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# ON THE CONVERGENCE OF RATIONAL APPROXIMATIONS OF SEMIGROUPS ON INTERMEDIATE SPACES

MIHÁLY KOVÁCS

Abstract. We generalize a result by Brenner and Thomée on the rate of convergence of rational approximation schemes for semigroups. Using abstract interpolation techniques we obtain convergence on a continuum of intermediate spaces between the Banach space $X$ and the domain of a certain power of the generator of the semigroup. The sharpness of the results is also discussed.

## 1. Introduction

At the core of this paper is the study of time-discretization methods for differential equations $\dot{u}(t) = Au(t)$, where $A : X \supset \mathcal{D}(A) \to X$ is a linear operator with domain $\mathcal{D}(A)$ in a Banach space $X$. Throughout the paper it is assumed that $A$ generates a strongly continuous semigroup ($C_0$-semigroup) $T(\cdot)$ of type $M$; that is, there exists $M \geq 1$ such that $||T(t)|| \leq M$ for all $t \geq 0$.[1] Many of the basic methods used to analyse time-discretization schemes in a Banach-space setting go back to Lax and Richtmyer [18] (see also [17], [21]). Often, the semigroup $T(t)$ is approximated by a product of operators $\prod_{i=1}^{n} r(\tau_i A)$, $\sum_{i=1}^{n} \tau_i = t$, where $r$ is a *rational approximation method of order* $q \geq 1$; that is, $r$ *approximates the exponential function to order* $q \geq 1$; that is, $r(z) = e^z + O(z^{q+1})$ as $z \to 0$ and $r$ is *A-stable*; that is, $|r(z)| \leq 1$ for $\operatorname{Re} z \leq 0$. For example, the Backward Euler approximation method $r_{BE}(z) = \frac{1}{1-z}$ is of order 1 and, for every $C_0$-semigroup $T(\cdot)$, $T(t)x = \lim_{n\to\infty} r_{BE}(\frac{t}{n}A)^n x = \lim_{n\to\infty}(I - \frac{t}{n}A)^{-n}x$ for all $x \in X$. The basis for our investigations are the papers by Hersh and Kato [11] as well as by Brenner and Thomée [3], where the following result is proved.

**Theorem 1.1.** *If $r$ is a rational approximation method of order $q \geq 1$, then there is a constant $K > 0$ such that for $k = 0, 1, \dots, q+1$, $k \neq \frac{q+1}{2}$, we have*[2]

$$||r^n(\tfrac{t}{n}A)x - T(t)x|| \leq KMt^{k-\eta_q(k)}\left(\tfrac{t}{n}\right)^{\eta_q(k)}||A^k x||,\ t \geq 0,\ n \in \mathbb{N},\ x \in \mathcal{D}(A^k),$$

*where $\eta_q(k)$ is defined as*

$$\eta_q(k) := \begin{cases} k - \frac{1}{2} & \text{if } 0 \leq k < \frac{q+1}{2}, \\ k\frac{q}{q+1} & \text{if } \frac{q+1}{2} \leq k \leq q+1. \end{cases}$$

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[1]For simplicity we consider bounded semigroups only. The general situation can be handled by employing an appropriate shifting procedure in the proof of the bounded case (see [3]).

[2]Throughout the paper we denote the set $\{1, 2, 3, \dots\}$ by $\mathbb{N}$.

*If* $k = \frac{q+1}{2}$, *then* $||r^n(\frac{t}{n}A)x - T(t)x|| \leq KMt^{k-\eta_q(k)}\left(\frac{t}{n}\right)^{\eta_q(k)} \ln(n+1)||A^k x||$ *for all* $n \in \mathbb{N}$ *and* $t \geq 0$. $\square$

Since $\eta_q(0) = -\frac{1}{2}$, Theorem 1.1 suggests that for general $x \in X$ one cannot expect convergence (rather a growth proportional to $\sqrt{n}$) and that for $x \in \mathcal{D}(A^{q+1})$ the order of convergence is optimal (and proportional to $\frac{1}{n^q}$ since $\eta_q(q+1) = q$). In [2] and [4, Chapter 5] it is shown that for $k \neq \frac{q+1}{2}$ the above rates are sharp for the left-translation semigroup on $L_1(\mathbb{R})$ which means that the convergence rates in Theorem 1.1 cannot be improved in general. However, the set of initial data that corresponds to a certain speed of convergence in Theorem 1.1 is not optimal. In Theorem 3.2 we will show that the estimates in Theorem 1.1 remain valid if the initial data is taken from the Favard space of order $k$ instead of $\mathcal{D}(A^k)$ and if $||A^k x||$ is replaced by the appropriate Favard norm of $x$. If $X$ is not reflexive, then the Favard spaces are usually significantly larger than $\mathcal{D}(A^k)$.

As proposed in [3] and [11], we use the Hille–Phillips (H-P) functional calculus in our analysis (for the original approach to the H-P functional calculus via regular Borel measures, see [12]; for the reformulation in terms of functions of bounded variation, see [14]). We recall the following basic facts about the H-P functional calculus.[3] A function $\alpha : [0, R] \to \mathbb{C}$ is in $NBV[0, R]$ if it is of *bounded variation* and *normalized*; i.e., $\alpha(0) = 0$, and $\alpha(u) = \frac{\alpha(u+)+\alpha(u-)}{2}$ for all $u \in (0, R)$. The space $NBV := \{\alpha \in \bigcap_{R>0} NBV[0, R] : \ \sup_{t>0} V_\alpha(t) < +\infty\}$ is a commutative Banach algebra with multiplication defined by the *Stieltjes convolution* $(\alpha * \beta)(t) := \int_0^t \alpha(t-u)\, d\beta(u)$ and norm $||\alpha||_{TV} := \sup_{t>0} V_\alpha(t)$, where $V_\alpha(t)$ denotes the total variation of $\alpha$ on $[0, t]$. Let $\mathbb{C}_0 := \{z \in \mathbb{C} : \operatorname{Re} z \leq 0\}$ and $\mathcal{G} := \{f_\alpha : \ f_\alpha(z) = \int_0^\infty e^{zt}\, d\alpha(t), \ z \in \mathbb{C}_0, \ \alpha \in NBV\}$. Then the operator $\Phi : NBV \to \mathcal{G}$ defined by $\Phi(\alpha) := f_\alpha$ is an algebra isomorphism, and if we set $||f_\alpha|| := ||\alpha||_{TV}$, then $\mathcal{G}$ becomes a Banach algebra. If a rational function $r$ satisfies $|r(z)| \leq M$ $(z \in \mathbb{C}_0)$ for some $M > 0$ (in particular, if $r$ is A-stable), then $r \in \mathcal{G}$. Indeed, constant functions and functions $z \to \frac{1}{a-z}$ belong to the algebra $\mathcal{G}$ for $\operatorname{Re} a > 0$. By developing $r$ into partial fractions, we see that $r \in \mathcal{G}$. Another important example of a function in $\mathcal{G}$ is $z \mapsto e^{zt}$ for fixed $t \geq 0$ since $e^{zt} = \int_0^\infty e^{zs}\, dH_t(s)$, where the normalized Heaviside function $H_t$ for $t > 0$ is defined as

