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A GENERALIZED BPX MULTIGRID FRAMEWORK COVERING NONNESTED V-CYCLE METHODS

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Abstract. More than a decade ago, Bramble, Pasciak and Xu developed a framework in analyzing the multigrid methods with nonnested spaces or noninherited quadratic forms. It was subsequently known as the BPX multigrid framework, which was widely used in the analysis of multigrid and domain decomposition methods. However, the framework has an apparent limit in the analysis of nonnested V-cycle methods, and it produces a variable V-cycle, or nonuniform convergence rate V-cycle methods, or other nonoptimal results in analysis thus far.

This paper completes a long-time effort in extending the BPX multigrid framework so that it truly covers the nonnested V-cycle. We will apply the extended BPX framework to the analysis of many V-cycle nonnested multigrid methods. Some of them were proven previously only for two-level and W-cycle iterations. Some numerical results are presented to support the theoretical analysis of this paper.

1. Introduction

The multigrid method, consisting of the fine-level smoothing and the coarse-level correction, is an effective iterative method for solving the linear system arising from, e.g., the finite element discretization of boundary-value problems. The multigrid method provides the optimal-order computation in such a case, in the sense that the number of arithmetic operations is proportional to the number of unknowns in the system of linear equations; cf. [1], [5], [28], [31], [33]. The constant rate of W-cycle multigrid iterations was proved in several early papers, one of them is [1], which is generalized to many nonnested cases, for example, [13], [15], [42], [43].

The multigrid method is often nonnested because the multilevel discrete spaces may not be nested, or discrete bilinear forms may be different on different levels. For example, the nonnestedness may be caused by bubble elements [43], composite elements [18], nonconforming elements [4], [13], nonnested meshes [42], the mortar method [2], [25], numerical integrations [24], or other situations (cf. [11]), such as finite difference equations. The multigrid methods with noninherited forms but nested spaces, other than the cases in [11], are studied in [26], [27], [30] for the discontinuous Galerkin method and the edge element. Many earlier two-level and W-cycle nonnested multigrid iterations were analyzed by extending the method of [1]. However, a generalized framework [11], referred as the BPX multigrid framework, is widely used in the analysis of multigrid iterations; e.g., [5], [6], [8], [9], [10],

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[12], [23], [25], [27], [32], [34], [39], [40], [37]. The framework is rooted in [7] and [38].

Although the convergence theory for the W-cycle was established [1], [11], [15], [4], the problem of how to establish the convergence rate for the V-cycle nonnested multigrid method is subtle, and is still an active research subject; see [3], [11], [10], [9], [7], [16], [28], [20], [21], [29], [6], [12], etc. The BPX framework [11] was generalized to allow nonsymmetric smoothings and can be applied to some nonnested multigrid methods. In particular, it provides a constant convergence rate for the nonnested V-cycle under the assumption that

$$A_k(I_k u, I_k u) \le A_{k-1}(u,u) \quad \forall u \in U_{k-1}, \forall k, \tag{1}$$

where $I_k : U_{k-1} \to U_k$ is the coarse-to-fine intergrid transfer operator and A_k is the bilinear form on U_k. However, (1) does not hold for most nonnested multigrid methods. Thus the BPX framework produces some nonoptimal mathematics results, such as the variable V-cycle, nonuniform convergence rate, and multigrid preconditioners, (cf. [11], [37]) though most of these methods provide the optimal order of computation. The question has remained open for a long time whether one can lift this obvious limit, the inequality (1), from the BPX framework.

This question will be answered in this paper. We will extend the BPX framework so that the number of smoothings can play its important role in the V-cycle analysis so that the BPX framework can provide a uniform convergence rate without the nearly nested bound (1). We will then apply the extended BPX framework to show the uniform convergence rates of several common nonnested multigrid methods. Some of them were proven previously for two-level and W-cycle iterations only.

So far, we still require the full elliptic regularity assumption in our applications of the BPX framework. Brenner recently gave a proof in [17] for the nonconforming V-cycle multigrid method applied to the second-order elliptic problem, under a lower regularity requirement. It is a better result. In addition, the analysis [17] can be extended to some other nonnested multigrid methods; cf. [44]. However, it is not straightforward to apply Brenner's analysis to different cases in general, due to its lengthy analysis and its long list of approximation properties and inverse estimates. For example, the standard inverse estimate fails to hold on the combined space of finite element functions on two nonnested grids such as the ones in Figure 1. In contrast to [17], our extended BPX framework is simple in analysis and can be applied to all common nonnested cases.

The outline of this paper is as follows. In Section 2 we recall the V-cycle multigrid method. The convergence analysis is given in Section 3. In Section 4 we provide a proof for the regularity-approximation assumption for various nonnested methods. Some numerical results will be provided in Section 5 to support the theoretical analysis in this paper.

2. The V-cycle multigrid method

For $k \ge 0$, let U_k be a sequence of finite-dimensional vector spaces, along with coarse-to-fine intergrid transfer operators $I_k : U_{k-1} \to U_k$. Let $A_k(\cdot,\cdot)$ and $(\cdot,\cdot)_k$ be symmetric positive definite discrete bilinear forms on $U_k \times U_k$. We solve the following linear system of equations. Given $f \in U_k$, find $v \in U_k$ satisfying

$$A_k(v,\phi) = (f,\phi)_k \quad \forall \phi \in U_k. \tag{2}$$

To define a V-cycle multigrid method for (2), following the notations in [11], we introduce operators $A_k : U_k \to U_k$, $P_{k-1} : U_k \to U_{k-1}$ and $P^0_{k-1} : U_k \to U_{k-1}$ as:

$$\begin{aligned}(A_k w, \phi)_k &= A_k(w, \phi) && \forall \phi \in U_k,\\ A_{k-1}(P_{k-1}w, \phi) &= A_k(w, I_k\phi) && \forall \phi \in U_{k-1},\\ (P^0_{k-1}w, \phi)_{k-1} &= (w, I_k\phi)_k && \forall \phi \in U_{k-1}.\end{aligned}$$

We also introduce linear smoothing operators $R_k : U_k \to U_k$, along with the adjoint operators R^t_k with respect to the inner product $(\cdot,\cdot)_k$. We define

$$R^{(l)}_k = \begin{cases} R_k & \text{if } l \text{ is odd},\\ R^t_k & \text{if } l \text{ is even}.\end{cases}$$

Now we define the standard (symmetric) V-cycle multigrid method [11].

