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A POSTERIORI ERROR ANALYSIS FOR LOCALLY CONSERVATIVE MIXED METHODS

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Abstract. In this work we present a theoretical analysis for a residual-type error estimator for locally conservative mixed methods. This estimator was first introduced by Braess and Verfürth for the Raviart–Thomas mixed finite element method working in mesh-dependent norms. We improve and extend their results to cover any locally conservative mixed method under minimal assumptions, in particular, avoiding the saturation assumption made by Braess and Verfürth. Our analysis also takes into account discontinuous coefficients with possibly large jumps across interelement boundaries. The main results are applied to the $P1$ nonconforming finite element method and the interior penalty discontinuous Galerkin method as well as the mixed finite element method.

1. Introduction

In this paper we concern ourselves with a posteriori error analysis for locally conservative mixed methods of the second order elliptic boundary value problem. Over the last decades many locally conservative mixed methods for this problem have been developed and extensively utilized in practical applications due to their desirable properties such as good approximation of the velocity and the local mass conservation. The most popular one among such mixed methods is the mixed finite element method; see, for example, [8, 18] for an extensive discussion on this topic (and the references therein). There are also finite volume type methods such as mixed covolume methods [10, 12] and mixed finite volume methods on nonstaggered grids [13, 16].

On the other hand, it is now well recognized that a posteriori error estimators are an indispensable tool for assessing numerical solutions of partial differential equations. They can provide a systematic way for controlling numerical errors locally and thus performing adaptive mesh refinements, which leads to more efficient numerical computations. For a survey on various types of error estimators and their analysis, we refer to the books of Ainsworth and Oden [1] and Verfürth [20].

In recent years there have been several attempts to develop a posteriori error estimators for mixed finite element methods. For residual-type error estimators the first result was obtained by Braess and Verfürth [7]. They could circumvent some technical difficulties in applying the conventional strategy for the standard

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primal finite element methods by utilizing the mesh-dependent norms which are similar to the $H(\mathrm{div})$-norm for the vector and the H^1-like norm for the scalar but at the expense of a saturation assumption. In [2, 9] the Helmholtz decomposition was used to develop an error estimator for the vector in the natural $H(\mathrm{div})$-norm, without any saturation assumption. In particular, the duality argument was used in [9] to derive an error estimator for the L^2-norm of the scalar error, which requires a certain regularity on the given problem. For other types of error estimators and their comparison we refer to [22].

The aim of this paper is to improve and extend the results of [6, 7] to cover *any* locally conservative mixed method which produce $H(\mathrm{div})$-conforming vector approximations. Although the same mesh-dependent norm of [6, 7] is employed for the scalar, our approach is quite different from that of previous works and has the advantage of being independent of any particular discretization, as long as the vector approximation satisfies the local conservation law. Moreover, our analysis is based on minimal assumptions, without requiring additional regularity of the true solution nor any kind of saturation assumption.

Our results are applied to three locally conservative methods, namely, the mixed finite element method, the $P1$ nonconforming finite element method and the interior penalty discontinuous Galerkin method. For the first method, let us point out that the scalar error in the mesh-dependent H^1-like norm was overestimated in [7], and it does not converge at all in the lowest order case. We resolve this difficulty by following the idea proposed in [6] which employs a higher order scalar approximation computed through a suitable postprocessing scheme.

Our analysis also takes into account discontinuous coefficients with possibly large jumps across interelement boundaries. By incorporating the proper weights for element and edge residuals, as in [5], we show that the error estimator is robust with respect to the jumps under a suitable hypothesis (like Hypothesis 2.7 in [5]).

The rest of the paper is organized as follows. In the next section we introduce some notation and function spaces. In Section 3 the model problem is stated with the main assumption on its coefficient, and the locally conservative discretization is introduced. In Section 4 we perform a rigorous analysis to establish the reliability and efficiency of the Braess–Verfürth error estimator (up to higher order terms) in a general setting. In Section 5 we discuss the application to the mixed finite element method the $P1$ nonconforming finite element method, and the interior penalty discontinuous Galerkin method. Finally in Section 6 numerical experiments are carried out to illustrate our results.

2. Preliminaries

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, $\Gamma_D \cap \Gamma_N = \emptyset$. Suppose that $\{\mathcal{T}_h\}_{h>0}$ is a family of regular triangulations of Ω into triangular elements such that the intersection of any two elements is either empty, a vertex or a complete edge. We define the usual mesh parameters

$$h = \max_{T\in\mathcal{T}_h} h_T, \qquad h_T = \mathrm{diam}(T), \qquad |T| = \mathrm{meas}(T).$$

Let $\mathcal{E}_h$ be the collection of all edges of $\mathcal{T}_h$ which is split into three disjoint parts $\mathcal{E}_I$, $\mathcal{E}_D$, and $\mathcal{E}_N$, according to whether the edge belongs to Ω, Γ_D, or Γ_N, respectively. The notation $\mathcal{E}_T$ is used to denote the set of edges of an element T. With each edge $E \in \mathcal{E}_h$ we associate a fixed unit normal n_E which is taken outward to Ω for

$E \in \mathcal{E}_D \cup \mathcal{E}_N$. For $E \in \mathcal{E}_I$ shared by two elements T^+ and T^- with n_E being directed from T^+ to T^-, we define the average and the jump of v on E by

$$\{v\} = \frac{v^+ + v^-}{2}, \qquad [\![v]\!] = v^+ - v^-,$$

where v^+ (resp. v^-) denotes the trace of $v|_{T^+}$ (resp. $v|_{T^-}$). For the boundary edge $E \in \mathcal{E}_D \cap \mathcal{E}_T$, we set $\{v\} = [\![v]\!] = v|_T$.

Let $\mathcal{P}_r(T)$ be the standard space of all polynomials on T whose degrees are less than or equal to r. There are many well-known spaces for the vector approximation (cf. [8, 18]), for example, the Raviart–Thomas (RT) space defined by

$$\mathcal{RT}_r(T) = (\mathcal{P}_r(T))^2 + \boldsymbol{x}\mathcal{P}_r(T) \qquad (\boldsymbol{x} = (x, y)),$$

or the Brezzi–Douglas–Marini (BDM) space defined by

$$\mathcal{BDM}_r(T) = (\mathcal{P}_r(T))^2.$$

The global finite element spaces are then defined to be

$$\begin{aligned}
\mathcal{P}_r(\mathcal{T}_h) &= \{v \in L^2(\Omega) : v|_T \in \mathcal{P}_r(T) \ \ \forall T \in \mathcal{T}_h\},\\
\mathcal{RT}_r(\mathcal{T}_h) &= \{\tau \in H(\mathrm{div};\Omega) : \tau|_T \in \mathcal{RT}_r(T) \ \ \forall T \in \mathcal{T}_h\},\\
\mathcal{BDM}_r(\mathcal{T}_h) &= \{\tau \in H(\mathrm{div};\Omega) : \tau|_T \in \mathcal{BDM}_r(T) \ \ \forall T \in \mathcal{T}_h\},
\end{aligned}$$

where

$$H(\mathrm{div};\Omega) = \{\tau \in (L^2(\Omega))^2 : \mathrm{div}\,\tau \in L^2(\Omega)\}.$$

We also need the finite element space on interelement boundaries defined by

$$\mathcal{P}_r(\mathcal{E}_h) = \{v \in L^2(\bigcup \partial T) : v|_E \in \mathcal{P}_r(E) \ \ \forall E \in \mathcal{E}_h\}.$$

The L^2-projections onto $\mathcal{P}_r(\mathcal{T}_h)$ and $\mathcal{P}_r(\mathcal{E}_h)$ are denoted by P_h^r and Q_h^r, respectively. Similar definitions can be made for subsets of $\mathcal{T}_h$ or $\mathcal{E}_h$ as well.

The standard notation $H^l(G)$ is adopted for the Sobolev space on a domain $G \subset \mathbb{R}^2$, with its norm and seminorm denoted by $\|\cdot\|_{l,G}$ and $|\cdot|_{l,G}$ (simply written as $\|\cdot\|_l$ and $|\cdot|_l$ if $G = \Omega$), and we set

$$H_D^1(\Omega) = \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}.$$

It will also be convenient to define the broken Sobolev spaces for $l \geq 0$ as

$$\begin{aligned}
H^l(\mathcal{T}_h) &= \{v \in L^2(\Omega) : v|_T \in H^l(T) \ \ \forall T \in \mathcal{T}_h\},\\
H^l(\mathcal{E}_D) &= \{v \in L^2(\Gamma_D) : v|_E \in H^l(E) \ \ \forall E \in \mathcal{E}_D\},\\
H^l(\mathcal{E}_N) &= \{v \in L^2(\Gamma_N) : v|_E \in H^l(E) \ \ \forall E \in \mathcal{E}_N\}.
\end{aligned}$$

3. Model problem and locally conservative discretization

Throughout the paper we are concerned with the following second order elliptic problem with mixed boundary conditions

$$\left\{\begin{array}{l}\sigma = \kappa\nabla u, \quad \mathrm{div}\,\sigma + f = 0 \quad \text{in } \Omega,\\ u|_{\Gamma_D} = u_D \quad \text{and} \quad \sigma\cdot n|_{\Gamma_N} = g_N,\end{array}\right. \tag{3.1}$$

where $f \in L^2(\Omega)$, $u_D \in H^{\frac{1}{2}}(\Gamma_D) \cap C^0(\overline{\Gamma}_D)$, and $g_N \in L^2(\Gamma_N)$ are the given data, and n is the outward unit normal to $\partial\Omega$. The SPD matrix-valued coefficient $\kappa = \kappa(x)$ may be discontinuous with possibly large jumps across interelement boundaries, but is assumed to be smooth within each element $T \in \mathcal{T}_h$. This means that there exist positive constants λ_T and Λ_T such that for all $\boldsymbol{x} \in T$ and $\xi \in \mathbb{R}^2$,

$$\lambda_T \|\xi\|^2_{\mathbb{R}^2} \le (\kappa(\boldsymbol{x})\xi, \xi)_{\mathbb{R}^2} \le \Lambda_T \|\xi\|^2_{\mathbb{R}^2},$$

with the ratio Λ_T/λ_T being uniformly bounded over the whole family $\{\mathcal{T}_h\}_{h>0}$.

For the error analysis in the next section, we need to impose some restriction on the distribution of κ. For a vertex $\boldsymbol{z}$ of $\mathcal{T}_h$, let $\omega_{\boldsymbol{z}}$ be the collection of all elements sharing $\boldsymbol{z}$. Obviously, given a pair of elements T, T' in $\omega_{\boldsymbol{z}}$, there is always at least one connected path from T to T' through adjacent elements in $\omega_{\boldsymbol{z}}$. We denote by $T_{\boldsymbol{z}}$ a fixed element of $\omega_{\boldsymbol{z}}$ for which Λ_T is maximal, that is,

$$\Lambda_{T_{\boldsymbol{z}}} = \max_{T \in \omega_{\boldsymbol{z}}} \Lambda_T,$$

and we make the following assumption for every vertex $\boldsymbol{z}$.