$$H_t(s) := \begin{cases} 0 & \text{if } 0 \leq s < t, \\ \frac{1}{2} & \text{if } s = t, \\ 1 & \text{if } t < s, \end{cases}$$

and, similarly, $H_0(s) := 0$ for $s = 0$ and $H_0(s) := 1$ for $s > 0$. Now let $A$ generate a $C_0$-semigroup $T(\cdot)$ of type $M$ on a Banach space $X$. For $f \in \mathcal{G}$ with $f(z) := \int_0^\infty e^{zt}\ d\alpha(t)$ $(z \in \mathbb{C}_0)$, let $f(A)x := \int_0^\infty T(t)x\, d\alpha(t)$. Then the map $\Psi : \mathcal{G} \to \mathcal{B}(X)$ defined by $\Psi(f) := f(A)$ is an algebra homomorphism and $||f(A)|| \leq M||\alpha||_{TV}$. This is, in essence, the H-P functional calculus.

In addition to the H-P functional calculus, the proof of the error estimates for time-discretization schemes on integer order Favard spaces given in Section 3 relies on the study of the following Laplace-Stieltjes transform problem which is discussed in [16] and [15, Chapter 2].

[3] We recall the H-P functional calculus for bounded semigroups only, since the general version of it is not needed in this paper.

Assume that the functions $r_n(z) := \int_0^\infty e^{zs}\, d\alpha_n(s)$, $\alpha_n \in NBV$, converge pointwise to $v(z) := \int_0^\infty e^{zs}\, d\alpha(s)$, $\alpha \in NBV$, as $n \to \infty$. Does this imply the convergence of $\alpha_n$ to $\alpha$ and, if yes, in what sense? If we assume that the speed of convergence of $r_n$ to $v$ is known pointwise, can we say something about the speed of convergence of $\alpha_n$ to $\alpha$ in various norms? Answers to these questions are given in Sections 2 and 3 for cases in which $r_n(z) := r^n(\frac{z}{n})$ and $r$ is a rational approximation method of order $q \geq 1$.

In Section 4, the Brenner–Thomée estimates are extended to standard intermediate spaces between $X$ and $\mathcal{D}(A^{q+1})$. Using abstract interpolation techniques, optimal order of convergence is obtained for almost all Favard spaces $\mathcal{F}_\alpha$, $0 \leq \alpha \leq q+1$. A similar procedure was carried out in [2] and [4] for interpolation spaces based on $L_p(\mathbb{R})$. The latter studies inspired us to use the $K$-method when constructing the various intermediate spaces, since when $X = L_p(\mathbb{R})$ a family of them coincides with the appropriate Besov spaces considered in [2] and [4]. In Corollary 4.3 we prove a new stability result for the discrete orbits $r^n(\frac{t}{n}A)x$ if $x$ is taken from an intermediate space of order $\frac{1}{2}$ between $X$ and $\mathcal{D}(A)$. We also show how interpolation results can be applied to obtain optimal error estimates for stable schemes. In the latter case our result improves the estimate in [3, Thm. 4] on $\mathcal{D}(A^{\frac{q+1}{2}})$ and generalizes [9, Thm. 1.7]. The sharpness of the estimates is discussed at the end of Section 4. Finally, we mention that in a recent paper by Hausenblas [10] interpolation theory and intermediate spaces are used to obtain various convergence results for spatial discretizations of anlytic semigroups.

## 2. Stability

In this section we discuss a basic stability result due to Brenner and Thomée [3]. For a proof of the next theorem, see [15, Thms. 2.2.2, 2.2.5] or [16].

**Theorem 2.1.** *Let $r(z) := \int_0^\infty e^{zs}\, d\alpha(s),\ z \in \mathbb{C}_0$, be an A-stable rational function with $\alpha \in NBV$. Then there is a constant $K > 0$ such that $||\alpha^{n*}||_{TV} \leq K\sqrt{n}$ for all $n \in \mathbb{N}$, where $\alpha^{n*}$ denotes the $n$-times Stieltjes convolution of $\alpha$ with itself. Moreover, if $|r(is)| = 1$ for all $s \in \mathbb{R}$, then there is a constant $L > 0$ such that $||\alpha^{n*}||_{TV} \geq L\sqrt{n}$ for all $n \in \mathbb{N}$.* □

The first statement of the following theorem, due to Brenner and Thomée [3, Thm. 1], is an immediate consequence of Theorem 2.1 (for the variable step-size version, see [1]). The second statement is proved in [7] using Fourier multipliers. Here, a simple and more elementary proof is presented. In the following we denote by $C(\mathbb{R})$, $(C_0(\mathbb{R}),\ C_b(\mathbb{R}))$ the space of all continuous (continuous vanishing at $\pm\infty$, continuous and bounded) functions $f:\ \mathbb{R} \to \mathbb{C}$.

