived from ones of the cubic tetrahedral element. Set

$$(3.10) \quad \tilde{p}_{ij} = \lambda_i^2 \lambda_j - \frac{2\|B_j\|}{9} q_j + \sum_{\substack{1 \leq k \leq 4 \\ k \neq i, k \neq j}} \frac{2(B_i - B_j)B_k^\top}{9\|B_k\|} q_k, \quad 0 \leq i \neq j \leq 3,$$

where $\{q_i\}$ are given in (3.2). We can verify that these $\{\tilde{p}_{ij}\}$, together with the $\{p_i\}$ given in (3.2), form a basis for the incomplete cubic tetrahedral element. The corresponding interpolation operator Π_T can be written by

$$(3.11) \quad \Pi_T v = \sum_{0 \leq i \leq 3} p_i v(a_i) + \sum_{0 \leq i \neq j \leq 3} \tilde{p}_{ij} Dv(a_i)(a_j - a_i), \quad \forall v \in C^1(T).$$

For the incomplete cubic tetrahedral element, we can define the corresponding finite element space V_{h0} as follows: $V_{h0} = \{v \in L^2(\Omega) \mid v|_T \in P_3''(T), \forall T \in \mathcal{T}_h, v \text{ and } Dv \text{ are continuous at all vertices of elements in } \mathcal{T}_h \text{ and vanish at all vertices belonging to } \partial\Omega\}$.

Similar to the cubic tetrahedral element, V_{h0} here is still not a subspace of $C^0(\bar{\Omega})$.

3.3. The three-dimensional Zienkiewicz element. The incomplete tetrahedral element above is reminiscent to the three-dimensional Zienkiewicz element, the element of Hermite tetrahedron of type (3') in [8]. We shall now discuss this element and its relevant questions.

For $0 \leq i < j < k \leq 3$, define

$$\psi_{ijk}(v) = 6v(a_{ijk}) - 2 \sum_{l=i,j,k} v(a_l) + \sum_{l=i,j,k} Dv(a_l)(a_l - a_{ijk}).$$

Define

$$P_3'(T) = \{p \in P_3(T) \mid \psi_{ijk}(p) = 0, 0 \leq i < j < k \leq 3\}.$$

For the three-dimensional Zienkiewicz element, (T, P_T, Φ_T) is given as follows:

- 1) The element T is a tetrahedron.
- 2) The shape function space $P_T = P_3'(T)$.
- 3) For $v \in C^1(T)$, its degree of freedom vector $\Phi_T(v)$ is given by

$$\Phi_T(v) = \left(v(a_0), Dv(a_0), v(a_1), Dv(a_1), v(a_2), Dv(a_2), v(a_3), Dv(a_3) \right)^\top.$$

The corresponding interpolation operator Π_T is defined by

$$(3.12) \quad \begin{aligned} \Pi_T v = & \sum_{i=0}^3 \left(3\lambda_i^2 - 2\lambda_i^3 + 2\lambda_i \sum_{\substack{0 \leq j < k \leq 3 \\ j, k \neq i}} \lambda_j \lambda_k \right) v(a_i) \\ & + \frac{1}{2} \sum_{0 \leq i \neq j \leq 3} \lambda_i \lambda_j (1 + \lambda_i - \lambda_j) Dv(a_i)(a_j - a_i), \quad \forall v \in C^1(T). \end{aligned}$$

For the three-dimensional Zienkiewicz element, we can define the corresponding finite element space V_{h0} as follows: $V_{h0} = \{v \in L^2(\Omega) \mid v|_T \in P'_3(T), \forall T \in \mathcal{T}_h, v \text{ and } Dv \text{ are continuous at all vertices of elements in } \mathcal{T}_h \text{ and vanish at all vertices belonging to } \partial\Omega\}$. From [8], we know that $V_{h0} \subset H^1(\Omega)$.

3.4. The quasi-conforming tetrahedral element. The Zienkiewicz element is not convergent in general. We will show in next section that the three-dimensional Zienkiewicz element is also divergent for a special tetrahedral grid. In the two-dimensional case, a convergent element was proposed by the so-called quasi-conforming element technique in [25, 7]. Now we use the technique to give a new element by modifying the three-dimensional Zienkiewicz element.