Main Assumption. *For every element T in $\omega_{\boldsymbol{z}}$, there exists a connected path $\{T_j\}_{j=0}^J \subset \omega_{\boldsymbol{z}}$ from T to $T_{\boldsymbol{z}}$ with*

$$T_0 = T, \quad T_J = T_{\boldsymbol{z}}, \quad \text{and} \quad \partial T_j \cap \partial T_{j+1} \ne \emptyset, \tag{3.2}$$

such that

$$\Lambda_T \le C \min_{0 \le j \le J} \Lambda_{T_j}, \tag{3.3}$$

with C depending only on the minimum angle of $\{\mathcal{T}_h\}_{h>0}$.

This assumption is satisfied; e.g., if Λ_T increases monotonically along the path $\{T_j\}_{j=0}^J$ (Hypothesis 2.7 in [5]). From now on C will denote a generic positive constant depending only on the minimum angle of $\{\mathcal{T}_h\}_{h>0}$. Further dependence of C on the polynomial degree (to be specified later) will be stated if necessary.

Suppose that we are given a locally conservative method of the mixed system (3.1) which produces the $H(\mathrm{div})$-conforming vector approximation σ_h in V_h and the possibly discontinuous scalar approximation u_h in W_h. By *locally conservative* we mean that σ_h should satisfy

$$\int_T (\operatorname{div} \sigma_h + f)\, d\boldsymbol{x} = 0 \quad \text{and} \quad \int_E (\sigma_h \cdot n - g_N)\, ds = 0 \tag{3.4}$$

for all $T \in \mathcal{T}_h$ and $E \in \mathcal{E}_N$. In fact, (σ_h, u_h) is not necessarily the exact solution of the given mixed method, as long as it satisfies the local conservation law (3.4).

The error measure used in this paper is given by

$$\|\kappa^{-1/2}(\sigma - \sigma_h)\|_0 + |\!|\!| u - u_h |\!|\!|, \tag{3.5}$$

where we define for $w \in H^1(\mathcal{T}_h)$

$$|\!|\!| w |\!|\!|^2 = \sum_{T \in \mathcal{T}_h} \|\kappa^{1/2} \nabla w\|^2_{0,T} + \sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D} W_E h_E^{-1} \|[\![w]\!]\|^2_{0,E}.$$

The mesh-dependent norm $|\!|\!| \cdot |\!|\!|$ is the energy norm typically encountered in the analysis of discontinuous Galerkin methods (cf. [3, 15, 17]). The weight factor W_E is taken to be the simple or the harmonic average

$$W_E = \frac{\Lambda_T + \Lambda_{T'}}{2} \quad \text{or} \quad \frac{2\Lambda_T \Lambda_{T'}}{\Lambda_T + \Lambda_{T'}} \tag{3.6}$$

for an interior edge $E \in \mathcal{E}_T \cap \mathcal{E}_{T'}$. For a boundary edge $E \in \mathcal{E}_T$, we simply set

$$W_E = \Lambda_T. \tag{3.7}$$

We will frequently use the following inequalities for these averages:

$$\frac{1}{2}\max\{\Lambda_T,\Lambda_{T'}\} \le \frac{\Lambda_T+\Lambda_{T'}}{2} \le \max\{\Lambda_T,\Lambda_{T'}\}$$

and

$$\min\{\Lambda_T,\Lambda_{T'}\} \le \frac{2\Lambda_T\Lambda_{T'}}{\Lambda_T+\Lambda_{T'}} \le 2\min\{\Lambda_T,\Lambda_{T'}\}.$$

For the optimality of the error (3.5) for smooth σ and u, it is natural to seek u_h in a finite element space of one order higher than that of σ_h; that is,

$$(\mathcal{P}_k(T))^2 \subseteq V_h(T) \subsetneq (\mathcal{P}_{k+1}(T))^2, \qquad \mathcal{P}_{k+1}(T) \subseteq W_h(T) \subsetneq \mathcal{P}_{k+2}(T),$$

where $V_h(T)$ and $W_h(T)$ are the local spaces. We can then expect the following optimal a priori error estimate for $1 \le s \le k+1$:

$$\|\kappa^{-1/2}(\sigma-\sigma_h)\|_0 + |||u-u_h||| \le Ch^s(\|\sigma\|_s + \|u\|_{s+1}). \tag{3.8}$$

4. Braess–Verfürth error estimator

The residual-type error estimator of Braess and Verfürth [7] is defined by

$$\eta^2 = \eta_c^2 + \eta_{nc}^2,$$

where

$$\eta_c^2 := \sum_{T\in\mathcal{T}_h}\left(\|\kappa^{-1/2}(\sigma_h-\kappa\nabla u_h)\|_{0,T}^2 + \Lambda_T^{-1}h_T^2\|\operatorname{div}\sigma_h + f\|_{0,T}^2\right) + \sum_{E\in\mathcal{E}_N} W_E^{-1}h_E\|\sigma_h\cdot n - g_N\|_{0,E}^2, \tag{4.1}$$

$$\eta_{nc}^2 := \sum_{E\in\mathcal{E}_I} W_E h_E^{-1}\|[\![u_h]\!]\|_{0,E}^2 + \sum_{E\in\mathcal{E}_D} W_E h_E^{-1}\|u_h-u_D\|_{0,E}^2. \tag{4.2}$$

Remark 4.1. The first component η_c can be viewed as the residuals of the equations of (3.1) and the Neumann boundary condition, while the second component η_{nc} represent the nonconformity of $u_h \notin H^1(\Omega)$ and $u_h \neq u_D$.

4.1. Reliability of η.

The following theorem is the first part of the main results of this paper which establishes the reliability of the estimator η.

Theorem 4.2. *There exists a constant $C>0$ depending only on the minimum angle of $\{\mathcal{T}_h\}_{h>0}$ and the polynomial degree k such that*

$$\|\kappa^{-1/2}(\sigma-\sigma_h)\|_0 + |||u-u_h||| \le C(\eta+\eta_{hot}), \tag{4.3}$$

where η_{hot} is defined by

$$\eta_{hot}^2 := \sum_{E\in\mathcal{E}_D} W_E\left(h_E^{-1}\|u_D-\tilde{u}_D\|_{0,E}^2 + h_E|u_D-\tilde{u}_D|_{1,E}^2\right).$$

Here $\tilde{u}_D$ is the standard Lagrange interpolant of u_D from $\mathcal{P}_{k+1}(\mathcal{E}_D)\cap C^0(\overline{\Gamma}_D)$.

Remark 4.3. As the subscript indicates, the extra term η_{hot} is of higher order than η for $u_D \in H^{k+2}(\mathcal{E}_D)$

$$\eta_{hot} \le C\left(\sum_{E\in\mathcal{E}_D} W_E h_E^{2k+3}\|u_D\|_{k+2,E}^2\right)^{1/2},$$

and is thus negligible.

In order to prove Theorem 4.2, it is crucial to note that

$$\|\kappa^{-1/2}(\sigma - \sigma_h)\|_0 \le \left(\sum_{T\in\mathcal{T}_h} \|\kappa^{1/2}\nabla(u-u_h)\|_{0,T}^2\right)^{1/2} + \left(\sum_{T\in\mathcal{T}_h} \|\kappa^{-1/2}(\kappa\nabla u_h - \sigma_h)\|_{0,T}^2\right)^{1/2}$$

and that

$$\left(\sum_{E\in\mathcal{E}_I\cup\mathcal{E}_D} W_E h_E^{-1} \|[\![u-u_h]\!]\|_{0,E}^2\right)^{1/2} = \eta_{nc},$$

since $[\![u]\!] = 0$ on $E \in \mathcal{E}_I$ and $u|_{\Gamma_D} = u_D$. Thus the proof of Theorem 4.2 is reduced to showing that

$$\left(\sum_{T\in\mathcal{T}_h} \|\kappa^{1/2}\nabla(u-u_h)\|_{0,T}^2\right)^{1/2} \le C(\eta + \eta_{hot}). \tag{4.4}$$

For the proof of (4.4), we make use of the following decomposition of $u - u_h$ which is similar to the Helmholtz decomposition of $\kappa\nabla(u-u_h)$.

Lemma 4.4. *Let ϕ be the function in $H^1(\Omega)$ such that $\phi|_{\Gamma_D} = u_D$ and*

$$\int_\Omega \kappa\nabla\phi\cdot\nabla v\,d\boldsymbol{x} = \sum_{T\in\mathcal{T}_h}\int_T \kappa\nabla u_h\cdot\nabla v\,d\boldsymbol{x} \qquad \forall v\in H_D^1(\Omega). \tag{4.5}$$

Then the decomposition

$$u - u_h = (u-\phi) + (\phi - u_h)$$

satisfies the Pythagorean identity

$$\sum_{T\in\mathcal{T}_h} \|\kappa^{1/2}\nabla(u-u_h)\|_{0,T}^2 = \|\kappa^{1/2}\nabla(u-\phi)\|_0^2 + \sum_{T\in\mathcal{T}_h} \|\kappa^{1/2}\nabla(\phi-u_h)\|_{0,T}^2.$$

Proof. This is an immediate consequence of the orthogonality relation

$$\sum_{T\in\mathcal{T}_h}\int_T \kappa\nabla(u-\phi)\cdot\nabla(\phi-u_h)\,d\boldsymbol{x} = 0,$$

which is obtained by taking $v = u - \phi$. □

Remark 4.5. It is easy to see that the second term $\sum_{T\in\mathcal{T}_h}\|\kappa^{1/2}\nabla(\phi-u_h)\|_{0,T}^2$ measures the nonconformity of $u_h \notin H^1(\Omega)$ and $u_h \ne u_D$, as it will vanish if u_h belongs to $H^1(\Omega)$ and $u_h|_{\Gamma_D} = u_D$.

The next two lemmas establish the upper bound of the "conforming part" $\|\kappa^{1/2}\nabla(u-\phi)\|_0$ and the "nonconforming part" $\left(\sum_{T\in\mathcal{T}_h}\|\kappa^{1/2}\nabla(\phi-u_h)\|_{0,T}^2\right)^{1/2}$, respectively, from which the estimate (4.4) follows directly.