Let m be a positive integer, the number of fine-level smoothings. The multigrid operator $B_k : U_k \to U_k$ is defined by induction as follows. Set $B_0 = A_0^{-1}$. Assume that B_{k-1} has been defined, and define $B_k g \in U_k$ for $g \in U_k$ as follows.

(i) Set $x^0 = 0$.

(ii) Define x^l for $l = 1, 2, \ldots, m$ by

$$x^l = x^{l-1} + R^{(l+m)}_k (g - A_k x^{l-1}).$$

(iii) Define $y^m = x^m + I_k q^1$, where q^1 is defined by

$$q^1 = B_{k-1} P^0_{k-1}(g - A_k x^m).$$

(iv) Define y^l for $l = m+1, m+2, \ldots, 2m$ by

$$y^l = y^{l-1} + R^{(l+m)}_k (g - A_k y^{l-1}).$$

(v) Set $B_k g = y^{2m}$.

3. The convergence analysis

To analyze the convergence, we set $J_k = I - R_k A_k$ and $J^*_k = I - R^t_k A_k$, where J^*_k denotes the adjoint of J_k with respect to $A_k(\cdot,\cdot)$ and I is the identity operator. Set

$$\tilde{J}^{(m)}_k = \begin{cases} (J^*_k J_k)^{m/2} & \text{if } m \text{ is even},\\ (J^*_k J_k)^{(m-1)/2} J^*_k & \text{if } m \text{ is odd}.\end{cases}$$

We then have the following recursive relation among the multigrid operators (cf. [11])

$$I - B_k A_k = (\tilde{J}^{(m)}_k)^* [(I - I_k P_{k-1}) + I_k(I - B_{k-1}A_{k-1})P_{k-1}] \tilde{J}^{(m)}_k.$$

We make two standard hypotheses (cf. [11]) as follows:

(**C1**) Regularity-approximation assumption

$$|A_k((I - I_k P_{k-1})u, u)| \le C_1 \frac{||A_k u||^2_k}{\lambda_k} \quad \forall u \in U_k,$$

where λ_k is the largest eigenvalue of A_k, C_1 is independent of k, and $||\cdot||_k$ is the norm corresponding to $(\cdot,\cdot)_k$. In addition, we require that (see remarks below)

$$(3) \qquad (A_k((I - I_k P_{k-1})u, (I - I_k P_{k-1})u))^{1/2} \le C_Q\,(A_k(u,u))^{1/2} \quad \forall u \in U_k,$$

where C_Q is independent of k.

(**C2**)

$$\frac{||u||_k^2}{\lambda_k} \le C_R(\tilde{R}_k u, u)_k \quad \forall u \in U_k,$$

where $\tilde{R}_k = (I - J_k^* J_k) A_k^{-1}$ and C_R is independent of k.

Remark 3.1. The smoothing hypothesis (**C2**) can be easily verified for point, line, and block versions of the Jacobi and Gauss–Seidel iterations (cf. [8], for example). The verification of the regularity-approximation hypothesis (**C1**) will be carried out in the next section for many examples. The requirement (3) can be verified easily for all practical cases. Inequality (3) is a simple corollary (cf. [43] for example) of the stability estimate (see (1))

$$A_k(I_k u, I_k u) \le C\, A_{k-1}(u,u) \quad \forall u \in U_{k-1}. \tag{4}$$

Theorem 3.1. *Assume that* (**C1**) *and* (**C2**) *hold. Then, for all* $k \ge 0$*,*

$$|A_k((I - B_k A_k)u, u)| \le \delta\, A_k(u,u) \quad \forall u \in U_k, \tag{5}$$

where

$$\delta = \frac{C_1 C_R}{m - C_1 C_R} \tag{6}$$

with $m > 2\, C_1 C_R$*.*

Proof. The method here is motivated by [11],[7], reasoning by mathematical induction. For $k = 0$, we have a zero on the left-hand side of (5), and (5) holds. It is assumed that (5) and (6) hold for $k-1$. In what follows, we show that (5) and (6) hold for k too.

In view of (**C1**), we have

$$|A_k((I - I_k P_{k-1})\tilde{J}_k^{(m)} u, \tilde{J}_k^{(m)} u)| \le C_1 \frac{||A_k \tilde{J}_k^{(m)} u||_k^2}{\lambda_k}. \tag{7}$$

Define

$$\bar{J}_k = \begin{cases} J_k^* J_k & \text{if } m \text{ is even,} \\ J_k J_k^* & \text{if } m \text{ is odd.} \end{cases}$$

By (**C2**) we have

$$\frac{||A_k \tilde{J}_k^{(m)} u||_k^2}{\lambda_k} \le C_R\, A_k((I - \bar{J}_k)\bar{J}_k^m u, u). \tag{8}$$

Since the spectrum of $\bar{J}_k$ is in $[0,1]$, as shown in [11],[7], we have

$$A_k((I-\bar{J}_k)\bar{J}_k^m u, u) \le \frac{1}{m}\sum_{i=0}^{m-1} A_k((I-\bar{J}_k)\bar{J}_k^i u, u) = \frac{1}{m}\{A_k(u,u) - A_k(\bar{J}_k^m u, u)\}. \tag{9}$$

Note that $A_k(\bar{J}_k^m u, u) = A_k(\tilde{J}_k^{(m)} u, \tilde{J}_k^{(m)} u)$. We then get, by (7)–(9), that

$$|A_k((I - I_k P_{k-1})\tilde{J}_k^{(m)} u, \tilde{J}_k^{(m)} u)| \le \frac{C_1\, C_R}{m}\{A_k(u,u) - A_k(\tilde{J}_k^{(m)} u, \tilde{J}_k^{(m)} u)\}. \tag{10}$$