**Theorem 2.2.** *If $r$ is an A-stable rational function, then there is a constant $K > 0$ such that for all $\tau > 0$ and $n \in \mathbb{N}$ we have $||r^n(\tau A)|| \leq KM\sqrt{n}$. If $A$ is the derivative operator on $C_0(\mathbb{R})$ with maximal domain and $|r(is)| = 1$ for all $s \in \mathbb{R}$, then there is a constant $L > 0$ such that $||r^n(\tau A)|| \geq L\sqrt{n}$ for all $\tau > 0$ and $n \in \mathbb{N}$.*

*Proof.* Since $r$ is A-stable it follows that $r(z) = \int_0^\infty e^{zs} d\alpha(s),\ z \in \mathbb{C}_0$, for some $\alpha \in NBV$ and therefore $r^n(z) = \int_0^\infty e^{zs}\, d\alpha^{n*}(s),\ n \in \mathbb{N},\ z \in \mathbb{C}_0$. By the H-P functional calculus and Theorem 2.1,

$$||r^n(\tau A)x|| \leq \int_0^\infty ||T(\tau s)x||\, dV_{\alpha^{n*}}(s) \leq M||\alpha^{n*}||_{TV}||x|| \leq KM\sqrt{n}||x||.$$

To prove the second statement, let $A$ be the derivative operator on $C_0(\mathbb{R})$ with maximal domain and let $|r(is)| = 1$ for all $s \in \mathbb{R}$. Define

$$\tilde{\alpha}_n(t) := \begin{cases} \alpha^{n*}(t) & \text{if } t > 0, \\ \frac{\alpha^{n*}(0+)}{2} & \text{if } t = 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Then

$$(r^n(\tau A)f)(s) = \left[\int_0^\infty T(\tau t) f \, d\alpha^{n*}(t)\right](s) = \int_0^\infty f(s+\tau t)\, d\alpha^{n*}(t)$$
$$= \int_{-\infty}^\infty f(s+\tau t)\, d\tilde{\alpha}_n(t).$$

The Riesz representation theorem for $C_0(\mathbb{R})$ asserts that[4] $C_0(\mathbb{R})^* = NBV(\mathbb{R})$. Therefore, $||\tilde{\alpha}||_{C_0(\mathbb{R})^*} = V_{-\infty}^{\infty}(\tilde{\alpha}_n) = ||\alpha^{n*}||_{TV}$. Since the total variation of a function on $\mathbb{R}$ is independent of shifting and positive scaling,

$$||r^n(\tau A)|| = \sup_{\substack{f\in C_0(\mathbb{R})\\ ||f||\le 1}} \sup_{s\in R} |\int_{-\infty}^\infty f(s+\tau t)\, d\tilde{\alpha}_n(t)| = \sup_{\substack{f\in C_0(\mathbb{R})\\ ||f||\le 1}} \sup_{s\in \mathbb{R}} |\int_{-\infty}^\infty f(v)\, d\tilde{\alpha}_n(\frac{v-s}{\tau})|$$
$$= \sup_{s\in\mathbb{R}} \sup_{\substack{f\in C_0(\mathbb{R})\\ ||f||\le 1}} |\int_{-\infty}^\infty f(v)\, d\tilde{\alpha}_n(\frac{v-s}{\tau})| = \sup_{s\in\mathbb{R}} \sup_{\substack{f\in C_0(\mathbb{R})\\ ||f||\le 1}} |\langle f, \tilde{\alpha}_n(\frac{(\cdot)-s}{\tau})\rangle|$$
$$= \sup_{s\in\mathbb{R}} ||\tilde{\alpha}_n(\frac{(\cdot)-s}{\tau})||_{C_0(\mathbb{R})^*} = \sup_{s\in\mathbb{R}} V_{-\infty}^{\infty}(\tilde{\alpha}_n(\frac{(\cdot)-s}{\tau})) = \sup_{s\in\mathbb{R}} V_{-\infty}^{\infty}(\tilde{\alpha}_n) = ||\alpha^{n*}||_{TV}.$$

Now, Theorem 2.1 gives the desired estimate from below. □

## 3. Convergence on integer order Favard spaces

For $\alpha > 0$, $\alpha = l + \beta$, $\beta \in (0,1]$, $l \in \mathbb{N}\cup\{0\}$, the space

$$\mathcal{F}_\alpha := \{x \in \mathcal{D}(A^l) : \sup_{t>0} \left|\left|\frac{(T(t)-I)A^l x)}{t^\beta}\right|\right| < +\infty\} \tag{3.1}$$

is called the *Favard space of order* $\alpha$ (see, for example, [8]). An easy application of the uniform boundedness principle shows that, for bounded semigroups, the Favard space $\mathcal{F}_k$ ($k \in \mathbb{N}$) consists of $x \in \mathcal{D}(A^{k-1})$ for which $t \mapsto \langle T(t)A^{k-1}x, x^*\rangle$ is Lipschitz continuous for all $x^* \in X^*$. From the definition it is also clear that $\mathcal{D}(A^k) \subset \mathcal{F}_k$ for $k \in \mathbb{N}$. If $X$ is reflexive, then $\mathcal{D}(A^k) = \mathcal{F}_k$ (see, for example, [8, Cor. 5.21] and [5, Cor. 3.4.11]). However, if $X$ is not reflexive, then $\mathcal{D}(A)$ might be significantly smaller than $\mathcal{F}_1$. An example is the left-translation semigroup on $X := C_0(\mathbb{R})$. In this case $A = \frac{d}{dx}$, $\mathcal{D}(A) = \{f \in C_0(\mathbb{R}) : f' \in C_0(\mathbb{R})\}$ and $\mathcal{F}_1 = \{f \in C_0(\mathbb{R}) : f \text{ is of bounded total variation on } \mathbb{R}\}$. Similarly, if $X = L_1(\mathbb{R})$, then $\mathcal{D}(A) = \{f \in L_1(\mathbb{R}) : f \text{ is absolutely continuous on } \mathbb{R} \text{ and } f' \in L_1(\mathbb{R})\}$, while $\mathcal{F}_1 = \{f \in L_1(\mathbb{R}) : f \text{ is uniformly Lipschitz continuous on } \mathbb{R}\}$. Another example is the multiplication operator $(M_q, \mathcal{D}(M_q))$ on $X := C_0(\mathbb{R})$ defined by $(M_q f)(s) = q(s)f(s)$, where $q \in C(\mathbb{R})$ with $\frac{1}{q} \in C_b(\mathbb{R})$. In this case it is shown in [20, Prop.