Let (T, P_T, Φ_T) be the three-dimensional Zienkiewicz element. Given a tetrahedron T , and let Π_T^1 be the linear interpolation operator with the function values at four vertices as degrees of freedom. We define

$$N_{ij} = \text{span} \left\{ 1, \frac{1}{2}(\lambda_i + \lambda_j) \right\}, \quad 1 \leq i, j \leq 3.$$

For $p \in P_T$, define $\partial_T^{ij} p \in N^{ij}$ ($1 \leq i, j \leq 3$) such that for any $q \in N_{ij}$,

$$(3.13) \quad \int_T q \partial_T^{ij} p = \frac{1}{2} \int_T q \left(\Pi_T^1 \frac{\partial p}{\partial x_i} \nu_j + \Pi_T^1 \frac{\partial p}{\partial x_j} \nu_i \right) - \frac{1}{2} \int_T \left(\frac{\partial q}{\partial x_i} \frac{\partial p}{\partial x_j} + \frac{\partial q}{\partial x_j} \frac{\partial p}{\partial x_i} \right).$$

Set

$$(3.14) \quad E_T(p) = \left(\partial_T^{11} p, \partial_T^{22} p, \partial_T^{33} p, \partial_T^{12} p, \partial_T^{13} p, \partial_T^{23} p \right)^\top.$$

For the quasi-conforming tetrahedral element, we use $E_T(p)$ to approximate $E(p)$.

Define

$$N = \begin{pmatrix} \tilde{N}_{11} & & & & & \\ & \tilde{N}_{22} & & & & \\ & & \tilde{N}_{33} & & & \\ & & & \tilde{N}_{12} & & \\ & 0 & & & \tilde{N}_{13} & \\ & & & & & \tilde{N}_{23} \end{pmatrix},$$

where $\tilde{N}_{ij} = (1, \frac{1}{2}(\lambda_i + \lambda_j))$, $1 \leq i, j \leq 3$,

$$H_T = \begin{pmatrix} \xi_{11}^2 & \xi_{21}^2 & \xi_{31}^2 & 2\xi_{11}\xi_{21} & 2\xi_{11}\xi_{31} & 2\xi_{21}\xi_{31} \\ \xi_{12}^2 & \xi_{22}^2 & \xi_{32}^2 & 2\xi_{12}\xi_{22} & 2\xi_{12}\xi_{32} & 2\xi_{22}\xi_{32} \\ \xi_{13}^2 & \xi_{23}^2 & \xi_{33}^2 & 2\xi_{13}\xi_{23} & 2\xi_{13}\xi_{33} & 2\xi_{23}\xi_{33} \\ \xi_{11}\xi_{12} & \xi_{21}\xi_{22} & \xi_{31}\xi_{32} & \xi_{12}\xi_{21} + \xi_{11}\xi_{22} & \xi_{12}\xi_{31} + \xi_{11}\xi_{32} & \xi_{22}\xi_{31} + \xi_{21}\xi_{32} \\ \xi_{11}\xi_{13} & \xi_{21}\xi_{23} & \xi_{31}\xi_{33} & \xi_{13}\xi_{21} + \xi_{11}\xi_{23} & \xi_{13}\xi_{31} + \xi_{11}\xi_{33} & \xi_{23}\xi_{31} + \xi_{21}\xi_{33} \\ \xi_{12}\xi_{13} & \xi_{22}\xi_{23} & \xi_{32}\xi_{33} & \xi_{13}\xi_{22} + \xi_{12}\xi_{23} & \xi_{13}\xi_{32} + \xi_{12}\xi_{33} & \xi_{23}\xi_{32} + \xi_{22}\xi_{33} \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & -32 & 0 & 0 & 0 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 32 & 0 & 4 & 4 & -32 & 24 & -4 & -4 & 0 & 4 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & -32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 & 0 & 0 \\ 32 & 4 & 0 & 4 & 0 & 0 & 4 & 0 & -32 & -4 & 24 & -4 & 0 & 0 & 4 \\ 0 & 0 & 0 & -32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 \\ 32 & 4 & 4 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 & -32 & -4 & -4 \\ 0 & -16 & -16 & 0 & 0 & 0 & 16 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 \\ 16 & -1 & -1 & 2 & -8 & 4 & 4 & -1 & -8 & 4 & 4 & -1 & 0 & 1 & 1 \\ 0 & -16 & 0 & -16 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ 16 & -1 & 2 & -1 & -8 & 4 & -1 & 4 & 0 & 1 & 0 & 1 & -8 & 4 & -1 \\ 0 & 0 & -16 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 & 16 \\ 16 & 2 & -1 & -1 & 0 & 0 & 1 & 1 & -8 & -1 & 4 & 4 & -8 & -1 & 4 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 4 & -10 \\ -10 & 40 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 9 & -30 \\ -30 & -120 \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & & & \\ & A_1 & & 0 \\ & & A_1 & \\ & & & A_2 \\ & 0 & & A_2 & A_2 \end{pmatrix},$$