Set

$$t := \frac{A_k(\tilde{J}_k^{(m)} u, \tilde{J}_k^{(m)} u)}{A_k(u,u)} \quad \forall u \ne 0, u \in U_k, \tag{11}$$

or $t := 0$ for $u = 0$. Clearly, $t \in [0, 1]$. We now rewrite (10) as

$$|A_k((I - I_k P_{k-1})\tilde{J}_k^{(m)} u, \tilde{J}_k^{(m)} u)| \le \frac{C_1\, C_R(1-t)}{m} A_k(u, u). \tag{12}$$

On the other hand, from the Cauchy–Schwarz inequality and (3) we have

$$\begin{aligned} &|A_k((I - I_k P_{k-1})\tilde{J}_k^{(m)} u, \tilde{J}_k^{(m)} u)| \\ \le& \{A_k((I - I_k P_{k-1})\tilde{J}_k^{(m)} u, (I - I_k P_{k-1})\tilde{J}_k^{(m)} u)\}^{\frac{1}{2}}\, \{A_k(\tilde{J}_k^{(m)} u, \tilde{J}_k^{(m)} u)\}^{\frac{1}{2}} \\ \le& C_Q\, A_k(\tilde{J}_k^{(m)} u, \tilde{J}_k^{(m)} u) = C_Q\, t\, A_k(u, u). \end{aligned} \tag{13}$$

Combining (12) and (13), we get

$$|A_k((I - I_k P_{k-1})\tilde{J}_k^{(m)} u, \tilde{J}_k^{(m)} u)| \le \min\{C_Q\, t, \frac{C_1 C_R}{m}(1-t)\}\, A_k(u, u). \tag{14}$$

By the relation

$$A_{k-1}(P_{k-1}\tilde{J}_k^{(m)} u, P_{k-1}\tilde{J}_k^{(m)} u) = A_k(\tilde{J}_k^{(m)} u, \tilde{J}_k^{(m)} u) - A_k(\tilde{J}_k^{(m)} u, (I - I_k P_{k-1})\tilde{J}_k^{(m)} u),$$

the induction hypothesis and the symmetry of A_k, we get

$$\begin{aligned} &|A_k((I - B_k A_k)u, u)| \\ &\quad \le |A_k((I - I_k P_{k-1})\tilde{J}_k^{(m)} u, \tilde{J}_k^{(m)} u)| \\ &\quad\quad + |A_{k-1}((I - B_{k-1}A_{k-1})P_{k-1}\tilde{J}_k^{(m)} u, P_{k-1}\tilde{J}_k^{(m)} u)| \\ &\quad \le (1+\delta)|A_k((I - I_k P_{k-1})\tilde{J}_k^{(m)} u, \tilde{J}_k^{(m)} u)| + \delta A_k(\tilde{J}_k^{(m)} u, \tilde{J}_k^{(m)} u) \\ &\quad \le (1+\delta)\, \min\{C_Q\, t, \frac{C_1 C_R}{m}(1-t)\}\, A_k(u, u) + \delta\, t\, A_k(u, u). \end{aligned}$$

Now, to show that (5) and (6) for k, we only need to verify

$$(1+\delta)\, \min\{C_Q\, t, \frac{C_1\, C_R}{m}(1-t)\} + \delta\, t \le \frac{C_1 C_R}{m - C_1 C_R} \qquad \forall t \in [0, 1]. \tag{15}$$

When $t = 0$, the left-hand side of (15) is zero. When $t = 1$, (15) is the induction hypothesis. Next, we consider the case of $t \in (0, 1)$. To show (15), by the hypothesis (6) on level $k - 1$, it suffices to show that

$$(1+\delta)\, C_Q\, \min(\frac{t}{1-t}, \frac{C_1\, C_R}{C_Q\, m}) \le \delta. \tag{16}$$

We consider two cases. First,

$$\frac{C_1 C_R}{C_Q\, m + C_1 C_R} \le t < 1,$$

i.e.,

$$\frac{t}{1-t} \ge \frac{C_1 C_R}{C_Q\, m}.$$

Thus

$$\min(\frac{t}{1-t}, \frac{C_1 C_R}{C_Q\, m}) = \frac{C_1 C_R}{C_Q\, m},$$

and

$$(1+\delta)\, C_Q\, \min(\frac{t}{1-t}, \frac{C_1\, C_R}{C_Q\, m}) = \frac{C_1 C_R}{m - C_1 C_R}. \tag{17}$$

For the second case,

$$0 < t \leq \frac{C_1 C_R}{C_Q\, m + C_1 C_R},$$

i.e.,

$$\frac{t}{1-t} \leq \frac{C_1 C_R}{C_Q\, m},$$

we have

$$\min\Big(\frac{t}{1-t}, \frac{C_1 C_R}{C_Q\, m}\Big) = \frac{t}{1-t},$$

and

$$(1+\delta)\, C_Q \, \min\Big(\frac{t}{1-t}, \frac{C_1\, C_R}{C_Q\, m}\Big) = \frac{m\, C_Q}{m - C_1\, C_R}\, \frac{t}{1-t} \leq \frac{C_1\, C_R}{m - C_1 C_R}. \tag{18}$$

Thus, equation (16) holds for both cases (17) and (18). □

Remark 3.2. Note that from (14) we can get

$$|A_k((I - I_k P_{k-1})\tilde{J}_k^{(m)} u, \tilde{J}_k^{(m)} u)| \leq \frac{C_1 C_R}{m + \frac{C_1 C_R}{C_Q}} A_k(u,u), \tag{19}$$

which indicates that the number of smoothings, m, has to be large enough for the convergence rate in the interval $(0,1)$ in general, even for the two-level method.

On the other hand, if $C_Q = 1$, we get a convergence rate in $(0,1)$ by (19), for the two-level method, for any $m \geq 1$. Note that if

$$A_k(I_k P_{k-1} v, I_k P_{k-1} v) \leq 2\, A_{k-1}(P_{k-1} v, P_{k-1} v) \quad \forall v \in U_k, \forall k, \tag{20}$$

then $C_Q = 1$. In some cases, (20) holds (see [11] and [22]). Note that (20) is a generalization of (1).