Lemma 4.6. *There exists a constant $C > 0$ depending only on the minimum angle of $\{\mathcal{T}_h\}_{h>0}$ such that*

$$\|\kappa^{1/2}\nabla(u-\phi)\|_0 \le C\eta_c. \tag{4.6}$$

Proof. The left-hand side of (4.6) can be written as

$$\|\kappa^{1/2}\nabla(u-\phi)\|_0^2 = \sum_{T\in\mathcal{T}_h}\int_T \kappa\nabla(u-u_h)\cdot\nabla(u-\phi)\,d\boldsymbol{x} = I_1 + I_2,$$

where

$$I_1 := \int_\Omega(\sigma-\sigma_h)\cdot\nabla(u-\phi)\,d\boldsymbol{x},$$
$$I_2 := \sum_{T\in\mathcal{T}_h}\int_T(\sigma_h-\kappa\nabla u_h)\cdot\nabla(u-\phi)\,d\boldsymbol{x}.$$

Since $u-\phi\in H_D^1(\Omega)$, we get, via the integration by parts,

$$I_1 = \sum_{T\in\mathcal{T}_h}\left(\int_T(f+\operatorname{div}\sigma_h)(u-\phi)\,d\boldsymbol{x} + \int_{\Gamma_N\cap\partial T}(g_N-\sigma_h\cdot n)(u-\phi)\,ds\right).$$

By using a constant approximation c_T of $u-\phi$ on T such that

$$\|(u-\phi)-c_T\|_{0,T} + h_T^{1/2}\|(u-\phi)-c_T\|_{0,\partial T} \le Ch_T\|\nabla(u-\phi)\|_{0,T}$$

and the local conservation law (3.4), one can deduce that

$$I_1 \le C\left(\sum_{T\in\mathcal{T}_h}\Lambda_T^{-1}h_T^2\|\operatorname{div}\sigma_h+f\|_{0,T}^2 + \sum_{E\in\mathcal{E}_N}W_E^{-1}h_E\|\sigma_h\cdot n-g_N\|_{0,E}^2\right)^{1/2} \times \|\kappa^{1/2}\nabla(u-\phi)\|_0.$$

The other term I_2 can be bounded in a trivial way:

$$I_2 \le \left(\sum_{T\in\mathcal{T}_h}\|\kappa^{-1/2}(\sigma_h-\kappa\nabla u_h)\|_{0,T}^2\right)^{1/2}\|\kappa^{1/2}\nabla(u-\phi)\|_0.$$

Combining these two results, we obtain the desired estimate (4.6). □

Lemma 4.7. *There exists a constant $C>0$ depending only on the minimum angle of $\{\mathcal{T}_h\}_{h>0}$ and the polynomial degree k such that*

$$\left(\sum_{T\in\mathcal{T}_h}\|\kappa^{1/2}\nabla(\phi-u_h)\|_{0,T}^2\right)^{1/2} \le C(\eta_{nc}+\eta_{hot}). \tag{4.7}$$

Proof. By the definition (4.5) of ϕ, it is easy to verify that

$$\sum_{T\in\mathcal{T}_h}\|\kappa^{1/2}\nabla(\phi-u_h)\|_{0,T}^2 = \min_{\substack{\chi\in H^1(\Omega),\\ \chi|_{\Gamma_D}=u_D}}\sum_{T\in\mathcal{T}_h}\|\kappa^{1/2}\nabla(\chi-u_h)\|_{0,T}^2,$$

which gives a further splitting of the left-hand side of (4.7) as

$$\left(\sum_{T\in\mathcal{T}_h}\|\kappa^{1/2}\nabla(\phi-u_h)\|_{0,T}^2\right)^{1/2} \le \min_{\substack{\chi\in H^1(\Omega),\\ \chi|_{\Gamma_D}=u_D}}\|\kappa^{1/2}\nabla(\chi-\tilde{u}_h)\|_0 + \left(\sum_{T\in\mathcal{T}_h}\|\kappa^{1/2}\nabla(\tilde{u}_h-u_h)\|_{0,T}^2\right)^{1/2}$$

for every choice of $\tilde{u}_h\in H^1(\Omega)$.

The next step is to construct an appropriate function $\tilde{u}_h \in H^1(\Omega)$ from u_h and adapt the argument of [15] (cf. Theorem 2.2), taking into account the large jumps of κ across interelement boundaries. For this sake, we denote by $\mathcal{N}(S)$ the set of all Lagrangian nodes associated with the space $\mathcal{P}_{k+1}(\mathcal{T}_h) \cap C^0(\overline{\Omega})$ which are contained in a domain S. The definitions of $\omega_{\boldsymbol{z}}$ and $T_{\boldsymbol{z}}$ immediately carry over to $\boldsymbol{z} \in \mathcal{N}(\overline{\Omega})$. We now define $\tilde{u}_h$ to be the unique function in $\mathcal{P}_{k+1}(\mathcal{T}_h) \cap C^0(\overline{\Omega})$ interpolating

$$\tilde{u}_h(\boldsymbol{z}) = \begin{cases} u_D(\boldsymbol{z}) & \text{for } \boldsymbol{z} \in \mathcal{N}(\overline{\Gamma}_D), \\ u_h|_{T_{\boldsymbol{z}}}(\boldsymbol{z}) & \text{for } \boldsymbol{z} \in \mathcal{N}(\overline{\Omega}) \setminus \mathcal{N}(\overline{\Gamma}_D). \end{cases}$$

It then suffices to establish the estimates

$$\min_{\substack{\chi \in H^1(\Omega),\\ \chi|_{\Gamma_D} = u_D}} \|\kappa^{1/2} \nabla(\chi - \tilde{u}_h)\|_0 = \min_{\substack{\chi \in H^1(\Omega),\\ \chi|_{\Gamma_D} = u_D - \tilde{u}_D}} \|\kappa^{1/2} \nabla \chi\|_0 \leq C\eta_{hot} \tag{4.8}$$

and

$$\left(\sum_{T \in \mathcal{T}_h} \|\kappa^{1/2} \nabla(\tilde{u}_h - u_h)\|_{0,T}^2 \right)^{1/2} \leq C(\eta_{nc} + \eta_{hot}). \tag{4.9}$$

To prove (4.8), we define $\chi_T \in H^1(T)$ on those elements T with $\partial T \cap \Gamma_D \neq \emptyset$ to be the harmonic extension of the boundary values

$$\chi_T = \begin{cases} u_D - \tilde{u}_D & \text{on } \partial T \cap \Gamma_D, \\ 0 & \text{on } \partial T \setminus \Gamma_D. \end{cases}$$

Noting that $u_D - \tilde{u}_D$ vanishes at the endpoints of each edge $E \in \mathcal{E}_D$, we obtain

$$\begin{aligned} \|\kappa^{1/2} \nabla \chi_T\|_{0,T}^2 &\leq C \sum_{E \in \mathcal{E}_T \cap \mathcal{E}_D} W_E \|u_D - \tilde{u}_D\|^2_{H^{\frac{1}{2}}_{00}(E)} \\ &\leq C \sum_{E \in \mathcal{E}_T \cap \mathcal{E}_D} W_E \|u_D - \tilde{u}_D\|_{0,E} |u_D - \tilde{u}_D|_{1,E} \\ &\leq C \sum_{E \in \mathcal{E}_T \cap \mathcal{E}_D} W_E \big(h_E^{-1} \|u_D - \tilde{u}_D\|_{0,E}^2 + h_E |u_D - \tilde{u}_D|_{1,E}^2 \big), \end{aligned}$$

where the interpolation result was used in the second inequality. Hence, if we set

$$\chi = \begin{cases} \chi_T & \text{for those elements } T \text{ with } \partial T \cap \Gamma_D \neq \emptyset, \\ 0 & \text{elsewhere}, \end{cases}$$

we get $\chi \in H^1(\Omega)$, $\chi|_{\Gamma_D} = u_D - \tilde{u}_D$ and $\|\kappa^{1/2} \nabla \chi\|_0 \leq C\eta_{hot}$.

To derive the remaining estimate (4.9), we observe that $(\tilde{u}_h - u_h)(\boldsymbol{z}) = 0$ for all $\boldsymbol{z} \in \mathcal{N}(T^\circ)$ (T° denotes the interior of T), and thus

$$\|\kappa^{1/2} \nabla(\tilde{u}_h - u_h)\|_{0,T}^2 \leq C \Lambda_T h_T^{-2} \|\tilde{u}_h - u_h\|_{0,T}^2 \leq C \sum_{\boldsymbol{z} \in \mathcal{N}(\partial T)} \Lambda_T |(\tilde{u}_h - u_h|_T)(\boldsymbol{z})|^2,$$

with C depending on the polynomial degree k. Now fix $\boldsymbol{z} \in \mathcal{N}(\partial T)$. If $\boldsymbol{z} \in \mathcal{N}(\overline{\Gamma}_D)$, then

$$|(\tilde{u}_h - u_h)(\boldsymbol{z})| = |(\tilde{u}_D - u_h|_T)(\boldsymbol{z})|.$$

If $\boldsymbol{z} \notin \mathcal{N}(\overline{\Gamma}_D)$, we consider three different cases.

(i) If $T_{\boldsymbol{z}} = T$, then

$$|(\tilde{u}_h - u_h|_T)(\boldsymbol{z})| = 0.$$

(ii) If $T_{\boldsymbol{z}} \neq T$ and $\boldsymbol{z} \in \mathcal{N}(E^\circ)$ for some $E \in \mathcal{E}_T$, then $T_{\boldsymbol{z}}$ is adjacent to T and so we obtain

$$\Lambda_T |(\tilde{u}_h - u_h|_T)(\boldsymbol{z})|^2 \leq W_E |[\![u_h]\!]|_E(\boldsymbol{z})|^2.$$

for either choice of W_E in (3.6).

(iii) If $T_{\boldsymbol{z}} \neq T$ and $\boldsymbol{z}$ is a vertex of T, then we choose a path $\{T_j\}_{j=0}^J \subset \omega_{\boldsymbol{z}}$ from T to $T_{\boldsymbol{z}}$ satisfying the Main Assumption (3.2)–(3.3) to obtain

$$\begin{aligned}
\Lambda_T |(\tilde{u}_h - u_h|_T)(\boldsymbol{z})|^2 &= \Lambda_T |(u_h|_{T_{\boldsymbol{z}}} - u_h|_T)(\boldsymbol{z})|^2 \\
&\leq C \sum_{j=0}^{J-1} \Lambda_T |(u_h|_{T_j} - u_h|_{T_{j+1}})(\boldsymbol{z})|^2 \\
&\leq C \sum_{j=0}^{J-1} W_{E_j} |[\![u_h]\!]|_{E_j}(\boldsymbol{z})|^2,
\end{aligned}$$

where $E_j = \partial T_j \cap \partial T_{j+1}$.