and

$$M_T = \begin{pmatrix} 1 & 0 & & & \\ 0 & B_T^\top & & & 0 \\ & & 1 & 0 & \\ & & 0 & B_T^\top & \\ & & & & 1 & 0 \\ & & & & 0 & B_T^\top \\ & 0 & & & & & 1 & 0 \\ & & & & & & 0 & B_T^\top \end{pmatrix};$$

then

$$(3.15) \quad E_T(p) = \frac{1}{48} H_T N A Q M_T \Phi_T(p), \quad \forall p \in P_T.$$

Now let V_{h0} be the finite element space corresponding to the three-dimensional Zienkiewicz element. Define

$$(3.16) \quad \bar{a}_h(v_h, w_h) = \sum_{T \in \mathcal{T}_h} \int_T E_T(w_h)^\top K E_T(v_h), \quad \forall v_h, w_h \in V_{h0}.$$

Instead of solving problem (2.6), the new element finds $\bar{u}_h \in V_{h0}$ such that

$$(3.17) \quad \bar{a}_h(\bar{u}_h, v_h) = (f, v_h), \quad \forall v_h \in V_{h0}.$$

The quasi-conforming tetrahedral element is a three-dimensional analogue of the element proposed in [25, 7] (see also [33]).

For $v_h \in V_{h0}$ and $i, j \in \{1, 2, 3\}$, define $\partial_h^{ij} v_h$ by

$$\partial_h^{ij} v_h|_T = \partial_T^{ij}(v_h|_T), \quad \forall T \in \mathcal{T}_h.$$

Let Π_T be the interpolation operator of the three-dimensional Zienkiewicz element.

Lemma 3.2. *The quasi-conforming tetrahedral element has the following properties:*

(1) $E_T(p) = E(p)$, $\forall p \in P_2(T)$.

(2) There exist positive constants c_1 and c_2 independent of h such that

$$(3.18) \quad c_1 |p|_{2,T} \leq \sum_{1 \leq i,j \leq 3} |\partial_T^{ij} p|_{0,T} \leq c_2 |p|_{2,T}, \quad \forall p \in P_T.$$

(3) There exists a constant C independent of h such that

$$(3.19) \quad \sum_{1 \leq i,j \leq 3} \left| \frac{\partial^2 v}{\partial x_i \partial x_j} - \partial_T^{ij} \Pi_T v \right|_{0,T} \leq Ch |v|_{3,T}, \quad \forall v \in H^3(T).$$

Proof. For $1 \leq i, j \leq 3$, any $p \in P_T$ and any $q \in N_{ij}$, Green's formula gives

$$(3.20) \quad \int_T q \frac{\partial^2 p}{\partial x_i \partial x_j} = \frac{1}{2} \int_T q \left(\frac{\partial p}{\partial x_i} \nu_j + \frac{\partial p}{\partial x_j} \nu_i \right) - \frac{1}{2} \int_T \left(\frac{\partial q}{\partial x_i} \frac{\partial p}{\partial x_j} + \frac{\partial q}{\partial x_j} \frac{\partial p}{\partial x_i} \right).$$

If $p \in P_2(T)$, then $E(p)$ is uniquely determined by (3.20). On the other hand,

$$\Pi_T \frac{\partial p}{\partial x_i} = \frac{\partial p}{\partial x_i}, \quad \forall p \in P_2(T).$$

By (3.13) we obtain that $E_T(p) = E(p)$, $\forall p \in P_2(T)$.