Remark 3.3. The key step in our proof is the introduction of a variable t in (11). By it, we extend the BPX framework from the very limited case (1) to the general case (3), or just (4).

4. Verification of (C1)

In this section, we provide a proof for the regularity-approximation assumption (**C1**) in solving the symmetric and positive definite second-order elliptic problems by various nonnested methods.

Let Ω be a bounded, connected domain in $\mathbb{R}^n$, $n = 2$ or 3, with Lipschitz continuous boundary $\partial\Omega$. We will use the Sobolev space $H^l(\Omega)$, $l \geq 0$, with the norm and seminorm $||\cdot||_{H^l(\Omega)}$ and $|\cdot|_{H^l(\Omega)}$. The $L^2(\Omega)(= H^0(\Omega))$ inner product is denoted by $(\cdot,\cdot)_{L^2(\Omega)}$.

We set $U = H_0^1(\Omega)$ and $A(u,v) = \sum_{|\alpha|,|\beta|\leq 1} \int_\Omega a_{\alpha\beta}(x) \partial^\alpha u\, \partial^\beta v$, where α, β are n-indexes and $a_{\alpha\beta}(x) \in L^\infty(\Omega)$. Let $\mathcal{J}_k$, $k \geq 0$ denote a sequence of shape-regular triangulations of Ω, with the mesh-size h_k; cf. [19]. On $\mathcal{J}_k$, let U_k be a finite-dimensional space and $A_k(\cdot,\cdot)$ a discrete form on $U_k \times U_k$.

We first list some general hypotheses.

H1) We require that $A(\cdot,\cdot)$ is symmetric and positive definite, and that for any given $f \in L^2(\Omega)$ there is a unique solution $u \in U$ such that

$$A(u,v) = (f,v)_{L^2(\Omega)} \quad \forall v \in U,$$

and that $u \in H^2(\Omega)$ satisfies

$$||u||_{H^2(\Omega)} \leq C\,||f||_{L^2(\Omega)}.$$

H2) For all k, we require that $A_k(\cdot,\cdot)$ is a symmetric, positive definite and bounded bilinear form. We set

$$|||v|||_{1,k} := \sqrt{A_k(v,v)} \quad \forall v \in U_k,\ \forall k.$$

H3) Let $\Pi_j u \in U_j$ denote the standard interpolant to $u \in H^2(\Omega)$. For all k, we require that

$$||u - \Pi_j u||_{L^2(\Omega)} + h_k\,|||u - \Pi_j u|||_{1,j} \leq C\,h_k^2\,||u||_{H^2(\Omega)}, \quad j = k-1, k.$$

H4) Let $I_k : U_{k-1} \to U_k$ denote the coarse-to-fine intergrid transfer operator. For all k, we require that

$$||I_k v||_{L^2(\Omega)} \leq C\,||v||_{L^2(\Omega)} \quad \forall v \in U_{k-1}.$$

H5) For all k, we require that

$$C^{-1}\,||v||_{L^2(\Omega)} \leq ||v||_k \leq C\,||v||_{L^2(\Omega)} \quad \forall v \in U_k.$$

H6) For all k, we require that the following inverse inequality holds,

$$|||v|||_{1,k} \leq C\,h_k^{-1}\,||v||_{L^2(\Omega)} \quad \forall v \in U_k.$$

H7) Let $u_j \in U_j$ be a finite-element approximation to u, the exact solution for a given $f \in L^2(\Omega)$, i.e.

$$A(u,v) = (f,v)_{L^2(\Omega)} \quad \forall v \in U, \quad A_j(u_j,v) = (f,v)_{L^2(\Omega)} \quad \forall v \in U_j.$$

For all k, we require that

$$||u - u_j||_{L^2(\Omega)} + h_k\,|||u - u_j|||_{1,j} \leq C\,h_k^2\,||f||_{L^2(\Omega)}, \quad j = k-1, k.$$

Remark 4.1. For C^0 conforming elements and nonconforming elements such as the Crouzeix–Raviart element, usually **H7**) results from **H1**)-**H3**) (cf. [19], [35] and [18]).

Theorem 4.1. *Assume* **H2**), **H5**) *and* **H6**). *If the following assumption,*
(C1)'

$$||(I - I_k P_{k-1})v||_k \leq C\,h_k^2\,||A_k v||_k \quad \forall v \in U_k$$

holds, then (**C1**) *holds.*

Proof. By **H2**), the eigenvalues $\lambda_{k,i}$ and eigenvectors $\psi_{k,i}$, $1 \leq i \leq N_k$, satisfy

$$A_k(\psi_{k,i}, v) = \lambda_{k,i}(\psi_{k,i}, v)_k \quad \forall v \in U_k,$$
$$0 < \lambda_{k,1} \leq \lambda_{k,2} \leq \cdots \leq \lambda_{k,N_k},$$
$$(\psi_{k,i}, \psi_{k,j})_{0,k} = \delta_{ij}, \quad A_k(\psi_{k,i}, \psi_{k,j}) = \lambda_{k,i}\delta_{ij},$$

where δ_{ij} is the Kronecker symbol. It can be easily seen that

$$||A_k^{1/2} w||_k^2 = (A_k w, w)_k \leq \lambda_{k,N_k}\,(w,w)_k \quad \forall w \in U_k.$$

Set $\lambda_k := \lambda_{k,N_k}$. From **H5**) and **H6**) we see that

$$\lambda_k \leq C\,h_k^{-2}.$$

By **(C1)'** we can conclude that (**C1**) holds, since

$$|A_k((I - I_k P_{k-1})v, v)| \leq ||(I - I_k P_{k-1})v||_k\,||A_k v||_k \leq C\,h_k^2\,||A_k v||_k^2 \leq C_1\,\frac{||A_k v||_k^2}{\lambda_k}$$

and

$$\begin{aligned}(A_k((I-I_kP_{k-1})v,(I-I_kP_{k-1})v))^{1/2} &= |||(I-I_kP_{k-1})v|||_{1,k}\\ &\le C\,h_k^{-1}\,||(I-I_kP_{k-1})v||_k\\ &\le C\,h_k\,||A_kv||_k = C\,h_k\,||A_k^{1/2}\,A_k^{1/2}\,v||_k\\ &\le C\,h_k\,\lambda_k^{1/2}\,(A_k(v,v))^{1/2}\\ &\le C_Q\,(A_k(v,v))^{1/2}. \qquad \square\end{aligned}$$