Note that for $\boldsymbol{z} \in \mathcal{N}(\overline{\Gamma}_N)$, we have either case (i) in which there is no contribution at all or case (iii) in which the jumps across only some interior edges are involved. By collecting the above results and summing over $T \in \mathcal{T}_h$, it follows that

$$\begin{aligned}
\sum_{T \in \mathcal{T}_h} \|\kappa^{1/2} \nabla(\tilde{u}_h - u_h)\|_{0,T}^2 &\leq C \Bigg(\sum_{E \in \mathcal{E}_I} W_E \sum_{\boldsymbol{z} \in \mathcal{N}(\overline{E})} |[\![u_h]\!]|_E(\boldsymbol{z})|^2 \\
&\qquad + \sum_{E \in \mathcal{E}_D} W_E \sum_{\boldsymbol{z} \in \mathcal{N}(\overline{E})} |(u_h - \tilde{u}_D)(\boldsymbol{z})|^2 \Bigg) \\
&\leq C \Bigg(\sum_{E \in \mathcal{E}_I} W_E h_E^{-1} \|[\![u_h]\!]\|_{0,E}^2 \\
&\qquad + \sum_{E \in \mathcal{E}_D} W_E h_E^{-1} \|u_h - \tilde{u}_D\|_{0,E}^2 \Bigg) \\
&\leq C(\eta_{nc}^2 + \eta_{hot}^2),
\end{aligned}$$

with C depending on the polynomial degree k. This completes the proof. □

4.2. **Efficiency of η.** To establish the converse result of Theorem 4.2, we follow closely the argument of Verfürth [20] which is based on the weighted norms by the bubble functions and the inverse inequalities. For this sake let us introduce the cubic bubble function b_T on $T \in \mathcal{T}_h$ and the quadratic bubble function b_E associated with $E \in \mathcal{E}_N$ satisfying

$$0 \leq b_T \leq 1 = \max b_T, \qquad 0 \leq b_E \leq 1 = \max b_E.$$

Then there exists a constant $C > 0$ depending only on the minimum angle of $\{\mathcal{T}_h\}_{h>0}$ and the polynomial degree r such that

$$\|v\|_{0,T} \leq C \|b_T^{1/2} v\|_{0,T} \qquad \forall v \in \mathcal{P}_r(T), \tag{4.10}$$

$$\|v\|_{0,E} \leq C \|b_E^{1/2} v\|_{0,E} \qquad \forall v \in \mathcal{P}_r(E). \tag{4.11}$$

Also, there exists an operator P_E that extends any function defined on $E \in \mathcal{E}_T$ to the element T and satisfies

$$C_1 h_E^{1/2} \|v\|_{0,E} \leq \|b_E P_E v\|_{0,T} \leq C_2 h_E^{1/2} \|v\|_{0,E} \qquad \forall v \in \mathcal{P}_r(E). \tag{4.12}$$

Now we are ready to prove the efficiency of the estimator η. To avoid unnecessary technical details, the data f and g_N are assumed to be piecewise polynomials of degree r. This restriction can be relaxed, which causes some extra higher order approximation error for piecewise smooth data.

Theorem 4.8. *There exists a constant $C > 0$ depending only on the minimum angle of $\{\mathcal{T}_h\}_{h>0}$ and the polynomial degrees k, r such that*

$$\eta \leq C(\|\kappa^{-1/2}(\sigma - \sigma_h)\|_0 + \|\!|u - u_h|\!\|). \tag{4.13}$$

Proof. From the identity

$$\sigma_h - \kappa\nabla u_h = (\sigma_h - \sigma) + \kappa\nabla(u - u_h),$$

it follows immediately that

$$\|\kappa^{-1/2}(\sigma_h - \kappa\nabla u_h)\|_0 + \eta_{nc} \leq C(\|\kappa^{-1/2}(\sigma - \sigma_h)\|_0 + \|\!|u - u_h|\!\|).$$

To deal with the remaining terms of η_c, we note that for all $w \in H^1_D(\Omega)$,

$$\int_\Omega (\sigma - \sigma_h)\cdot\nabla w\, d\boldsymbol{x} = \int_\Omega (f + \operatorname{div}\sigma_h)w\, d\boldsymbol{x} + \int_{\Gamma_N}(g_N - \sigma_h\cdot n)w\, ds.$$

Fix $T \in \mathcal{T}_h$ and set $w = b_T(f + \operatorname{div}\sigma_h)$. Then, by (4.10) and the inverse inequality, we obtain

$$\begin{aligned}\|f + \operatorname{div}\sigma_h\|^2_{0,T} &\leq C\|\kappa^{-1/2}(\sigma - \sigma_h)\|_{0,T}\|\kappa^{1/2}\nabla w\|_{0,T}\\ &\leq C\|\kappa^{-1/2}(\sigma - \sigma_h)\|_{0,T}\,\Lambda_T^{1/2}h_T^{-1}\|w\|_{0,T},\end{aligned}$$

which yields

$$\Lambda_T^{-1/2}h_T\|f + \operatorname{div}\sigma_h\|_{0,T} \leq C\|\kappa^{-1/2}(\sigma - \sigma_h)\|_{0,T}. \tag{4.14}$$

Similarly, fixing $E \in \mathcal{E}_N \cap \mathcal{E}_T$ and setting $w = b_E P_E(g_N - \sigma_h\cdot n)$, we obtain by (4.11), (4.14) and the inverse inequality

$$\begin{aligned}\|g_N - \sigma_h\cdot n\|^2_{0,E} &\leq C(\|\kappa^{-1/2}(\sigma - \sigma_h)\|_{0,T}\|\kappa^{1/2}\nabla w\|_{0,T} + \|f + \operatorname{div}\sigma_h\|_{0,T}\|w\|_{0,T})\\ &\leq C\|\kappa^{-1/2}(\sigma - \sigma_h)\|_{0,T}\,W_E^{1/2}h_T^{-1}\|w\|_{0,T},\end{aligned}$$

which yields by (4.12)

$$W_E^{-1/2}h_E^{1/2}\|g_N - \sigma_h\cdot n\|_{0,E} \leq C\|\kappa^{-1/2}(\sigma - \sigma_h)\|_{0,T}.$$

Collecting all results together proves the desired assertion. □

5. Applications

In this section we apply the main results established in the previous section to three locally conservative mixed methods, namely, the mixed finite element method, the $P1$ nonconforming finite element method, and the interior penalty discontinuous Galerkin method. The latter two methods can be considered to be mixed methods since we can recover the $H(\text{div})$-conforming vector approximations from the scalar solution in a local and inexpensive way.

5.1. **Mixed finite element methods.** Given a pair of finite element subspaces $V_h \times W_h \subset H(\mathrm{div};\Omega)\times L^2(\Omega)$ satisfying the LBB stability condition, the mixed finite element method for problem (3.1) is defined as follows: find $(\sigma_h, u_h) \in V_h \times W_h$ such that $\sigma_h \cdot n|_{\Gamma_N} = Q_h^k g_N$ and

$$\int_\Omega \kappa^{-1}\sigma_h \cdot \tau \, d\boldsymbol{x} + \int_\Omega u_h \operatorname{div} \tau \, d\boldsymbol{x} = \int_{\Gamma_D} u_D \, \tau \cdot n \, ds \qquad \forall \tau \in V_h^0, \tag{5.1}$$

$$\int_\Omega \operatorname{div} \sigma_h \, w \, d\boldsymbol{x} = -\int_\Omega f w \, d\boldsymbol{x} \qquad \forall w \in W_h, \tag{5.2}$$

where

$$V_h^0 = \{\tau \in V_h : \tau \cdot n|_{\Gamma_N} = 0\}.$$

For clarity of exposition, we restrict ourselves to the Raviart–Thomas space

$$V_h = \mathcal{RT}_k(\mathcal{T}_h), \qquad W_h = \mathcal{P}_k(\mathcal{T}_h).$$

It is clear that the local conservation law (3.4) holds for this method. We also have the following optimal a priori error estimate in the L^2-norm for $1 \le s \le k+1$

$$\|\kappa^{-1/2}(\sigma - \sigma_h)\|_0 + \|u - u_h\|_0 \le C h^s (\|\sigma\|_s + \|u\|_s).$$

However, it is not difficult to see that the error bound for $u - u_h$ in the norm $|\!|\!| \cdot |\!|\!|$ is suboptimal:

$$|\!|\!| u - u_h |\!|\!| \le C h^{s-1} (\|\sigma\|_s + \|u\|_s).$$

This leads us to conclude that the following combination of norms chosen by Braess and Verfürth [7],

$$\|\kappa^{-1/2}(\sigma - \sigma_h)\|_0 + |\!|\!| u - u_h |\!|\!|,$$

is inappropriate as the second component would give an overestimated estimator. As was proposed in [6], this incompatibility can be resolved by employing a higher order scalar approximation computed through a suitable postprocessing scheme. Our choice is the postprocessing scheme due to Stenberg [19] which is general enough to cover all existing mixed finite elements, and moreover, does not require the use of Lagrange multipliers.

Stenberg's Postprocessing Scheme. *Compute $u_h^* \in \mathcal{P}_{k+1}(\mathcal{T}_h)$ on each element $T \in \mathcal{T}_h$ as the solution of the local Neumann problem*

$$\int_T \kappa \nabla u_h^* \cdot \nabla v_h \, d\boldsymbol{x} = \int_T f v_h \, d\boldsymbol{x} + \int_{\partial T} \sigma_h \cdot n_T \, v_h \, ds \qquad \forall v_h \in \mathcal{P}_{k+1}(T), \tag{5.3}$$

$$\int_T u_h^* \, d\boldsymbol{x} = \int_T u_h \, d\boldsymbol{x}. \tag{5.4}$$

Now we apply the results of the previous section to the approximation (σ_h, u_h^*). Since we have

$$\operatorname{div} \sigma_h = -P_h^k f, \qquad \sigma_h \cdot n|_{\Gamma_N} = Q_h^k g_N,$$

it follows that for $f \in H^{k+1}(\mathcal{T}_h)$ and $g_N \in H^{k+1}(\mathcal{E}_N)$,

$$h_T \|\operatorname{div} \sigma_h + f\|_{0,T} \le C h_T^{k+2} \|f\|_{k+1,T} \qquad \forall T \in \mathcal{T}_h,$$
$$h_E^{1/2} \|\sigma_h \cdot n - g_N\|_{0,E} \le C h_E^{k+3/2} \|g_N\|_{k+1,E} \qquad \forall E \in \mathcal{E}_N,$$

which are of higher order than the remaining terms in η. This leads to the following simpler error estimator than η:

$$\hat\eta^2 = \hat\eta_c^2 + \eta_{nc}^2 \qquad \text{where} \quad \hat\eta_c^2 := \sum_{T\in\mathcal{T}_h} \|\kappa^{-1/2}(\sigma_h - \kappa\nabla u_h^*)\|_{0,T}^2. \tag{5.5}$$

The following proposition states that Stenberg's scheme possesses the property of minimizing the component $\hat\eta_c$ up to a higher order term.