It can be verified that the rank of matrix Q is 12. Thus the rank of AQM_T is 12, too. Let S be the subspace of R^{16} such that $AQM_T d = 0, \forall d \in S$. Then the dimension of S is 4. By conclusion (1) of the lemma, we have

$$S = \text{span} \{ \Phi_T(1), \Phi_T(x_1), \Phi_T(x_2), \Phi_T(x_3) \}.$$

If $E_T(p) = 0$ for some $p \in P_T$, then $AQM_T \Phi_T(p) = 0$. It follows that $p \in P_1(T)$. Therefore, for all $T \in \mathcal{T}_h$,

$$(3.21) \quad \alpha_{1T} |p|_{2,T} \leq \sum_{1 \leq i,j \leq 3} |\partial_T^{ij} p|_{0,T} \leq \alpha_{2T} |p|_{2,T}, \quad \forall p \in P_T,$$

where α_{1T} and α_{2T} are positive constants perhaps dependent on T . Now define

$$g(T) = \sup_{\substack{p \in P_T \\ |p|_{2,T} \neq 0}} \frac{1}{|p|_{2,T}} \sum_{1 \leq i,j \leq 3} |\partial_T^{ij} p|_{0,T}.$$

By (3.15), (3.1) and the quality

$$\begin{aligned} p &= \sum_{i=0}^3 \left(3\lambda_i^2 - 2\lambda_i^3 + 2\lambda_i \sum_{\substack{0 \leq j < k \leq 3 \\ j, k \neq i}} \lambda_j \lambda_k \right) p(a_i) \\ &\quad + \frac{1}{2} \sum_{0 \leq i \neq j \leq 3} \lambda_i \lambda_j (1 + \lambda_i - \lambda_j) Dp(a_i)(a_j - a_i), \quad \forall p \in P_T, \end{aligned}$$

we can treat $|\partial_T^{ij} p|_{0,T}$ and $|p|_{2,T}$ as continuous functions with respect to B_T and $\Phi_T(p)$. Thus function $g(T)$ is a continuous function with respect to matrix B_T , say

$g(B_T)$. By (3.21) and the property of continuous function, there exist two positive constants c_1 and c_2 such that

$$(3.22) \quad \max_{\|B_T\|=1} g(B_T) \leq c_2, \quad \min_{\|B_T\|=1} g(B_T) \geq c_1,$$

where $\|B_T\|$ is a norm of matrix B_T .

For $T \in \mathcal{T}_h$, define $\tilde{T} = \{ \tilde{x} \mid \tilde{x} = \|B_T\|^{-1}x, \forall x \in T \}$. Then $\|B_{\tilde{T}}\| = 1$ and

$$(3.23) \quad c_1 |\tilde{p}|_{2,\tilde{T}} \leq \sum_{1 \leq i,j \leq 3} |\partial_{\tilde{T}}^{ij} \tilde{p}|_{0,\tilde{T}} \leq c_2 |\tilde{p}|_{2,\tilde{T}}, \quad \forall \tilde{p} \in P_{\tilde{T}}.$$

Given $p \in P_T$, let $\tilde{p}(\tilde{x}) = p(\|B_T\|\tilde{x})$, $\forall \tilde{x} \in \tilde{T}$. Then

$$(3.24) \quad |\tilde{p}|_{2,\tilde{T}} = \|B_T\|^{1/2} |p|_{2,T}.$$

On the other hand, by (3.13) we have, $\forall \tilde{q} \in N_{ij}$, $1 \leq i, j \leq 3$,