Theorem 4.2. *Let hypotheses* **H1**)–**H5**) *and* **H7**) *hold. If*

$$||w_k - I_kw_{k-1}||_{L^2(\Omega)} \le C\,h_k^2\,||g||_{L^2(\Omega)} \tag{21}$$

holds, then **(C.1)'** *holds. Here $w_j \in U_j$ denotes the finite-element solution on the j-th level for a given $g \in L^2(\Omega)$; i.e.*

$$A_j(w_j,q) = (g,q)_{L^2(\Omega)} \quad \forall q \in U_j.$$

Proof. The proof is divided into two steps. In the first step, we show that (21) implies

$$||w_{k-1} - P_{k-1}w_k||_{L^2(\Omega)} \le C\,h_k^2\,||g||_{L^2(\Omega)}. \tag{22}$$

To do so, we consider a dual problem: Find $z \in U$ such that

$$A(z,q) = (w_{k-1} - P_{k-1}w_k, q)_{L^2(\Omega)} \quad \forall q \in U. \tag{23}$$

Denote by $z_j \in U_j$ the finite element solution to (23); i.e.

$$A_j(z_j,q) = (w_{k-1} - P_{k-1}w_k, q)_{L^2(\Omega)} \quad \forall q \in U_j.$$

Applying (21) with a right-hand side function $w_{k-1} - P_{k-1}w_k \in L^2(\Omega)$, by **H7**) and the triangle inequality, we have

$$\begin{aligned}&||z_{k-1} - I_kz_{k-1}||_{L^2(\Omega)}\\ &\le ||z_{k-1} - z_k||_{L^2(\Omega)} + ||z_k - I_kz_{k-1}||_{L^2(\Omega)}\\ &\le ||z_{k-1} - z||_{L^2(\Omega)} + ||z - z_k||_{L^2(\Omega)} + ||z_k - I_kz_{k-1}||_{L^2(\Omega)}\\ &\le C\,h_k^2\,||w_{k-1} - P_{k-1}w_k||_{L^2(\Omega)}.\end{aligned}$$

Therefore,

$$\begin{aligned}||w_{k-1} - P_{k-1}w_k||^2_{L^2(\Omega)} &= A_{k-1}(z_{k-1}, w_{k-1} - P_{k-1}w_k)\\ &= A_{k-1}(z_{k-1}, w_{k-1}) - A_{k-1}(z_{k-1}, P_{k-1}w_k)\\ &= A_{k-1}(z_{k-1}, w_{k-1}) - A_k(I_kz_{k-1}, w_k)\\ &= (g, z_{k-1} - I_kz_{k-1})_{L^2(\Omega)}\\ &\le C\,h_k^2\,||g||_{L^2(\Omega)}\,||w_{k-1} - P_{k-1}w_k||_{L^2(\Omega)}.\end{aligned}$$

It follows that (22) holds.

Now we take the second step, showing **(C1)'**. To do so, set $E_k := I - I_kP_{k-1}$. Again, we consider a dual problem: Find $z \in U$ such that

$$A(z,q) = (E_kv, q)_{L^2(\Omega)} \quad \forall q \in U.$$

Let $z_j \in U_j$ be the finite-element solution, approximating z; i.e.,

$$A_j(z_j,q) = (E_kv, q)_{L^2(\Omega)} \quad \forall q \in U_j.$$

From (21) and (22) we have

$$||z_k - I_kz_{k-1}||_{L^2(\Omega)} + ||z_{k-1} - P_{k-1}z_k||_{L^2(\Omega)} \le C\,h_k^2\,||E_kv||_{L^2(\Omega)}.$$

Therefore, in view of **H4**) and **H5**),

$$\begin{aligned}||E_k v||^2_{L^2(\Omega)} &= A_k(E_k v, z_k)\\ &= A_k(v, z_k) - A_{k-1}(P_{k-1}v, P_{k-1}z_k)\\ &= A_k(v, z_k - I_k z_{k-1}) + A_k(v, I_k(z_{k-1} - P_{k-1}z_k))\\ &\le ||A_k v||_k\, ||z_k - I_k z_{k-1}||_k + ||A_k v||_k\, ||I_k(z_{k-1} - P_{k-1}z_k)||_k\\ &\le C\, ||A_k v||_k \,\{||z_k - I_k z_{k-1}||_{L^2(\Omega)} + ||I_k(z_{k-1} - P_{k-1}z_k)||_{L^2(\Omega)}\}\\ &\le C\, ||A_k v||_k \,\{||z_k - I_k z_{k-1}||_{L^2(\Omega)} + ||z_{k-1} - P_{k-1}z_k||_{L^2(\Omega)}\}\\ &\le C\, h_k^2\, ||A_k v||_k\, ||E_k v||_{L^2(\Omega)},\end{aligned}$$

The proof is completed. □

Proposition 4.1. *Assume* **H1**)–**H4**) *and* **H7**). *If the following estimate holds,*

$$||\Pi_k w - I_k \Pi_{k-1} w||_{L^2(\Omega} \le C\, h_k^2\, ||w||_{H^2(\Omega)} \quad \forall w \in H^2(\Omega), \tag{24}$$

then (21) *holds.*

Proof. With a right-hand side function $g \in L^2(\Omega)$, let $w \in H^2(\Omega)$ be the solution to

$$A(w, q) = (g, q)_{L^2(\Omega)} \quad \forall q \in U. \tag{25}$$

Let $w_j \in U_j$ be the finite-element solution to w. From **H7**) we know that

$$||w - w_j||_{L^2(\Omega)} \le C\, h_k^2\, ||g||_{L^2(\Omega)}, \quad j = k-1, k. \tag{26}$$