[4] We say that a function $\alpha$ is a normalized function of bounded variation on $\mathbb{R}$ ($\alpha \in NBV(\mathbb{R})$) if the total variation $V_{-\infty}^{\infty}(\alpha)$ of $\alpha$ on $\mathbb{R}$ is finite and $\alpha$ is normalized; i.e., $\alpha(-\infty) = 0$ and $\alpha(t) = \frac{\alpha(t+)+\alpha(t-)}{2}$ for $t \in (-\infty, \infty)$ (see [6, p. 10]).

3.1] that $\mathcal{D}(M_q) := \{f \in C_0(\mathbb{R}) : fq \in C_0(\mathbb{R})\}$ and $\mathcal{F}_1 = \{f \in C_0(\mathbb{R}) : fq \in C_b(\mathbb{R})\}$. For further concrete examples we refer to [20].

If $x$ is in an integer order Favard space $\mathcal{F}_k$, $(k \in \mathbb{N})$, then

$$M_x^k := \limsup_{t\to 0+} \left|\left| \frac{(T(t)-I)A^{k-1}x}{t} \right|\right|. \tag{3.2}$$

Observe that $M_x^k = ||A^k x||$ if $x \in \mathcal{D}(A^k)$. Also, if $x^* \in X^*$ and $x \in \mathcal{F}_k$, then $t \mapsto \langle T(t)A^{k-1}x, x^*\rangle$ is differentiable a.e. and $|\langle T(t)A^{k-1}x, x^*\rangle'| \le MM_x^k ||x^*||$. If $A$ has a bounded inverse on $X$, then $\mathcal{F}_k$ is a Banach space with norm $M_x^k$ and $\mathcal{D}(A^k) \hookrightarrow \mathcal{F}_k \hookrightarrow \mathcal{D}(A^{k-1}) \hookrightarrow X$ where the symbol $\hookrightarrow$ stands for continuous embedding and where $\mathcal{D}(A^k)$ is endowed with the graph norm $||x||_{\mathcal{D}(A^k)} := ||x|| + ||A^k x||$ (see [15, Thm. 3.2.3] and [13]). We note that $\mathcal{F}_k$ $(k \in \mathbb{N})$ is also a Banach space under the norms

$$||x||_{\mathcal{F}_k} := ||x||_{\mathcal{D}(A^{k-1})} + \sup_{t\in(0,\infty)} \left|\left| \frac{(T(t)-I)A^{k-1}x}{t} \right|\right|, \tag{3.3}$$

or, if $A$ has a bounded inverse, $|||x|||_{\mathcal{F}_k} := \sup_{t\in(0,\infty)} \left|\left| \frac{(T(t)-I)A^{k-1}x}{t} \right|\right|$ (see [5], [8]).

Let $r$ be a rational approximation method of order $q \ge 1$. Then $r$ is given by $r(z) = \int_0^\infty e^{zs}\, d\alpha(s)$ for some $\alpha \in NBV$, $z \in \mathbb{C}_0$, and if $t > 0$, then

$$r^n(\tfrac{t}{n}z) = \int_0^\infty e^{zs}\, d\alpha_n(s),\ z \in \mathbb{C}_0,\ n \in \mathbb{N}, \tag{3.4}$$

where $\alpha_n(s) := \alpha^{n*}(\frac{n}{t}s)$. Note that $\alpha_n$ depends on $t$. Since $r^n(\frac{t}{n}z) \to e^{tz}$ as $n \to \infty$ and $e^{tz} = \int_0^\infty e^{zs}\, dH_t(s)$, one may suspect that $\alpha_n$ converges to $H_t$ in some sense. Let $I^{(k)}[\alpha_n - H_t]$ denote the $k$th antiderivative of $\alpha_n - H_t$; that is,

$$I^{(k)}[\alpha_n - H_t](s) := \int_0^s \cdots \int_0^{s_3} \int_0^{s_2} (\alpha_n - H_t)(s_1)\, ds_1\, ds_2 \cdots ds_k,\ k \in \mathbb{N}. \tag{3.5}$$

We set $I^{(0)}[\alpha_n - H_t] := \alpha_n - H_t$ and define

$$\tilde{\eta}_q(k) := \eta_q(k+1) = \begin{cases} k + \frac{1}{2} & \text{if } k < \frac{q-1}{2}, \\ (k+1)\frac{q}{q+1} & \text{if } \frac{q-1}{2} \le k. \end{cases}$$

The following theorem shows the convergence of $\alpha_n$ and its antiderivatives to $H_t$ and its antiderivatives in $L_1(\mathbb{R}_+)$; see [15, Cor. 2.3.2, Thm. 2.3.4] and [16].

**Theorem 3.1.** *Let $r$ be a rational approximation method of order $q \ge 1$ given by (3.4). If $k = 0, 1, \ldots, q$, $k \ne \frac{q-1}{2}$, then there is a constant $K > 0$ such that*

$$||I^{(k)}[\alpha_n - H_t]||_{L_1(\mathbb{R}_+)} \le K t^{k+1-\tilde{\eta}_q(k)} \left(\tfrac{t}{n}\right)^{\tilde{\eta}_q(k)},\ n \in \mathbb{N},\ t > 0. \tag{3.6}$$

*If $k = \frac{q-1}{2}$, then*

$$||I^{(k)}[\alpha_n - H_t]||_{L_1(\mathbb{R}_+)} \le K t^{k+1-\tilde{\eta}_q(k)} \left(\tfrac{t}{n}\right)^{\tilde{\eta}_q(k)} \ln(n+1),\ n \in \mathbb{N},\ t > 0. \tag{3.7}$$

*Moreover, for $k = 0, 1, \ldots, q$, we have* $\lim_{s\to\infty} I^{(k)}[\alpha_n - H_t](s) = 0$. □

Theorem 3.1 provides the core of the proof of Theorem 3.2 below which is our first extension of Theorem 1.1. In Theorem 3.2 the same speed of convergence is obtained on $\mathcal{F}_k$ as Theorem 1.1 predicts on $\mathcal{D}(A^k)$. In particular, we obtain optimal order of convergence on $\mathcal{F}_{q+1}$ instead of on $\mathcal{D}(A^{q+1})$ and a convergence rate of $\frac{1}{\sqrt{n}}$ on $\mathcal{F}_1$ instead of on $\mathcal{D}(A)$. The basic difference between our approach (which leads to convergence estimates on Favard spaces $\mathcal{F}_k$ ) and the one of Brenner and Thomée in [3] (leading to convergence estimates on $\mathcal{D}(A^k)$) is that we use the convergence of $\alpha_n$ (and its antiderivatives) to $H_t$ (and its antiderivatives) in $L_1(\mathbb{R}_+)$, while Brenner and Thomée prove and use that the inverse Laplace–Stieltjes transforms of the functions

$$z \mapsto \frac{r^n(\frac{t}{n}z) - e^{tz}}{z^{k+1}}$$

converge to 0 in the total variation norm.