Proposition 5.1. *Let $u_h^* \in \mathcal{P}_{k+1}(\mathcal{T}_h)$ be defined by Stenberg's scheme* (5.3)–(5.4). *Then there exists a constant $C > 0$ depending only on the minimum angle of $\{\mathcal{T}_h\}_{h>0}$ such that for all $T \in \mathcal{T}_h$,*

$$\|\kappa^{-1/2}(\sigma_h - \kappa\nabla u_h^*)\|_{0,T} \le \min_{v_h\in\mathcal{P}_{k+1}(T)} \|\kappa^{-1/2}(\sigma_h - \kappa\nabla v_h)\|_{0,T} + C\Lambda_T^{-1/2} h_T \|\operatorname{div}\sigma_h + f\|_{0,T}.$$

Proof. Note that for $v_h \in \mathcal{P}_{k+1}(T)$,

$$\|\kappa^{-1/2}(\sigma_h - \kappa\nabla v_h)\|_{0,T}^2 = \|\kappa^{-1/2}(\sigma_h - \kappa\nabla u_h^*)\|_{0,T}^2 + \|\kappa^{1/2}\nabla(u_h^* - v_h)\|_{0,T}^2 + 2\int_T (\sigma_h - \kappa\nabla u_h^*)\cdot\nabla(u_h^* - v_h)\, d\boldsymbol{x}$$

and

$$\int_T (\sigma_h - \kappa\nabla u_h^*)\cdot\nabla(u_h^* - v_h)\, d\boldsymbol{x} = -\int_T (\operatorname{div}\sigma_h + f)(u_h^* - v_h)\, d\boldsymbol{x}.$$

The latter equality follows directly from the definition of u_h^*. By using (3.4) and then Young's inequality $ab \le \epsilon a^2 + \frac{1}{4\epsilon} b^2$ $(a, b\in\mathbb{R}, \epsilon > 0)$, we obtain

$$\begin{aligned}\int_T (\sigma_h - \kappa\nabla u_h^*)\cdot\nabla(u_h^* - v_h)\, d\boldsymbol{x} &\le C\|\operatorname{div}\sigma_h + f\|_{0,T}\lambda_T^{-1/2} h_T \|\kappa^{1/2}\nabla(u_h^* - v_h)\|_{0,T} \\ &\le C\Lambda_T^{-1} h_T^2 \|\operatorname{div}\sigma_h + f\|_{0,T}^2 + \frac{1}{4}\|\kappa^{1/2}\nabla(u_h^* - v_h)\|_{0,T},\end{aligned}$$

from which the desired result follows immediately. □

Remark 5.2. The above proposition shows that one may define $\hat\eta_c$ equivalently by the minimization quantity

$$\hat\eta_c^2 := \min_{v_h\in\mathcal{P}_{k+1}(\mathcal{T}_h)} \sum_{T\in\mathcal{T}_h} \|\kappa^{-1/2}(\sigma_h - \kappa\nabla v_h)\|_{0,T}^2$$

without losing reliability and efficiency. A similar statement was given in [9], where the estimator involves the minimization

$$\min_{v_h\in\mathcal{P}_k(\mathcal{T}_h)} \sum_{T\in\mathcal{T}_h} h_T^2 \|\kappa^{-1/2}(\sigma_h - \kappa\nabla v_h)\|_{0,T}^2.$$

Remark 5.3. For the lowest order case $k = 0$, it is possible to further simplify $\hat\eta_c$. Suppose, for simplicity, that κ is piecewise constant. It is easy to see that σ_h can be written as

$$\sigma_h|_T = \bar\sigma_h|_T - \frac{\bar f}{2}(\boldsymbol{x} - \boldsymbol{x}_T),$$

where $\bar f = P_h^0 f$, and $\boldsymbol{x}_T$ is the barycenter of T. Taking $v_h = x - x_T, y - y_T$ in (5.3), one can verify that

$$\bar\sigma_h|_T = \kappa\nabla u_h^*|_T - \frac{1}{|T|}\int_T (\operatorname{div}\sigma_h + f)(\boldsymbol{x} - \boldsymbol{x}_T)\, d\boldsymbol{x}.$$

Hence it follows that

$$\|\kappa^{-1/2}(\sigma_h - \kappa\nabla u^*)\|_{0,T} \le \left\|\kappa^{-1/2}\frac{\bar f}{2}(\boldsymbol{x}-\boldsymbol{x}_T)\right\|_{0,T} + \frac{1}{|T|}\left\|\kappa^{-1/2}\int_T (\operatorname{div}\sigma_h + f)(\boldsymbol{x}-\boldsymbol{x}_T)\,d\boldsymbol{x}\right\|_{0,T}.$$

Since the second term can be bounded in a trivial way,

$$\frac{1}{|T|}\left\|\kappa^{-1/2}\int_T (\operatorname{div}\sigma_h + f)(\boldsymbol{x}-\boldsymbol{x}_T)\,d\boldsymbol{x}\right\|_{0,T} \le C\Lambda_T^{-1/2}h_T\|\operatorname{div}\sigma_h + f\|_{0,T},$$

we obtain the estimator

$$\hat\eta^2 = \hat\eta_c^2 + \eta_{nc}^2 \qquad \text{where} \quad \hat\eta_c^2 := \sum_{T\in\mathcal{T}_h}\left\|\kappa^{-1/2}\frac{\bar f}{2}(\boldsymbol{x}-\boldsymbol{x}_T)\right\|_{0,T}^2. \tag{5.6}$$

Note that this $\hat\eta_c$ can be computed in a priori way without knowing (σ_h, u_h^*).

Remark 5.4. No specific choice for W_E was made, and thus all the results obtained above are valid for both the simple and the harmonic averages. We believe, however, that the harmonic average would be the proper choice because we have no explicit control of $[\![u_h^*]\!]$ in the discrete formulation, unlike the interior penalty discontinuous Galerkin method. This point is further illustrated for the $P1$ nonconforming finite element method in the next subsection.

5.2. $P1$ **nonconforming finite element method.** The $P1$ nonconforming space of Crouzeix and Raviart [14] is defined to be

$$\mathcal{P}_1^{nc}(\mathcal{T}_h) := \left\{v_h \in \mathcal{P}_1(\mathcal{T}_h) : \int_E [\![v_h]\!]\,ds = 0 \quad \forall E \in \mathcal{E}_I\right\}.$$

The $P1$ nonconforming finite element method then consists of finding $u_h \in \mathcal{P}_1^{nc}(\mathcal{T}_h)$ such that $Q_h^0 u_h|_{\Gamma_D} = Q_h^0 u_D$ and

$$\sum_{T\in\mathcal{T}_h}\int_T \kappa\nabla u_h\cdot\nabla v_h\,d\boldsymbol{x} = \int_\Omega f v_h\,d\boldsymbol{x} + \int_{\Gamma_N} g_N v_h\,ds \tag{5.7}$$

for all $v_h \in \mathcal{P}_1^{nc}(\mathcal{T}_h)$ with $Q_h^0 v_h|_{\Gamma_D} = 0$.

Recently, it was revealed in [13] that the discretization (5.7) can be interpreted as a mixed finite volume method if we replace f and g_N by $\bar f$ and $Q_h^0 g_N$, respectively. More specifically, the modified method of (5.7)

$$\sum_{T\in\mathcal{T}_h}\int_T \kappa\nabla u_h\cdot\nabla v_h\,d\boldsymbol{x} = \int_\Omega \bar f v_h\,d\boldsymbol{x} + \int_{\Gamma_N} Q_h^0(g_N) v_h\,ds \tag{5.8}$$

along with the explicit formula

$$\sigma_h|_T = \bar\kappa\nabla u_h|_T - \frac{\bar f}{2}(\boldsymbol{x}-\boldsymbol{x}_T) \tag{5.9}$$

is equivalent to the following mixed finite volume method on nonstaggered grids: find $(\sigma_h, u_h) \in \mathcal{RT}_0(\mathcal{T}_h)\times\mathcal{P}_1^{nc}(\mathcal{T}_h)$ such that $\sigma_h\cdot n|_{\Gamma_N} = Q_h^0 g_N$, $Q_h^0 u_h|_{\Gamma_D} = Q_h^0 u_D$, and

$$\int_T(\sigma_h - \kappa\nabla u_h)\,d\boldsymbol{x} = 0, \qquad \int_T(\operatorname{div}\sigma_h + f)\,d\boldsymbol{x} = 0 \tag{5.10}$$

for all $T \in \mathcal{T}_h$.

It is a simple matter to derive the same simple error estimator as obtained for the lowest order Raviart–Thomas mixed finite element method. Suppose again that κ is piecewise constant on $\mathcal{T}_h$. Then it follows obviously from (5.9) that

$$\|\kappa^{-1/2}(\sigma_h - \kappa\nabla u_h)\|_{0,T} = \left\|\kappa^{-1/2}\frac{\bar{f}}{2}(\boldsymbol{x} - \boldsymbol{x}_T)\right\|_{0,T},$$

which gives exactly the simple error estimator (5.6). Moreover, we can even prove that the local contribution $\hat{\eta}_T$ defined by

$$\begin{aligned}(5.11)\quad \hat{\eta}_T^2 := \left\|\kappa^{-1/2}\frac{\bar{f}}{2}(\boldsymbol{x} - \boldsymbol{x}_T)\right\|_{0,T}^2 &+ \frac{1}{2}\sum_{E\in\mathcal{E}_T\cap\mathcal{E}_I} W_E h_E^{-1}\|[\![u_h]\!]\|_{0,E}^2 \\ &+ \sum_{E\in\mathcal{E}_T\cap\mathcal{E}_D} W_E h_E^{-1}\|u_h - u_D\|_{0,E}^2\end{aligned}$$

provides a local lower bound for the energy norm of the scalar error only, if we choose the harmonic average for W_E.

Theorem 5.5. *Suppose that κ is piecewise constant on $\mathcal{T}_h$, and $\hat{\eta}_T$ is given by* (5.11) *with the weight factor*

$$W_E = \frac{2\Lambda_T\Lambda_{T'}}{\Lambda_T + \Lambda_{T'}} \qquad (E \in \mathcal{E}_T \cap \mathcal{E}_{T'}).$$

Then there exists a constant $C > 0$ depending only on the minimum angle of $\{\mathcal{T}_h\}_{h>0}$ such that for all $T \in \mathcal{T}_h$,

$$(5.12)\qquad \hat{\eta}_T \le C\Bigg(\sum_{T'\in\omega_T}\|\kappa^{1/2}\nabla(u-u_h)\|_{0,T'}^2 + \Lambda_T^{-1}h_T^2\|f-\bar{f}\|_{0,T}^2\Bigg)^{1/2},$$

where ω_T denotes the set of all elements sharing an edge with T.