Rewriting

$$w_k - I_k w_{k-1} = w_k - w + w - \Pi_k w + \Pi_k w - I_k \Pi_{k-1} w + I_k(\Pi_{k-1} w - w + w - w_{k-1}),$$

by (26), (24), **H3**) and **H4**), we get (21). □

Proposition 4.2. *Assume* **H1**)–**H4**) *and* **H7**). *If there exists a finite-dimensional space $\Sigma_{k-1} \subseteq U_k \cap U_{k-1}$, which has the same order of approximation as that of U_k and U_{k-1}, such that*

$$I_k w \equiv w \quad \forall w \in \Sigma_{k-1}, \tag{27}$$

or, if the following estimate holds,

$$||z - I_k q||_{L^2(\Omega)} \le C\, ||z - q||_{L^2(\Omega)} \quad \forall z \in U_k, \forall q \in U_{k-1}, \tag{28}$$

then (21) *holds.*

Proof. Inequality (21) trivially results from (28), the triangle inequality and **H7**).

Let us assume (27). Let $w_j \in U_j, j = k, k-1$ and $q_{k-1} \in \Sigma_{k-1}$ be the finite-element solutions to w, for $g \in L^2(\Omega)$; cf. (25) and (26).

$$\begin{aligned}w_k - I_k w_{k-1} &= w_k - q_{k-1} + q_{k-1} - I_k w_{k-1}\\ &= w_k - w + w - q_{k-1} + I_k(q_{k-1} - w + w - w_{k-1}).\end{aligned}$$

Inequality (21) follows. □

Remark 4.2. For $\mathcal{P}_1$ and Wilson's nonconforming elements, (27) is obviously true, with Σ_{k-1} being the conforming $\mathcal{P}_1$ and $\mathcal{Q}_1$ elements, respectively; see [13] and [41].

For C^0 elements with nonnested triangulations, (28) was shown in [42]. For other nonnested C^0 elements such as a bubble-enriching element and a composed element, (28) was shown in [23].

For nonconforming elements such as $\mathcal{Q}_1^{\rm rot}$ element and a discretely divergence-free $\mathcal{P}_1$ element, (24) was shown in [36], [4] and [14].

For the Mortar element, **(C1)'** was shown in [25].

Note that for all these nonnested cases there are different assumptions on the triangulations.

To avoid proliferation, here we leave out more detailed description and verification of assumptions (23), (26) and (27), associated with the assumption (20) for **(C1)'**, for various nonnested V-cycle methods. Readers can refer to the cited references for details.

Remark 4.3. To our best knowledge, all the existing intergrid transfer operators satisfy **H4**); cf. [43], [11], [4], [2], [14], [13], [41] and [23]. Other hypotheses **H1**)–**H3**) and **H5**)–**H7**) are often trivial and standard. We therefore do not insist on details here.

5. Numerical results

In our numerical tests, we studied P_1 linear triangles and P_1 nonconforming linear elements, where the nodal values are defined at the vertices or the midedge points, respectively. We tested both nested and nonnested, but uniform grids, (see Figure 1), on the unit square domain $\Omega = (0,1)^2$.

The bilinear form is the semi-H^1 product $A_k(u,v) = \int_\Omega u_x v_x + u_y v_y$. The discrete L^2 inner product is $(u,v)_k = h^2 \sum u_i v_i$ where the summation is over all nodal points and h is the grid size.

In Table 1 below, we listed the constants computed numerically for the P_1 conforming elements on nonnested grids, where C_1 and C_Q are used in the regularity-approximation assumption (C.1), the constant C_R is from the smoothing hypothesis (C.2), δ_k (less than the theoretic constant δ in (5)) is the error reduction factor of the V-cycle nonnested multigrid method, and δ'_k is the two-level error reduction factor. Here we solve the coarse-level correction problem exactly in the two-level multigrid method and δ'_k is the the spectral radius of such a multigrid operator:

$$\delta'_k = \rho\left((\tilde{J}_k^{(m)})^* (I - I_k P_{k-1}) \tilde{J}_k^{(m)} \right).$$

We note in particular that all constants are computed by Matlab as they are the maximum or the minimum of certain eigenvalues. Here the number of the smoothing parameter m is set to 8 and the Richardson iteration is used for the presmoothing and the postsmoothing. The same constants for the P_1 nonconforming elements on nonnested grids are listed in Table 2. Because of the nonnested grids, the intergrid transfer operator I_k is simply the nodal value interpolation operator Π_k as all fine-level nodal points are in the interior of some coarse-level triangles. This avoids

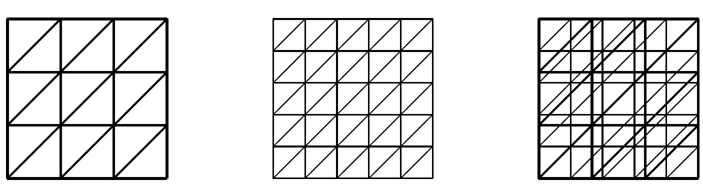

FIGURE 1. Nonnested grids. ($h = 1/3$ and $h = 1/5$)

TABLE 1. Constants for P_1 conforming elements on nonnested grids.

level k	grid	C_1	C_Q	C_R	δ_k	δ'_k
2	3×3	1.7676	5.3645	1.0000	0.0008	0.0000
3	5×5	3.3794	7.0620	1.0000	0.0470	0.0468
4	9×9	4.8324	7.7069	1.0000	0.0923	0.0787
5	17×17	6.0642	7.9184	1.0000	0.1289	0.1132
6	33×33	6.9498	7.9785	1.0000	0.1554	0.1358
7	65×65	7.5868	7.9945	1.0000	0.1705	0.1485

TABLE 2. Constants for P_1 nonconforming elements on nonnested grids.

level k	grid	C_1	C_Q	C_R	δ_k	δ'_k
2	3×3	10.9296	31.6111	1.0000	0.0965	0.0261
3	5×5	10.5700	26.3688	1.0000	0.0751	0.0741
4	9×9	17.0298	38.4648	1.0000	0.1539	0.1521
5	17×17	20.8358	44.6080	1.0000	0.2421	0.2051
6	33×33	23.3766	52.3772	1.0000	0.2862	0.2422

the trouble of defining nodal (midedge) values of $I_k v_{k-1}$, which is done usually by averaging the values of v_{k-1} at nearby nodes; cf. [4], [13], [14], [20], and [22].