Before we proceed, recall that if $f \in C[a,b]$ and $\alpha(s) = \int_c^s \phi(t)\,dt$ $(a \le c \le b)$ with $\phi \in L_1[a,b]$, then $\int_a^b f(s)\,d\alpha(s) = \int_a^b f(s)\phi(s)\,ds$ (see, for example, [23, Thm. I-6a]). More general, if $0 \le s_i < \infty$, $c_i \in \mathbb{R}$ for $i = 1,\dots,N$, and $f = g + \sum_{i=1}^N c_i H_{s_i}$ with $g \in C_b(\mathbb{R}_+)$ and $\alpha(s) = \int_c^s \phi(t)\,dt$ $(0 \le c < \infty)$ with $\phi \in L_1(\mathbb{R}_+)$, then

$$\int_0^\infty f(s)\,d\alpha(s) = \int_0^\infty f(s)\phi(s)\,ds. \tag{3.8}$$

Indeed, for $R > 0$, $\int_0^R g(s)\,d\alpha(t) = \int_0^R g(s)\phi(s)\,ds$ and hence $\int_0^\infty g(s)\,d\alpha(t) = \int_0^\infty g(s)\phi(s)\,ds$, since $g$ is bounded and $\alpha$ is of bounded variation on $[0,\infty)$. Finally,

$$\begin{aligned}\int_0^\infty H_{s_i}(s)\,d\alpha(s) &= \alpha(\infty) - \int_0^\infty \alpha(s)\,dH_{s_i}(s)\\ &= \alpha(\infty) - \alpha(s_i) = \int_{s_i}^\infty \phi(s)\,ds = \int_0^\infty H_{s_i}(s)\phi(s)\,ds.\end{aligned}$$

**Theorem 3.2.** *If $r$ is a rational approximation method of order $q \ge 1$, then there is a constant $K > 0$ such that for $k = 0,1,\dots,q+1$, $k \ne \frac{q+1}{2}$, we have*

$$\|r^n(\tfrac{t}{n}A)x - T(t)x\| \le KMt^{k-\eta_q(k)}\big(\tfrac{t}{n}\big)^{\eta_q(k)} M_x^k,\ t \ge 0,\ n \in \mathbb{N},\ x \in \mathcal{F}_k.$$

*If $k = \frac{q+1}{2}$, then*

$$\|r^n(\tfrac{t}{n}A)x - T(t)x\| \le KMt^{k-\eta_q(k)}\big(\tfrac{t}{n}\big)^{\eta_q(k)} \ln(n+1) M_x^k,\ t \ge 0,\ n \in \mathbb{N},\ x \in \mathcal{F}_k.$$

*Proof.* For $t = 0$ the statement is obvious. For $k = 0$ the estimate follows from Theorem 2.2. Let $k \ge 1$ and fix $t > 0$. The H-P functional calculus and (3.4) yields

$$\langle r^n(\tfrac{t}{n}A)x - T(t)x, x^*\rangle = \int_0^\infty \langle T(s)x, x^*\rangle\, d[\alpha_n(s) - H_t(s)].$$

Since $\langle T(\cdot)x, x^*\rangle \in C_b(\mathbb{R}_+)$ and $\alpha_n(\infty) = r^n(0-) = 1$, it follows that

$$\begin{aligned}&\langle r^n(\tfrac{t}{n}A)x - T(t)x, x^*\rangle = \int_0^\infty \langle T(s)x, x^*\rangle\, d[\alpha_n(s) - H_t(s)]\\ &= \langle T(s)x, x^*\rangle[\alpha_n(s) - H_t(s)]\big|_0^\infty - \int_0^\infty (\alpha_n(s) - H_t(s))\, d\langle T(s)x, x^*\rangle\\ &= -\int_0^\infty (\alpha_n(s) - H_t(s))\, d\langle T(s)x, x^*\rangle.\end{aligned}$$

Since $r(z) = \int_0^\infty e^{zt} d\alpha(t)$ is an A-stable rational function, the partial fraction decomposition of $r$ shows that $r^n(z) = \int_0^\infty e^{zs}\, d\alpha^{n*}(s)$, where $\alpha^{n*}(s) = r^n(\infty)H_0(s) + \beta_n(s)$ and $\beta_n(s) = \sum c_j s^{n_j} e^{-\lambda_j s}$. Therefore, by (3.8),

$$\begin{aligned}\langle r^n(\tfrac{t}{n}A)x - T(t)x, x^*\rangle &= \int_0^\infty (\alpha_n(s) - H_t(s))\, d\langle T(s)x, x^*\rangle \\ &= \int_0^\infty (\alpha_n(s) - H_t(s)) \frac{d}{ds}\big(\langle T(s)x, x^*\rangle\big)\, ds \\ &= \int_0^\infty I^{(0)}[\alpha_n - H_t](s) \frac{d}{ds}\big(\langle T(s)x, x^*\rangle\big)\, ds.\end{aligned}$$