Proof. The first term of $\hat{\eta}_T$ can be treated in a standard way. From the identity

$$\int_T \kappa\nabla(u-u_h)\cdot\nabla v\,d\boldsymbol{x} = \int_T fv\,d\boldsymbol{x} \qquad \forall v \in H_0^1(T),$$

we obtain by setting $v = b_T\bar{f}$

$$\begin{aligned}\int_T b_T|\bar{f}|^2\,d\boldsymbol{x} &= \int_T \kappa\nabla(u-u_h)\cdot\nabla v\,d\boldsymbol{x} - \int_T (f-\bar{f})v\,d\boldsymbol{x} \\ &\le (\|\kappa^{1/2}\nabla(u-u_h)\|_{0,T}\cdot C\Lambda_T^{1/2}h_T^{-1} + \|f-\bar{f}\|_{0,T})\|v\|_{0,T},\end{aligned}$$

which gives by (4.10)

$$\Lambda_T^{-1/2}h_T\|\bar{f}\|_{0,T} \le C(\|\kappa^{1/2}\nabla(u-u_h)\|_{0,T} + \Lambda_T^{-1/2}h_T\|f-\bar{f}\|_{0,T}).$$

For the second term of $\hat{\eta}_T$, we get for $E \in \mathcal{E}_T \cap \mathcal{E}_{T'}$

$$\begin{aligned}W_E h_E^{-1}\|[\![u_h]\!]\|_{0,E}^2 &= W_E h_E^{-1}\|[\![u-u_h]\!] - Q_h^0([\![u-u_h]\!])\|_{0,E}^2 \\ &\le CW_E(\|\nabla(u-u_h)\|_{0,T}^2 + \|\nabla(u-u_h)\|_{0,T'}^2) \\ &\le C(\Lambda_T\|\nabla(u-u_h)\|_{0,T}^2 + \Lambda_{T'}\|\nabla(u-u_h)\|_{0,T'}^2) \\ &\le C(\|\kappa^{1/2}\nabla(u-u_h)\|_{0,T}^2 + \|\kappa^{1/2}\nabla(u-u_h)\|_{0,T'}^2).\end{aligned}$$

The third term of $\hat{\eta}_T$ can be bounded in the same way. The proof is completed by combining all the results above. □

Remark 5.6. An error estimator similar to (5.11) was also derived in [21], where the simple average $W_E = \frac{\Lambda_T + \Lambda_{T'}}{2}$ was chosen. This choice is not a proper one for highly discontinuous coefficients, as we can only show that

$$W_E h_E^{-1} \|[\![u_h]\!]\|_{0,E}^2 \le \max\{\Lambda_T, \Lambda_{T'}\} h_E^{-1} \|[\![u_h]\!]\|_{0,E}^2$$
$$\le C \frac{\max\{\Lambda_T, \Lambda_{T'}\}}{\min\{\Lambda_T, \Lambda_{T'}\}} (\|\kappa^{1/2} \nabla (u - u_h)\|_{0,T}^2 + \|\kappa^{1/2} \nabla (u - u_h)\|_{0,T'}^2).$$

This bound is sharp, as illustrated by a simple example in the appendix.

5.3. **Interior penalty discontinuous Galerkin method.** The discontinuous Galerkin method is a class of numerical methods based on discontinuous finite elements for the trial and the test spaces. Very often, the interior penalty terms are added in order to take explicit control of the jumps of the discontinuous solution.

The interior penalty discontinuous Galerkin method considered in this subsection is a slight modification of the usual formulation which is presented as follows: find $u_h \in \mathcal{P}_{k+1}(\mathcal{T}_h)$ such that for all $v_h \in \mathcal{P}_{k+1}(\mathcal{T}_h)$,

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} \int_T \kappa \nabla u_h \cdot \nabla v_h \, d\boldsymbol{x} &- \sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D} \int_E \{\kappa \nabla u_h \cdot n_E\} Q_h^k([\![v_h]\!]) \, ds \\ &- \alpha \sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D} \int_E \{\kappa \nabla v_h \cdot n_E\} Q_h^k([\![u_h]\!]) \, ds \\ &+ \sum_{E \in \mathcal{E}_I \cup \mathcal{E}_D} \gamma W_E h_E^{-1} \int_E Q_h^k([\![u_h]\!]) Q_h^k([\![v_h]\!]) \, ds \\ = \int_\Omega f v_h \, d\boldsymbol{x} + \int_{\Gamma_N} g_N v_h \, ds &- \alpha \sum_{E \in \mathcal{E}_D} \int_E \kappa \nabla v_h \cdot n_E \, Q_h^k(u_D) \, ds \\ &+ \sum_{E \in \mathcal{E}_D} \gamma W_E h_E^{-1} \int_E Q_h^k(u_D) Q_h^k(v_h) \, ds. \end{aligned} \tag{5.13}$$

Here $\alpha = -1, 0$, or 1, and $\gamma > 0$ is a stabilization parameter to ensure the coercivity of the discrete system (5.13). Note that the jump terms are projected by the L^2-projection Q_h^k onto the finite element space $\mathcal{P}_k(\mathcal{E}_h)$. It can be shown that this does not affect the optimal order of convergence in the energy norm (cf. [16]).

The choice $\alpha = 1$ corresponds to the symmetric formulation studied in [3], while the choice $\alpha = -1$ was introduced and analyzed in [17]. The choice $\alpha = 0$ is less well understood, but it was shown in [16] that this choice is closely related to mixed finite volume methods on nonstaggered grids, especially when the data f and g_N are projected onto suitable finite element spaces.

One can easily see that for $\alpha = -1$, the method (5.13) yields a coercive system for all $\gamma > 0$ and $W_E > 0$. For $\alpha = 0$ or 1, we need to choose γ sufficiently large, and moreover, take W_E to be the simple average in order to ensure the coercivity of (5.13). This is shown in the following proposition.

Proposition 5.7. *For $\alpha = 0$ or 1, let W_E be given by the simple average*

$$W_E = \frac{\Lambda_T + \Lambda_{T'}}{2} \qquad (E \in \mathcal{E}_T \cap \mathcal{E}_{T'}).$$

Then there exists a constant $C > 0$ depending only on the minimum angle of $\{\mathcal{T}_h\}_{h>0}$ and the polynomial degree k such that the method (5.13) yields a coercive system for all $\gamma > C$.

Proof. It suffices to prove that for all $v_h \in \mathcal{P}_{k+1}(\mathcal{T}_h)$,

$$(1+\alpha) \sum_{E\in\mathcal{E}_I\cup\mathcal{E}_D} \int_E \{\kappa\nabla v_h \cdot n_E\} Q_h^k([\![v_h]\!])\, ds \le \frac{1}{2} \sum_{T\in\mathcal{T}_h} \|\kappa^{1/2}\nabla v_h\|_{0,T}^2 + C \sum_{E\in\mathcal{E}_I\cup\mathcal{E}_D} W_E h_E^{-1} \|[\![v_h]\!]\|_{0,E}^2.$$

For an interior edge $E \in \mathcal{E}_T \cap \mathcal{E}_{T'}$, we get

$$\begin{aligned}\int_E \{\kappa\nabla v_h \cdot n_E\} Q_h^k([\![v_h]\!])\, ds &\le \frac{1}{2}(\Lambda_T \|\nabla v_h \cdot n|_T\|_{0,E} + \Lambda_{T'} \|\nabla v_h \cdot n|_{T'}\|_{0,E}) \|[\![v_h]\!]\|_{0,E} \\ &\le C(\Lambda_T \|\nabla v_h\|_{0,T} + \Lambda_{T'} \|\nabla v_h\|_{0,T'}) h_E^{-1/2} \|[\![v_h]\!]\|_{0,E} \\ &\le \epsilon(\|\kappa^{1/2}\nabla v_h\|_{0,T}^2 + \|\kappa^{1/2}\nabla v_h\|_{0,T'}^2) + C W_E h_E^{-1} \|[\![v_h]\!]\|_{0,E}^2,\end{aligned}$$

where we used the inverse inequality

$$\|w_h\|_{0,\partial T} \le C h_T^{-1/2} \|w_h\|_{0,T} \qquad \forall w_h \in \mathcal{P}_k(T)$$

(with C depending on the polynomial degree k) and then Young's inequality. Similarly, for a boundary edge $E \in \mathcal{E}_D \cap \mathcal{E}_T$, we get

$$\int_E \kappa\nabla v_h \cdot n_E\, Q_h^k(v_h)\, ds \le \epsilon \|\kappa^{1/2}\nabla v_h\|_{0,T}^2 + C W_E h_E^{-1} \|v_h\|_{0,E}^2.$$

Summing over all $E \in \mathcal{E}_I \cup \mathcal{E}_D$ and taking $\epsilon = \frac{1}{6}$, we obtain the desired result. □

A locally conservative vector approximation $\sigma_h \in \mathcal{RT}_k(\mathcal{T}_h)$ can be constructed on each element $T \in \mathcal{T}_h$ from the scalar solution u_h by specifying its degrees of freedom as follows: (see, e.g., [4])

$$\sigma_h \cdot n_E|_E = \begin{cases} Q_h^k\big(\{\kappa\nabla u_h \cdot n_E\} - \gamma W_E h_E^{-1} [\![u_h]\!]\big) & \text{for } E \in \mathcal{E}_I, \\ Q_h^k\big(\kappa\nabla u_h \cdot n_E - \gamma W_E h_E^{-1}(u_h - u_D)\big) & \text{for } E \in \mathcal{E}_D, \\ Q_h^k g_N & \text{for } E \in \mathcal{E}_N, \end{cases} \tag{5.14}$$

and for $k \ge 1$

$$\int_T \sigma_h \cdot \tau\, d\boldsymbol{x} = \int_T \kappa\nabla u_h \cdot \tau\, d\boldsymbol{x} \qquad \forall \tau \in (\mathcal{P}_{k-1}(T))^2. \tag{5.15}$$

It is clear that this construction can be done locally and that σ_h satisfies the local conservation law (3.4). One remarkable thing is that, unlike the mixed finite element method, the second and third terms of η_c

$$\Big(\sum_{T\in\mathcal{T}_h} \Lambda_T^{-1} h_T^2 \|\operatorname{div}\sigma_h + f\|_{0,T}^2\Big)^{1/2} \quad \text{and} \quad \Big(\sum_{E\in\mathcal{E}_N} W_E^{-1} h_E \|\sigma_h \cdot n - g_N\|_{0,E}^2\Big)^{1/2}$$

are not higher order ones, and thus cannot be neglected, except for $k = 0$.