Comparing the data in Tables 1 and 2, we can see the constants C_1 and C_Q are much worse for the nonconforming P_1 elements. However, the V-cycle and the two-level convergence rates δ_k and δ'_k do not differ much between the conforming and nonconforming elements. We note that here the grid size ratio of the fine-to-coarse levels is more than 1/2—better than that in the nested multigrid method. So the nonnested multigrid convergence rate is better than that of the standard nested multigrid (shown in Table 3).

For Table 3, we have nested grids. We note that because we used one-sided value interpolation operator (since the fine-level midedge points are no longer inside coarse-level triangles) as the intergrid transfer operator, instead of some averaging operators (cf. [13]), the rates of the nonconforming multigrid method (listed in the last two columns) are much worse than that of the conforming method (listed in the middle two columns in Table 3.) Otherwise, the difference in rates should be small, as shown in the nonnested cases (listed in Tables 1 and 2). We further remark that, due to the perturbation to the subspace A_k-projection, the two-level nonconforming multigrid method is worse than its V-cycle multigrid version in terms of the rate of convergence. In other words, a more accurate coarse-level correction would produce a bigger error to the high-frequency components of the iterative solution on the finer grid. Therefore, if the fine-level smoothing number is not high enough, the multigrid iteration may even diverge. This phenomenon shows up clearly in the last numerical example in this paper.

In Figure 2 we plot the estimated rate δ_k of the V-cycle multigrid method in (6) and the estimated rate in (19) for the two-level multigrid method, against the actual (computed) rates. Here the estimates $\frac{C_1 C_R}{m - C_1 C_R}$ and $\frac{C_1 C_R}{m + C_1 C_R / C_Q}$ are both for P_1 conforming elements, and computed by numerical data C_1 and C_Q. The grid level is $k = 5$ in Figure 2.

TABLE 3. The convergence rate for nested-grid P_1 conforming and nonconforming elements.

level k	grid	δ_k(c)	δ'_k(c)	δ_k(nc)	δ'_k(nc)
2	4×4	0.0844	0.0186	0.3211	0.1088
3	8×8	0.1619	0.1332	0.3989	0.3775
4	16×16	0.2143	0.1663	0.4091	0.4469
5	32×32	0.2404	0.1791	0.4359	0.4678

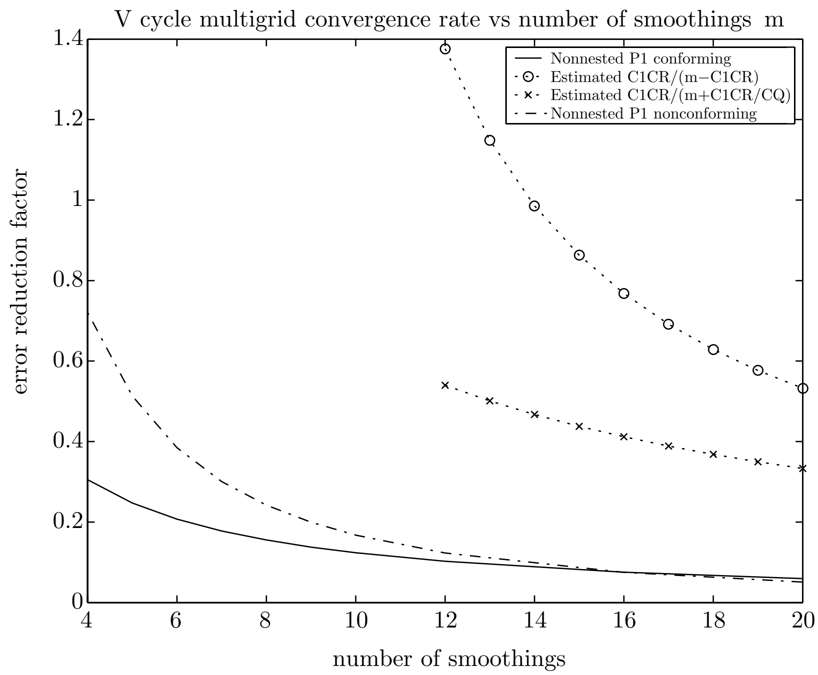

FIGURE 2. The convergence rate of the V-cycle multigrid

Finally, we would show a counterexample where the number of smoothings m must be sufficiently large, larger than one, depending on a parameter σ in the mesh perturbation, in order for the V-cycle nonnested multigrid methods to converge. In the example, we use the cubic Lagrange element to solve the Poisson equation with a homogeneous boundary condition. The domain is the unit square. On the first level, we have only two triangles. We then use the multigrid refinement to generate the higher level meshes. On the fifth level, we perturb the mesh by moving all internal nodes by the mapping $(x, y)/(r/1.5)^\sigma$. In Figure 3, the fine grid is plotted by solid lines and all the coarse grids are plotted (overlapped) by dash lines.