$$\begin{aligned} \int_{\tilde{T}} \tilde{q} \partial_{\tilde{T}}^{ij} \tilde{p} &= \frac{1}{2} \int_{\partial \tilde{T}} \tilde{q} \left(\Pi_{\tilde{T}}^1 \frac{\partial \tilde{p}}{\partial \tilde{x}_i} \nu_j + \Pi_{\tilde{T}}^1 \frac{\partial \tilde{p}}{\partial \tilde{x}_j} \nu_i \right) - \frac{1}{2} \int_{\tilde{T}} \left(\frac{\partial \tilde{q}}{\partial \tilde{x}_i} \frac{\partial \tilde{p}}{\partial \tilde{x}_j} + \frac{\partial \tilde{q}}{\partial \tilde{x}_j} \frac{\partial \tilde{p}}{\partial \tilde{x}_i} \right) \\ &= \frac{1}{2\|B_T\|} \int_{\partial T} \tilde{q} \left(\Pi_T^1 \frac{\partial p}{\partial x_i} \nu_j + \Pi_T^1 \frac{\partial p}{\partial x_j} \nu_i \right) \\ &\quad - \frac{1}{2\|B_T\|} \int_T \left(\frac{\partial \tilde{q}}{\partial x_i} \frac{\partial p}{\partial x_j} + \frac{\partial \tilde{q}}{\partial x_j} \frac{\partial p}{\partial x_i} \right) = \frac{1}{\|B_T\|} \int_T \tilde{q} \partial_T^{ij} p. \end{aligned}$$

Hence

$$\int_{\tilde{T}} \tilde{q} \partial_{\tilde{T}}^{ij} \tilde{p} = \|B_T\|^2 \int_{\tilde{T}} \widetilde{\partial_T^{ij} p}, \quad \forall \tilde{q} \in N_{ij}.$$

By the fact that $\widetilde{\partial_T^{ij} p} \in N_{ij}$, we obtain

$$\partial_{\tilde{T}}^{ij} \tilde{p} = \|B_T\|^2 \widetilde{\partial_T^{ij} p},$$

and it follows that

$$(3.25) \quad |\partial_{\tilde{T}}^{ij} \tilde{p}|_{0,\tilde{T}} = \|B_T\|^{1/2} |\partial_T^{ij} p|_{0,T}.$$

By (3.23), (3.24) and (3.25), we obtain (3.18).

Using the first two conclusions of the lemma and the interpolation theory, we can prove (3.19). \square

3.5. Remarks. In the coding for real computation, one prefers to use the degrees of freedom at the element vertices than ones at the relative interior of element edges and facets. In this sense, the incomplete cubic tetrahedral element, the three-dimensional Zienkiewicz element and the quasi-conforming tetrahedral element seem better than the cubic tetrahedral element. Although the interpolation error (or the approximation error) of the cubic tetrahedral element is one order higher than those of the incomplete cubic tetrahedral element and the quasi-conforming element, the error of these elements, to solve the boundary value problem of fourth order partial differential equation, are all the same order (see the next section). As for the three-dimensional Zienkiewicz element, it is divergent and cannot be used.

Using a nonconforming element in real computation for problem (2.1), one needs to derive the second order derivatives of the shape functions and to write these

derivatives in a form with respect to the vector of degrees of freedom. For the quasi-conforming tetrahedral element above, this has been given by (3.15), though the description of the element looks more complicated than ones of the cubic and incomplete cubic tetrahedral elements.

4. CONVERGENCE ANALYSIS

In this section, we discuss the convergence properties of the new elements in previous sections. Toward the end of this section, we show that the three-dimensional Zienkiewicz element is not convergent in general.

First, let us derive the error estimates for the interpolation operator.

Theorem 4.1. *Let Π_T be the interpolation operator corresponding to the cubic tetrahedral element, the incomplete cubic tetrahedral element, and the three-dimensional Zienkiewicz elements. Then there exists a constant C independent of h such that*

$$(4.1) \quad |v - \Pi_T v|_{m,T} \leq Ch^{r-m} |v|_{r,T}, \quad 0 \leq m \leq r, \quad \forall v \in H^r(T),$$

where $r = 4$ for the cubic tetrahedral element and $r = 3$ for the other two elements.

From Lemma 3.1 and the argument [21] for the Morley element, we can show the following lemma.

Lemma 4.2. *Let V_{h0} be the finite element space of the cubic tetrahedral element or the incomplete cubic tetrahedral element. Then there exists a constant C independent of h such that for $v \in H^3(\Omega) \cap H_0^2(\Omega)$ with $\Delta^2 v \in L^2(\Omega)$,*

$$(4.2) \quad |a_h(v, v_h) - (\Delta^2 v, v_h)| \leq Ch(|v|_{3,\Omega} + h\|\Delta^2 v\|_{0,\Omega}) |v_h|_{2,h}, \quad \forall v_h \in V_{h0}.$$

Now let u and u_h be the solutions of problems (2.4) and (2.6), respectively. Combining Theorem 4.1 and Lemma 4.2, we get the following theorem.