Let us now restrict ourselves to the lowest order $k = 0$, and suppose again that κ is piecewise constant on $\mathcal{T}_h$. In this case $\sigma_h \in \mathcal{RT}_0(\mathcal{T}_h)$ is determined by

$$\sigma_h \cdot n_E|_E = \begin{cases} \{\kappa\nabla u_h \cdot n_E\} - \gamma W_E h_E^{-1} Q_h^0([\![u_h]\!]) & \text{for } E \in \mathcal{E}_I, \\ \kappa\nabla u_h \cdot n_E - \gamma W_E h_E^{-1} Q_h^0(u_h - u_D) & \text{for } E \in \mathcal{E}_D, \\ Q_h^0 g_N & \text{for } E \in \mathcal{E}_N, \end{cases} \tag{5.16}$$

where we used the fact that $\kappa\nabla u_h$ is piecewise constant. As previously, we can write σ_h in the form

$$\sigma_h|_T = \bar{\sigma}_h|_T - \frac{\bar{f}}{2}(\boldsymbol{x} - \boldsymbol{x}_T).$$

To determine the constant part $\bar{\sigma}_h|_T$, we note that for all $v_h \in \mathcal{P}_1(T)$,

$$\begin{aligned}\int_T (\kappa\nabla u_h - \sigma_h)\cdot\nabla v_h\,d\boldsymbol{x} = \int_T (f + \operatorname{div}\sigma_h) v_h\,d\boldsymbol{x} + \int_{\partial T\cap\Gamma_N} (g_N - \sigma_h\cdot n) v_h\,ds \\ + \alpha \sum_{E\in\mathcal{E}_T\cap\mathcal{E}_I} \int_E \frac{1}{2}\kappa\nabla v_h\cdot n_T Q_h^0([\![u_h]\!])\,ds \\ + \alpha \sum_{E\in\mathcal{E}_T\cap\mathcal{E}_D} \int_E \kappa\nabla v_h\cdot n_T Q_h^0(u_h - u_D)\,ds.\end{aligned}$$

By taking $v_h = x - x_T, y - y_T$, we can obtain

$$\bar{\sigma}_h|_T = \kappa\nabla u_h|_T - (I_1 + I_2),$$

where

$$\begin{aligned} I_1 &= \frac{1}{|T|}\int_T (f + \operatorname{div}\sigma_h)(\boldsymbol{x} - \boldsymbol{x}_T)\,d\boldsymbol{x} + \frac{1}{|T|}\int_{\partial T\cap\Gamma_N}(g_N - \sigma_h\cdot n)(\boldsymbol{x} - \boldsymbol{x}_T)\,ds, \\ I_2 &= \frac{\alpha}{|T|}\sum_{E\in\mathcal{E}_T\cap\mathcal{E}_I}\int_E \frac{1}{2}\kappa n_T Q_h^0([\![u_h]\!])\,ds + \frac{\alpha}{|T|}\sum_{E\in\mathcal{E}_T\cap\mathcal{E}_D}\int_E \kappa n_T Q_h^0(u_h - u_D)\,ds.\end{aligned}$$

These two terms can be bound in a straightforward way:

$$I_1 \le C|T|^{-1/2}(h_T\|\operatorname{div}\sigma_h + f\|_{0,T} + h_T^{1/2}\|\sigma_h\cdot n - g_N\|_{0,\partial T\cap\Gamma_N})$$

and

$$I_2 \le C|T|^{-1/2}\Lambda_T\Bigg(\sum_{E\in\mathcal{E}_T\cap\mathcal{E}_I} h_E^{-1}\|[\![u_h]\!]\|_{0,E}^2 + \sum_{E\in\mathcal{E}_T\cap\mathcal{E}_D} h_E^{-1}\|u_h - u_D\|_{0,E}^2\Bigg)^{1/2}.$$

Therefore it follows that

$$\begin{aligned}\|\kappa^{-1/2}(\sigma_h - \kappa\nabla u_h)\|_{0,T} \le C(\hat{\eta}_T + \Lambda_T^{-1/2} h_T\|\operatorname{div}\sigma_h + f\|_{0,T} \\ + \Lambda_T^{-1/2} h_T^{1/2}\|\sigma_h\cdot n - g_N\|_{0,\partial T\cap\Gamma_N}),\end{aligned}$$

where $\hat{\eta}_T$ was defined by (5.11). Since the second and third terms are of higher order for $f \in H^1(\mathcal{T}_h)$ and $g \in H^1(\mathcal{E}_N)$, we conclude that the estimator $\hat{\eta}$ given by (5.6) or (5.11) is reliable for $k = 0$. The efficiency of $\hat{\eta}$ for the scalar error only can be established in a similar way to the $P1$ nonconforming finite element method:

$$\hat{\eta} \le C\Bigg(|\!|\!|u - u_h|\!|\!|^2 + \sum_{T\in\mathcal{T}_h}\Lambda_T^{-1}h_T^2\|f - \bar{f}\|_{0,T}^2\Bigg)^{1/2}.$$

Remark 5.8. When compared with the result of [15], the above result shows that for the lowest order $k = 0$, the contributions from the jumps of normal fluxes are redundant, and thus can be excluded.

6. Numerical results

In this section we present some numerical results to illustrate the performance of our error estimator. The test problems are posed on the domain $\Omega = (-1,1)^2$ with the piecewise constant coefficient $\kappa = R$ in the first and third quadrants and $\kappa = 1$ in the second and fourth quadrants for different values of $R > 1$. Note that this choice of κ does not fulfill the Main Assumption of Section 3. Indeed, it is found in the second example that the efficiency index tends to deteriorate for large values of R. Nevertheless, the first example indicates that we can get robust results under certain circumstances, e.g., for regular solutions.

Numerical experiments were carried out for all the three methods analyzed in Section 5, namely, the lowest order Raviart–Thomas mixed finite element method, the $P1$ nonconforming finite element method, and the lowest order interior penalty discontinuous Galerkin method with the parameters $\alpha = 1$ and $\gamma = 5$. The results are, however, presented for the interior penalty discontinuous Galerkin method only, as very similar behaviors were observed for the other two methods.

In each experiment below we report the scalar error measured in the broken H^1-norm given by

$$|u - u_h|_{1,h,\kappa} = \left(\sum_{T \in \mathcal{T}_h} \|\kappa^{1/2} \nabla (u - u_h)\|_{0,T}^2 \right)^{1/2}$$

and the error estimators η_{simple} or $\eta_{\text{harmonic}} = \left(\sum_{T\in\mathcal{T}_h} \hat{\eta}_T^2\right)^{1/2}$ whose local contribution $\hat{\eta}_T$ is given by (5.11) with the simple or harmonic average for W_E.

It should be noted that η_{simple} and η_{harmonic} differ significantly when the jump of the coefficient κ is large. This is the reason why we did not include the jump terms in the scalar error, as these terms (with the simple average for W_E) tend to be dominant over the gradient error in such cases; see the results in the following

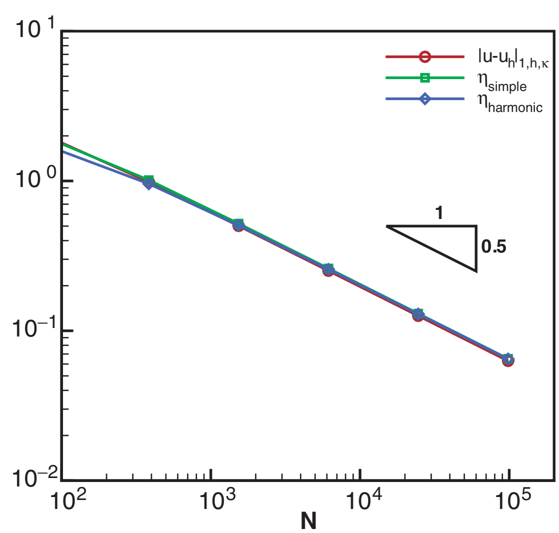

FIGURE 1. Scalar H^1-error and error estimators on uniform meshes for Example 6.1 with $R = 5$

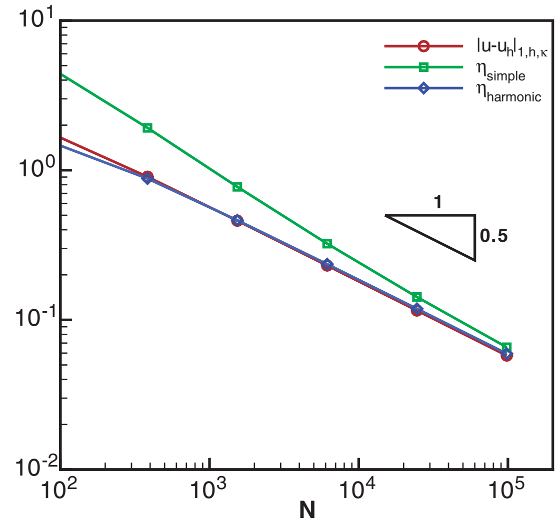

FIGURE 2. Scalar H^1-error and error estimators on uniform meshes for Example 6.1 with $R = 100$

examples. Moreover, the jump terms in the scalar error are unnecessary for the $P1$ nonconforming method and for the mixed finite element methods as well in the sense that we are more interested in the vector error (which was found to be slightly better than the scalar error in our experiments).

Example 6.1. In the first test problem we choose the solution

$$u(x, y) = \frac{1}{\kappa(x, y)} \sin(\pi x) \sin(\pi y)$$

with the corresponding right-hand side given by $f(x, y) = 2\pi^2 \sin(\pi x) \sin(\pi y)$ and the homogeneous Dirichlet boundary condition imposed on $\partial\Omega$. Note that u and $\kappa\nabla \cdot n$ are continuous across the lines of discontinuity of κ.

Since u is piecewise regular, we only consider a sequence of uniformly refined triangular meshes starting from the initial mesh composed of eight triangles generated by partitioning Ω into four equal squares and then dividing them by diagonals of slope one.

The numerical results are plotted in Figures 1–3 for $R = 5, 100$ and 10000, respectively, which display the optimal convergence behavior of the scalar errors and the error estimators as functions of N, where N represents the total number of degrees of freedom. The scalar errors are computed by using a high order Gaussian quadrature on each element. It is clearly observed that η_{harmonic} performs very well independently of the values of R (the efficiency index remaining close to one), while the performance of η_{simple} tends to deteriorate when R gets large. We remark that exactly the opposite result would be obtained if the jump terms with the simple average for W_E are included in the scalar error (as is typically done in a priori error analysis for discontinuous Galerkin methods).