In Table 4, we list the number of V-cycle iterations needed for the P_3-finite element iterative solution to reach its approximation accuracy. For the nested case, i.e., $\sigma = 0$ on the finest level, one smoothing is enough to make V-cycle iteration converge, as predicted by the standard multigrid theory. However, when the meshes are perturbed as shown in Figure 3, one smoothing is not enough for the nonnested V-cycle iteration to converge. As shown in Table 4, the number of smoothings m

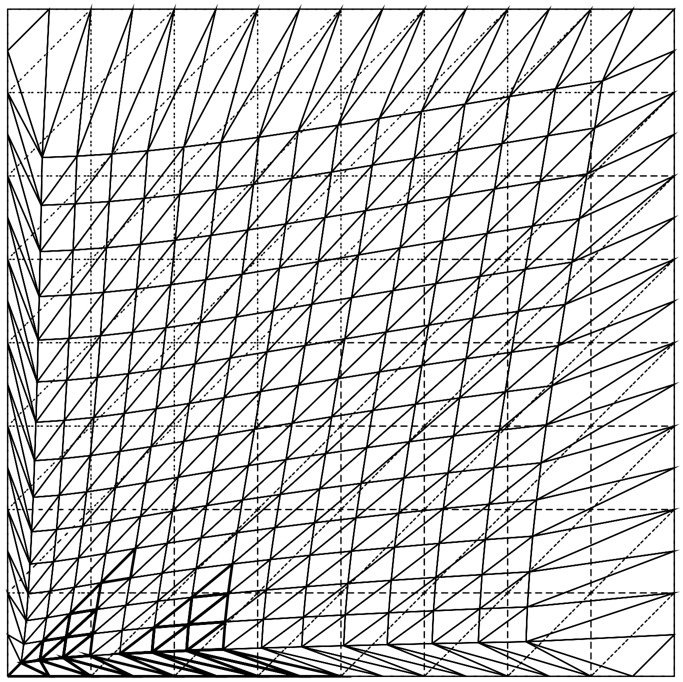

FIGURE 3. The fifth grid is perturbed by $(x, y)/(r/1.5)^{0.4}$.

must be larger than 4 (when $\sigma = 0.36$) or 7 (when $\sigma = 0.4$), respectively. Otherwise the V-cycle iterations would diverge. From Table 4, it seems that the when m is large, the converge rates of nested and nonnested V-cycles differ very little. We remark that the reduction rate δ_5 listed in Table 4 does not decrease monotonically when m increases. This is caused by the way we average the error reduction factors by the number of V-cycles:

$$\delta_5 = \frac{1}{n_v} \sum_{i=1}^{n_v} \frac{|u_h - u_i|_{H^1}}{|u_h - u_{i-1}|_{H^1}},$$

where n_v is the number of V-cycles.

In Figure 4, we plot the iterative error before doing a *nonnested* coarse-level correction and after doing such a correction. In the nonnested coarse-level correction, the low-frequency components of the iterative error are usually reduced well. However, due to nonnestedness, some high-frequency errors would be amplified. This can be seen by comparing the two graphs in Figure 4. Therefore, the fine-level smoothing has to be performed enough times in order for the nonnested multigrid method to converge.

We make a final remark on selecting the counterexample. The difficulty here arises when we use the multigrid refinement to generate meaningful, or likely practical, grids. With reasonable perturbations of the grids, we could not find a case where the $P1$ multigrid V-cycle diverges. After numerous successful tries, we turned to $P2$, $P3$ and high-order elements where one fine-level smoothing is not powerful enough to smooth out the non–$a(\cdot,\cdot)$-projection component of the coarse-level correction.

TABLE 4. The number of V-cycles and the error reduction rate for for nonnested P_3 elements.

m	# V-cycle,	$\delta_5(\sigma = 0)$	# V-cycle,	$\delta_5(\sigma = .36)$	# V-cycle,	$\delta_5(\sigma = .4)$
1	72	0.9035	∞	1.0266	∞	1.4101
2	36	0.8188	∞	1.2698	∞	1.7989
3	24	0.7481	∞	1.1905	∞	1.6841
4	18	0.6900	∞	1.0396	∞	1.4963
5	14	0.6215	> 40	0.9304	∞	1.3195
6	12	0.5848	> 40	0.8708	∞	1.1638
7	10	0.5247	13	0.6203	∞	1.0237
8	8	0.4374	11	0.5767	> 40	0.8833
9	7	0.3936	9	0.5107	> 40	0.8920
10	6	0.3358	8	0.4733	> 40	0.9015
11	6	0.3651	7	0.4251	10	0.5803
12	5	0.2767	7	0.4206	9	0.5615
13	5	0.3026	6	0.3673	7	0.4503
14	5	0.2821	6	0.3839	6	0.4033
15	4	0.2141	6	0.4006	5	0.2900
16	4	0.2315	5	0.2917	5	0.2966
17	4	0.2479	5	0.3013	5	0.3078
18	4	0.2699	5	0.3160	5	0.3148
19	4	0.2673	5	0.3341	5	0.3347
20	4	0.2690	5	0.3386	5	0.3743
21	4	0.2800	5	0.3608	4	0.2111
22	4	0.3340	5	0.3729	4	0.2143
23	3	0.1602	4	0.2117	4	0.2168
24	3	0.1645	4	0.2120	4	0.2217
25	3	0.1691	4	0.2159	4	0.2206

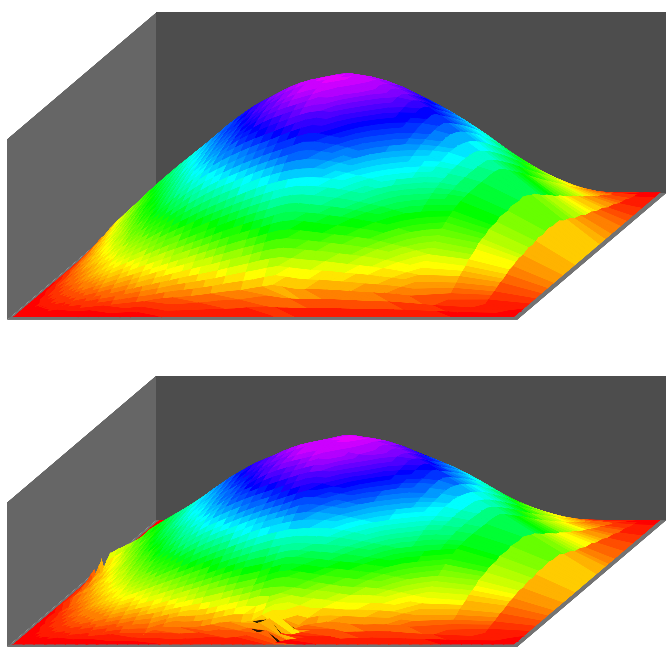

FIGURE 4. The iterative error before and after doing a coarse-level correction.

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