If $x \in \mathcal{F}_k$, then one can integrate by parts $(k-1)$-times. By Theorem 3.1,

$$\begin{aligned}&|\langle r^n(\tfrac{t}{n}A)x - T(t)x, x^*\rangle| = \left|\int_0^\infty I^{(k-1)}[\alpha_n - H_t](s) \frac{d^k}{ds^k}\left(\langle T(s)x, x^*\rangle\right)\, ds\right| \\ &= \left|\int_0^\infty I^{(k-1)}[\alpha_n - H_t](s) \frac{d}{ds}\left(\langle T(s)A^{k-1}x, x^*\rangle\right)\, ds\right| \\ &\le ||I^{(k-1)}[\alpha_n - H_t]||_{L_1(\mathbb{R}_+)} \text{ess sup}_{s>0} \left|\frac{d}{ds}\left(\langle T(s)A^{k-1}x, x^*\rangle\right)\right| \\ &\le ||I^{(k-1)}[\alpha_n - H_t]||_{L_1(\mathbb{R}_+)} M M_x^k ||x^*|| \\ &\le KMt^{k-\tilde{\eta}_q(k-1)}\big(\tfrac{t}{n}\big)^{\tilde{\eta}_q(k-1)} M_x^k ||x^*|| = Mt^{k-\eta_q(k)}\big(\tfrac{t}{n}\big)^{\eta_q(k)} M_x^k ||x^*||,\end{aligned}$$

where the last inequality holds if $k \neq \frac{q+1}{2}$. If $k = \frac{q+1}{2}$, then Theorem 3.1 shows that $|\langle r^n(\frac{t}{n}A)x - T(t)x, x^*\rangle| \le KMt^{k-\eta_q(k)}\big(\frac{t}{n}\big)^{\eta_q(k)} \ln(n+1) M_x^k ||x^*||$. Finally, the desired results follow from the Hahn–Banach theorem. □

As shown in [13, Prop. 1], one can derive Theorem 3.2 from Theorem 1.1 directly. However, the above proof of Theorem 3.2 includes a simple and transparent proof of Theorem 1.1 and does not require additional arguments.

## 4. Convergence on intermediate spaces generated by the $K$-method

In this section we show that Theorem 3.2 extends to all Favard spaces $\mathcal{F}_\alpha$ with $\alpha \in [0, q+1]$ (and not just $\alpha \in \mathbb{N}$). Let $p \in [1, \infty)$, $k \in \mathbb{N}$, and $\alpha \in (0, k)$. As in [5, Def. 3.1.1] define subspaces of $X$ by

$$X_{\alpha,k,p} := \{x : ||x||_{\alpha,k,p} := ||x|| + \left[\int_0^\infty \left(\frac{1}{t^\alpha}||[T(t) - I]^k x||\right)^p \frac{dt}{t}\right]^{\frac{1}{p}} < \infty\}. \tag{4.1}$$

Similarly, for $p = \infty$ and $\alpha \in (0, k]$,

$$X_{\alpha,k,\infty} := \{x : ||x||_{\alpha,k,\infty} := ||x|| + \sup_{t\in(0,\infty)} \left(\frac{1}{t^\alpha}||[T(t) - I]^k x||\right) < \infty\}. \tag{4.2}$$

The spaces $X_{\alpha,k,p}$ are Banach spaces and $\mathcal{D}(A^k) \hookrightarrow X_{\alpha,k,p} \hookrightarrow X$ (see [5, Prop. 3.1.3]). Moreover,

$$\mathcal{D}(A^l) \hookrightarrow X_{l,k,\infty} \quad (l = 1, \dots, k), \tag{4.3}$$

which can be seen as follows. For $x \in X$ we have $||[T(t) - I]^m x|| \le (M+1)^m ||x||$, $m \in \mathbb{N}$. If $x \in \mathcal{D}(A^l)$, then

$$[T(t) - I]^l x = \int_0^t \int_0^t \cdots \int_0^t T(s_1 + s_2 + \cdots + s_l) A^l x \, ds_1 \, ds_2 \cdots ds_l$$

and hence $||(T(t) - I)^l x|| \le M ||A^l x|| t^l$. This implies that

$$\begin{aligned} ||[T(t) - I]^k x|| \le ||[T(t) - I]^{k-l}|| \, ||[T(t) - I]^l x|| &\le (M+1)^{k-l} M ||A^l x|| t^l \\ &\le (M+1)^k ||A^l x|| t^l, \ l = 1, \dots, k. \end{aligned}$$

Therefore, by combining the estimates above,

$$\begin{aligned} ||[T(t) - I]^k x|| &\le (M+1)^k (||x|| + ||A^l x||) \min(1, t^l) \\ &= (M+1)^k \min(1, t^l) ||x||_{\mathcal{D}(A^l)}, \ x \in \mathcal{D}(A^l), \ l = 1, \dots, k. \end{aligned}$$

Thus, the statement (4.3) follows from

$$\begin{aligned} ||x||_{l,k,\infty} = ||x|| + \sup_{t \in (0,\infty)} (\frac{1}{t^l} ||[T(t) - l]^k x|| \\ \le (1 + (M+1)^k) ||x||_{\mathcal{D}(A^l)}, \ x \in \mathcal{D}(A^l), \ l = 1, \dots, k. \end{aligned}$$

In order to be able to state our main result, we recall from [5] (see also [19]) some definitions and relevant facts from the theory of intermediate spaces and interpolation.

If $X_1, \ X_2$ are Banach spaces continuously embedded in a Hausdorff topological vector space $\mathcal{X}$, then $(X_1, X_2)$ is called an *interpolation pair*. Moreover, $X_1 + X_2 := \{x = x_1 + x_2 : \ x_1 \in X_1, \ x_2 \in X_2\}$ with the norm

$$||x||_{X_1+X_2} := \inf_{\substack{x_1 \in X_1, x_2 \in X_2 \\ x_1 + x_2 = x}} (||x_1||_1 + ||x_2||_2),$$

the intersection $X_1 \cap X_2$ with the norm

$$||x||_{X_1 \cap X_2} = \max(||x||_1, ||x||_2)$$

are Banach spaces (see [5, Prop. 3.2.1]), and $X_1 \cap X_2 \hookrightarrow X_i \hookrightarrow X_1 + X_2 \hookrightarrow \mathcal{X}$ for $i = 1, 2$. Let $\tilde{X} \hookrightarrow \mathcal{X}$ be a Banach space satisfying $X_1 \cap X_2 \hookrightarrow \tilde{X} \hookrightarrow X_1 + X_2$. Then $\tilde{X}$ is called an *intermediate space* (of $X_1$ and $X_2$). In most applications we have $X_2 \hookrightarrow X_1 \hookrightarrow X = \mathcal{X}$. In this case $X_1 \cap X_2 = X_2$ and $X_1 + X_2 = X_1$ and an intermediate space $\tilde{X}$ of $X_1$ and $X_2$ satisfies $X_2 \hookrightarrow \tilde{X} \hookrightarrow X_1$.