Theorem 4.3. *Let V_{h0} be the finite element space of the cubic tetrahedral element or the incomplete cubic tetrahedral element. Then*

$$(4.3) \quad \lim_{h \rightarrow 0} \|u - u_h\|_{2,h} = 0,$$

and there exists a constant C independent of h such that

$$(4.4) \quad \|u - u_h\|_{2,h} \leq Ch(|u|_{3,\Omega} + h\|f\|_{0,\Omega})$$

when $u \in H^3(\Omega)$.

Proof. By the generalized Poincare-Friedrichs inequality (see [24]), we have

$$\|v_h\|_{2,h}^2 \leq Ca_h(v_h, v_h), \quad \forall v_h \in V_{h0}.$$

That is, $a_h(\cdot, \cdot)$ is uniformly V_{h0} -elliptic. From the well-known Strang Lemma (see [8] or [22]),

$$(4.5) \quad \|u - u_h\|_{2,h} \leq C \left(\inf_{w_h \in V_{h0}} \|u - w_h\|_{2,h} + \sup_{w_h \in V_{h0}, w_h \neq 0} \frac{|a_h(u, w_h) - (f, w_h)|}{\|w_h\|_{2,h}} \right).$$

When $u \in H^3(\Omega)$, we obtain (4.4) directly from (4.5), (4.1), and (4.2).

For general case, we obtain from Lemma 3.1 and Green's formula that

$$(4.6) \quad \sum_{T \in \mathcal{T}_h} \int_T \frac{\partial^2 v_h}{\partial x_i \partial x_j} = 0, \quad \forall v_h \in V_{h0}, \quad 1 \leq i, j \leq 3.$$

Then the patch test is passed on each \mathcal{T}_h . Because the assumption of Theorem 4.7 in [26] is satisfied, we obtain (4.3). \square

Now let Π_h^1 be the interpolation operator corresponding to the linear conforming element for the second order partial differential equation and \mathcal{T}_h . For the cubic tetrahedral element and the incomplete cubic tetrahedral elements, we can also consider another finite element method: to find $\tilde{u}_h \in V_{h0}$ such that

$$(4.7) \quad a_h(\tilde{u}_h, v_h) = (f, \Pi_h^1 v_h), \quad \forall v_h \in V_{h0}.$$

For the finite element solution \tilde{u}_h of problem (4.7), we can obtain its convergence and the error estimate.

Theorem 4.4. *Let V_{h0} be the finite element space of the cubic tetrahedral element or the incomplete cubic tetrahedral element. Then*

$$(4.8) \quad \lim_{h \rightarrow 0} \|u - \tilde{u}_h\|_{2,h} = 0,$$

and there exists a constant C independent of h such that

$$(4.9) \quad \|u - \tilde{u}_h\|_{2,h} \leq Ch|u|_{3,\Omega}$$

when $u \in H^3(\Omega)$.

For the convergence of the quasi-conforming tetrahedral element, we can follow the method used in [32] or [33]. We give the result without proof.

Theorem 4.5. *For the quasi-conforming tetrahedral element, problem (3.17) has a unique solution \bar{u}_h and*

$$(4.10) \quad \lim_{h \rightarrow 0} \|u - \bar{u}_h\|_{2,h} + \sum_{1 \leq i,j \leq 3} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} - \partial_h^{ij} \bar{u}_h \right|_{0,\Omega} = 0,$$

and there exists a constant C independent of h such that

$$(4.11) \quad \|u - \bar{u}_h\|_{2,h} + \sum_{1 \leq i,j \leq 3} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} - \partial_h^{ij} \bar{u}_h \right|_{0,\Omega} \leq Ch|u|_{3,\Omega}$$

when $u \in H^3(\Omega)$.

It is known that the Zienkiewicz element is not convergent for general meshes in two dimensions (see [20]). As an analogue in the three-dimensional case, the three-dimensional Zienkiewicz element has the same divergence property. Now we show that it is divergent for a special grid.