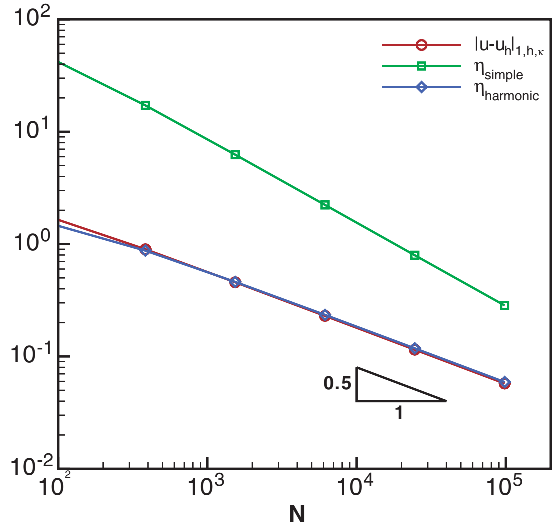

FIGURE 3. Scalar H^1-error and error estimators on uniform meshes for Example 6.1 with $R = 10000$

Example 6.2. The second test problem is the one from [11]. The exact solution for $f = 0$ is given in polar coordinates by $u(r, \theta) = r^\beta \mu(\theta)$, where

$$\mu(\theta) = \begin{cases} \cos((\pi/2 - \sigma)\beta) \cdot \cos((\theta - \pi/2 + \rho)\beta) & \text{if} \quad 0 \le \theta \le \pi/2, \\ \cos(\rho\beta) \cdot \cos((\theta - \pi + \sigma)\beta) & \text{if} \quad \pi/2 \le \theta \le \pi, \\ \cos(\sigma\beta) \cdot \cos((\theta - \pi - \rho)\beta) & \text{if} \quad \pi \le \theta \le 3\pi/2, \\ \cos((\pi/2 - \rho)\beta) \cdot \cos((\theta - 3\pi/2 - \sigma)\beta) & \text{if} \quad 3\pi/2 \le \theta \le 2\pi, \end{cases}$$

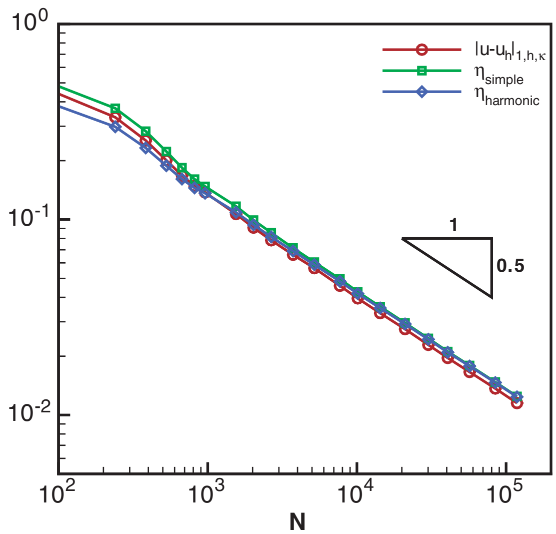

FIGURE 4. Scalar H^1-error and error estimators on adaptive meshes for Example 6.2 with $\beta = 0.5$ and $R \approx 5.83$

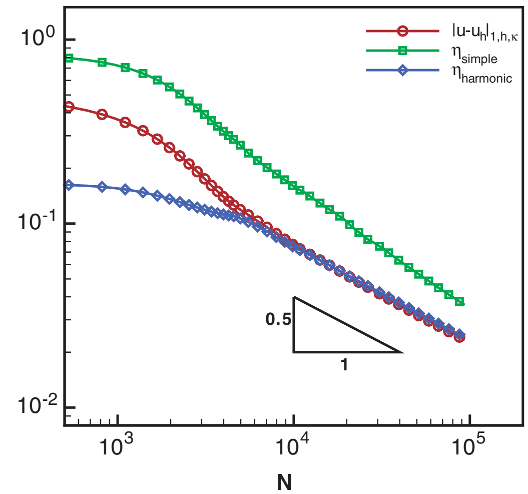

FIGURE 5. Scalar H^1-error and error estimators on adaptive meshes for Example 6.2 with $\beta = 0.1$ and $R \approx 161.45$

and the numbers β, R, ρ, σ satisfy the nonlinear relations

$$\begin{cases} R = -\tan((\pi/2-\sigma)\beta)\cdot\cot(\rho\beta), \\ 1/R = -\tan(\rho\beta)\cdot\cot(\sigma\beta), \\ R = -\tan(\sigma\beta)\cdot\cot((\pi/2-\rho)\beta), \\ 0 < \beta < 2, \\ \max(0, \pi\beta-\pi) < 2\beta\rho < \min(\pi\beta, \pi), \\ \max(0, \pi-\pi\beta) < -2\beta\sigma < \min(\pi, 2\pi-\pi\beta). \end{cases}$$

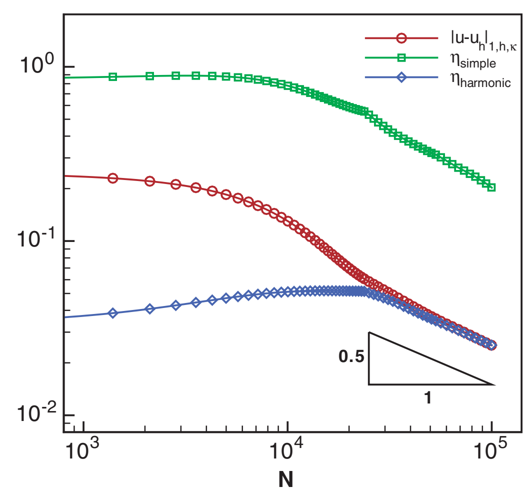

FIGURE 6. Scalar H^1-error and error estimators on adaptive meshes for Example 6.2 with $\beta = 0.02$ and $R \approx 4052.18$

We consider the three values $\beta = 0.5, 0.1$, and 0.02, respectively, which produce the following set of parameters R, ρ, σ:

$$\beta = 0.5, \quad R \approx 5.8284271247461907, \quad \rho = \pi/4, \quad \sigma \approx -2.3561944901923448,$$
$$\beta = 0.1, \quad R \approx 161.4476387975881, \quad \rho = \pi/4, \quad \sigma \approx -14.92256510455152,$$
$$\beta = 0.02, \quad R \approx 4052.1806954768103, \quad \rho = \pi/4, \quad \sigma \approx -77.754418176347386.$$

Since u belongs to $H^{1+\delta}(\Omega)$ for $\delta < \beta$ and thus is very singular for small values of β, we perform adaptive mesh refinement based on the error estimator η_{harmonic} by following the simple maximum strategy (cf. [20]): mark the element T for refinement if

$$\eta_T \geq \frac{1}{2} \max_{T' \in \mathcal{T}_h} \eta_{T'},$$

and further refine adjacent elements to avoid hanging nodes.

Numerical results are reported in Figures 4–6. The scalar errors are computed by applying high order Gaussian quadratures to the formula

$$|u - u_h|^2_{1,h,\kappa} = \int_{\partial\Omega} \kappa \nabla u \cdot n\, u\, ds - 2 \sum_{T \in \mathcal{T}_h} \int_{\partial T} \kappa \nabla u_h \cdot n\, u\, ds + \sum_{T \in \mathcal{T}_h} \|\kappa^{1/2} \nabla u_h\|^2_{0,T},$$

which is easily obtained via integration by parts. We see again that the performance of η_{simple} tends to deteriorate for large R or small β. The same phenomenon is observed for η_{harmonic} on coarse meshes with approximately $N \lesssim 10^3$ for $\beta = 0.1$ and $N \lesssim 10^4$ for $\beta = 0.02$, although the efficiency index gets close to one as the strong singularity at the origin is resolved by means of adaptive mesh refinement. In summary, we can still say that the overall performance of η_{harmonic} is better than that of η_{simple} and the optimality of adaptive mesh refinement based on η_{harmonic} is valid asymptotically.

Figure 7 presents an adaptive mesh for $\beta = 0.1$ which contains $N/3 = 31172$ elements. As expected, it is highly refined around the singularity at the origin, and the smallest mesh size is found to be of the order of 10^{-26}.

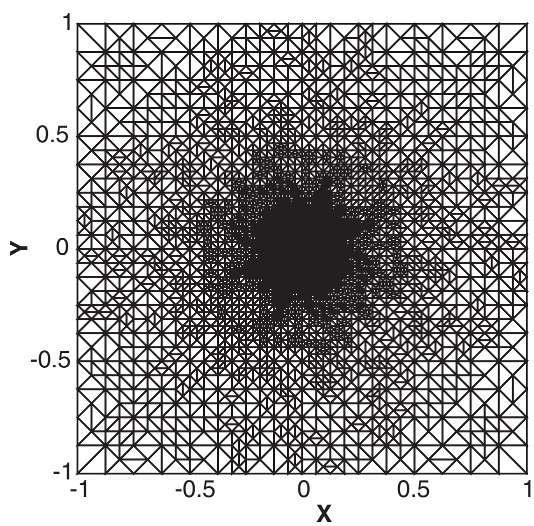

FIGURE 7. Adaptive mesh of $N = 93516$ for Example 6.2 with $\beta = 0.1$. The smallest mesh size is of the order of 10^{-26}.

7. Appendix

The purpose of this appendix is to illustrate by a simple example that the proper weight factor W_E for the $P1$ nonconforming finite element method is given by the harmonic average (see Theorem 5.5).

We consider the problem

$$\operatorname{div}(\kappa\nabla u) = 0 \quad \text{in } (0,1)^2$$

with the discontinuous coefficient ($\alpha > 0$)

$$\kappa = \begin{cases} 1 & \text{on } T_1, \\ \alpha^{-1} & \text{on } T_2, \end{cases}$$

and the Dirichlet boundary condition given by the true solution

$$u(x,y) = \begin{cases} 3(x^2-y^2) & \text{on } T_1, \\ 3\alpha(x^2-y^2) & \text{on } T_2. \end{cases}$$

The triangulation consists of two elements, T_1 and T_2, as shown in Figure 8. Note that both u and $\kappa\nabla u \cdot n_E$ are continuous across the interface $E = \partial T_1 \cap \partial T_2$.

Let ϕ_i be the $P1$ nonconforming basis function associated with the node $\boldsymbol{x}_i$. Then it is rather straightforward to derive that the $P1$ nonconforming finite element solution u_h is given by

$$u_h = \phi_1 + 2\phi_2 - 2\alpha\phi_3 - \alpha\phi_4.$$

This yields, by simple calculation,

$$h_E^{-1}\|[\![u_h]\!]\|_{0,E}^2 = \frac{1}{3}(1+\alpha)^2$$

and

$$\|\nabla(u-u_h)\|_{0,T_1}^2 + \alpha^{-1}\|\nabla(u-u_h)\|_{0,T_2}^2 = 2(1+\alpha).$$

Therefore we conclude that the harmonic average

$$W_E = \frac{2\alpha^{-1}}{1+\alpha^{-1}} = \frac{2}{1+\alpha}$$

gives a correct weight factor independently of the value of α.

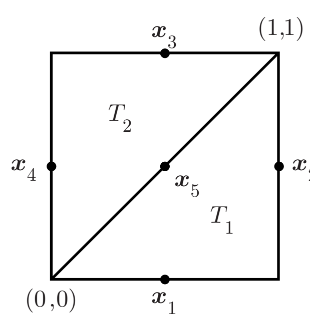

Figure 8. Domain geometry and triangulation

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