There are several ways to construct intermediate spaces. One of them is the *K-method* where $K : \ \mathbb{R}_+ \times (X_1 + X_2) \to \mathbb{R}_+ \cup \{0\}$ is defined by

$$K(t, x) := \inf_{\substack{x_1 \in X_1, x_2 \in X_2 \\ x_1 + x_2 = x}} (||x_1||_1 + t||x_2||_2).$$

Let $\theta \in (0, 1)$ and $p \in [1, \infty)$. Then

$$(X_1, X_2)_{\theta,p,K} := \{x \in X_1 + X_2 : \ ||x||_{\theta,p,K} := \left[\int_0^\infty \left(\frac{1}{t^\theta} K(t, x)\right)^p \frac{dt}{t}\right]^{\frac{1}{p}} < \infty\}$$

is an intermediate space of $X_1$ and $X_2$. If $\theta \in [0, 1]$, then

$$(X_1, X_2)_{\theta,\infty,K} := \{x \in X_1 + X_2 : \ ||x||_{\theta,\infty,K} := \text{ess sup}_{t>0} |\frac{1}{t^\theta} K(t, x)| < \infty\}$$

is also an intermediate space of $X_1$ and $X_2$ (see [5, Prop. 3.2.5]). We remark that the order of $X_1$ and $X_2$ is important, as we have $(X_1, X_2)_{\theta,p,K} = (X_2, X_1)_{1-\theta,p,K}$, $\theta \in (0, 1)$.

Let $X_{\alpha,q+1,p}$ be defined as in (4.1) and (4.2). It follows from [5, Thm. 3.4.2] that for $0 < \alpha < q + 1$ and $1 \le p \le \infty$ we have

$$X_{\alpha,q+1,p} = (X, \mathcal{D}(A^{q+1}))_{\frac{\alpha}{q+1},p,K}. \tag{4.4}$$

Let $[\alpha], \{\alpha\}$ denote the integer part and the fractional part of $\alpha \in \mathbb{R}$, respectively. If $\alpha \notin \mathbb{N}$ and $0 \le [\alpha] \le k \le q$ for some $k \in \mathbb{N}$, then

$$X_{\alpha,q+1,p} = (\mathcal{D}(A^{[\alpha]}), \mathcal{D}(A^{[\alpha]+1}))_{\{\alpha\},p,K} = (X, \mathcal{D}(A^{k+1}))_{\frac{\alpha}{k+1},p,K} = X_{\alpha,k+1,p}. \tag{4.5}$$

The equalities in (4.5) and (4.4) denote set equalities as well as isomorphisms of Banach spaces with equivalent norms (see the proof of [5, Thm. 3.4.6]). It is also shown there that the norm $||\cdot||_{\alpha,m,p}$ on $X_{\alpha,m,p}$ (where $m$ is either $k+1$ or $q+1$) is equivalent to the norm

$$|||x|||_{\alpha,p} := \begin{cases} ||x||_{\mathcal{D}(A^{[\alpha]})} + \left[\int_0^\infty \left(\frac{1}{t^{\{\alpha\}}}||[T(t)-I]A^{[\alpha]}x||\right)^p \frac{dt}{t}\right]^{\frac{1}{p}}, & p \in [1,\infty), \\ ||x||_{\mathcal{D}(A^{[\alpha]})} + \sup_{t\in(0,\infty)} \left(\frac{1}{t^{\{\alpha\}}}||[T(t)-I]A^{[\alpha]}x||\right), & p = \infty. \end{cases} \tag{4.6}$$

Observe that $|||\cdot|||_{\alpha,p}$ does not depend on $q$ or $k$. If $\alpha = 1, 2, \ldots, q$, then (4.5) is no longer valid. Instead,

$$X_{\alpha,q+1,p} = (\mathcal{D}(A^{\alpha-1}), \mathcal{D}(A^{\alpha+1}))_{\frac{1}{2},p,K}. \tag{4.7}$$

As before, the equality of the above spaces is understood as sets and as Banach spaces with equivalent norms. In this case we can also define a norm on $X_{\alpha,q+1,p}$, which is equivalent to $||\cdot||_{\alpha,q+1,p}$ and is independent of $q$, by

$$|||x|||_{\alpha,p} := ||x||_{\mathcal{D}(A^{\alpha-1})} + \left[\int_0^\infty \left(\frac{1}{t}||[T(t)-I]^2 A^{\alpha-1}x||\right)^p \frac{dt}{t}\right]^{\frac{1}{p}}, \ p \in [1,\infty),$$

with obvious modification for $p = \infty$ (see [5, Thm. 3.4.6]). The main point of the reduction equality (4.5) is that the spaces $X_{\alpha,q+1,p}$ can be viewed as intermediate spaces not only of $X$ and $\mathcal{D}(A^{q+1})$ but also of $X$ and $\mathcal{D}(A^{k+1})$ for $[\alpha] \le k$ or, most importantly, of $\mathcal{D}(A^{[\alpha]})$ and $\mathcal{D}(A^{[\alpha]+1})$. Since for each of these interpolation pairs one of the spaces is contained in the other, the spaces $X_{\alpha,q+1,p}$ are really intermediate spaces "between" the two. We also mention that for the noninteger order Favard spaces $\mathcal{F}_\alpha$, defined in (3.1), it follows from (4.5) and (4.6) that the set equality

$$\mathcal{F}_\alpha = X_{\alpha,[\alpha]+m,\infty} = (X, \mathcal{D}(A^{q+1}))_{\frac{\alpha}{q+1},\infty,K} \tag{4.8}$$

holds for $0 < \alpha < q+1$, $\alpha \notin \mathbb{N}$, $m \in \mathbb{N}$. The integer order Favard spaces $\mathcal{F}_\alpha$ cannot be regarded as intermediate spaces of $X$ and $\mathcal{D}(A^{q+1})$, except for $\alpha = q+1$. They can be identified with intermediate spaces of $X$ and $\mathcal{D}(A^\alpha)$. More precisely,

$$\mathcal{F}_\alpha = X_{\alpha,\alpha,\infty} = (X, \mathcal{D}(A^\alpha))_{1,\infty,K}, \ \alpha \in \mathbb{N}, \tag{4.9}$$

as [5, Thms. 3.4.3, 3.4.10] show.