In Figure 3, a cube is divided into eight sub-cubes, and then each sub-cube is divided six tetrahedrons, where the tetrahedrons not represented by dashed lines are symmetric with respect to the centric point of the cube.

Now let Ω be the cube $[-1, 1]^3$. For $k = 1, 2, \dots$, let \mathcal{T}_k be a triangulation of Ω defined as follows. First, Ω is subdivided into equal cubes with side length $h_k = 2/k$, and then each cube is subdivided into tetrahedrons such as the one shown in Figure 3. The cases of $k = 1$ and $k = 2$ are shown in Figures 3 and 4, respectively.

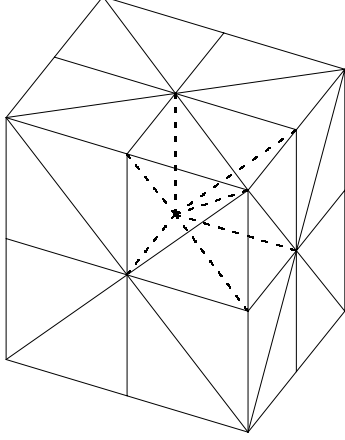


FIGURE 3

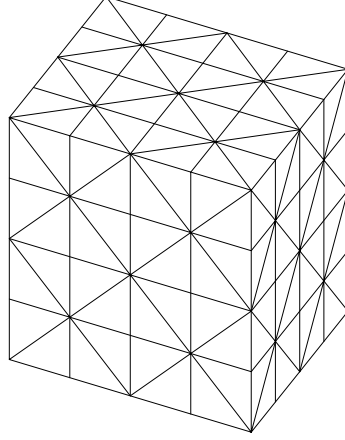


FIGURE 4

Theorem 4.6. *The three-dimensional Zienkiewicz element is divergent for triangulations \mathcal{T}_k .*

Proof. Let V_{10} be the finite element space of the three-dimensional Zienkiewicz element on \mathcal{T}_1 . Let $v_h \in V_{10}$ be the function such that v_h is 1 at the center point of Ω and vanishes at other vertices of all elements in \mathcal{T}_1 , and Dv_h vanishes at all vertices of all elements in \mathcal{T}_1 . It can be computed that

$$\sum_{T \in \mathcal{T}_1} \int_T \frac{\partial^2 v_h}{\partial x_i^2} = -\frac{8}{3}, \quad i = 1, 2, 3,$$

$$\sum_{T \in \mathcal{T}_1} \int_T \frac{\partial^2 v_h}{\partial x_i \partial x_j} = 0, \quad 1 \leq i \leq 3, \quad i < j \leq 3.$$

That is, the three-dimensional Zienkiewicz element does not pass the patch test. On the other hand, for \mathcal{T}_k the number of the patches reduced from \mathcal{T}_1 is k^3 , and the number of elements in \mathcal{T}_k is $48k^3$. By Theorem 6.1 in [26], we obtain the conclusion of the lemma. \square

5. CONCLUDING REMARKS

In this paper, we proposed and analyzed several tetrahedral complete or incomplete cubic finite elements for fourth order elliptic partial differential operators.

More work needs to be done for constructing other types of nonconforming elements. One noticeable element that is missing from our work is a three-dimensional extension of the Morley triangular element in two dimensions that only makes use of quadratic polynomials (although the cubic tetrahedral element, a P_3 -element in three-dimensions, may be viewed as a 3D extension of the P_2 -Morley element in 2D). As it turns out, in three and higher dimensions, the construction of the P_2 -Morley element is possible, but it no longer uses the element vertices as part of the degrees of freedom. We will report an extension of the Morley element in any dimensions in [28]. Another type of element is the hexahedral nonconforming

element that may be extended from rectangular nonconforming elements in two dimensions. We will report these extensions in any dimensions in [27].

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LMAM, SCHOOL OF MATHEMATICAL SCIENCES, PEKING UNIVERSITY, BEIJING, PEOPLE'S REPUBLIC OF CHINA

E-mail address: `mwang@math.pku.edu.cn`

THE SCHOOL OF MATHEMATICAL SCIENCES, PEKING UNIVERSITY; BEIJING, PEOPLE'S REPUBLIC OF CHINA; AND DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802

E-mail address: `xu@math.psu.edu`