Next we discuss the Riesz–Thorin interpolation theorem. Let $(X_1, X_2)$ and $(Y_1, Y_2)$ be two interpolation pairs (in $\mathcal{X}$ and $\mathcal{Y}$, respectively). Let

$$T \in \mathcal{B}(X_1 + X_2, Y_1 + Y_2)$$

such that the restriction of $T$ to $X_i$ $(i=1,2)$ belongs to $\mathcal{B}(X_i,Y_i)$, i.e,

$$||Tx_i||_i \le M_i||x_i||_i,\ i=1,2.$$

Then the restriction of $T$ to $(X_1,X_2)_{\theta,p,K}$ ($\theta\in(0,1)$ if $p\in[1,\infty)$ and $\theta\in[0,1]$ if $p=\infty$) belongs to $\mathcal{B}\big((X_1,X_2)_{\theta,p,K},(Y_1,Y_2)_{\theta,p,K}\big)$ and

$$||Tx||_{\theta,p,K}\le M_1^{1-\theta}M_2^{\theta}||x||_{\theta,p,K},\ x\in(X_1,X_2)_{\theta,p,K}. \tag{4.10}$$

For the proof see [5, Thm. 3.2.23].

Next we take the function $\eta_q(k)$ defined for integers $0\le k\le q+1$ and extend it to the interval $[0,q+1]$ using linear interpolation between the points $(k,\eta_q(k))$ and $(k+1,\eta_q(k+1))$.

If $q$ is *odd*, then we get

$$\eta_q(\alpha):=\begin{cases}\alpha-\frac{1}{2} & \text{if } 0\le\alpha<\frac{q+1}{2},\\ \alpha\frac{q}{q+1} & \text{if } \frac{q+1}{2}\le\alpha\le q+1.\end{cases}$$

If $q$ is *even*, then the number $\frac{q+1}{2}$ is not an integer, and therefore

$$\eta_q(\alpha):=\begin{cases}\alpha-\frac{1}{2} & \text{if } 0\le\alpha\le\frac{q}{2},\\ \frac{q}{2}-\frac{1}{2}+(\alpha-\frac{q}{2})\big(\frac{q+2}{2}\frac{q}{q+1}-(\frac{q}{2}-\frac{1}{2})\big) & \text{if } \frac{q}{2}<\alpha<\frac{q+2}{2},\\ \alpha\frac{q}{q+1} & \text{if } \frac{q+2}{2}\le\alpha\le q+1.\end{cases}$$

Now we have everything in place to extend Theorem 1.1 to the intermediate spaces $X_{\alpha,q+1,p}$ for $0<\alpha<q+1$.

**Theorem 4.1.** *If $r$ is a rational approximation method of order $q\ge1$, then there is a constant $\tilde{K}>0$ such that for $0<\alpha<q+1$ and $x\in X_{\alpha,q+1,p}$ we have*

$$||r^n(\tfrac{t}{n}A)x-T(t)x||\le\tilde{K}Mt^{\alpha}n^{-\eta_q(\alpha)}|||x|||_{\alpha,p},\ t\ge0,\ n\in\mathbb{N},\ 1\le p\le\infty,$$

*if $q$ is even and $\alpha\ne\frac{q}{2},\frac{q+2}{2}$ or if $q$ is odd but $\alpha\notin[\frac{q-1}{2},\frac{q+3}{2}]$. If $q$ is odd and $\alpha\in(\frac{q-1}{2},\frac{q+3}{2})\setminus\{\frac{q+1}{2}\}$, then*

$$||r^n(\tfrac{t}{n}A)x-T(t)x||\le\tilde{K}Mt^{\alpha}n^{-\eta_q(\alpha)}[\ln(n+1)]^{1-|\frac{q+1}{2}-\alpha|}|||x|||_{\alpha,p},\ t\ge0,\ n\in\mathbb{N}.$$

*Proof.* If $\alpha\ne1,2,\dots,q$, then $X_{\alpha,q+1,p}=(\mathcal{D}(A^{[\alpha]}),\mathcal{D}(A^{[\alpha]+1}))_{\{\alpha\},p,K}$ by (4.5). Using Theorem 3.2 on $\mathcal{D}(A^{[\alpha]})$ and on $\mathcal{D}(A^{[\alpha]+1})$ together with the observation that $M_x^k=||A^kx||\le||x||_{\mathcal{D}(A^k)}$ if $x\in\mathcal{D}(A^k)$, we apply the Riesz–Thorin inequality (4.10) to the intermediate spaces $(X_1,X_2)_{\{\alpha\},p,K}:=(\mathcal{D}(A^{[\alpha]}),\mathcal{D}(A^{[\alpha]+1}))_{\{\alpha\},p,K}$ and $(Y_1,Y_2)_{\{\alpha\},p,K}:=(X,X)_{\{\alpha\},p,K}=X$ with equivalence of the respective norms following from [5, Prop. 3.2.5]. This gives, for $[\alpha]\ne\frac{q\pm1}{2}$, and $x\in X_{\alpha,q+1,p}$,

$$\begin{aligned}&||r^n(\tfrac{t}{n}A)x-T(t)x||\le C||r^n(\tfrac{t}{n}A)x-T(t)x||_{\{\alpha\},p,K}\\&\le C\big(KMt^{[\alpha]-\eta_q([\alpha])}\big(\tfrac{t}{n}\big)^{\eta_q([\alpha])}\big)^{1-\{\alpha\}}\big(KMt^{[\alpha]+1-\eta_q([\alpha]+1)}\big(\tfrac{t}{n}\big)^{\eta_q([\alpha]+1)}\big)^{\{\alpha\}}|||x|||_{\alpha,p}\\&=\tilde{K}Mt^{\alpha}n^{-\eta_q([\alpha])+\{\alpha\}(\eta_q([\alpha])-\eta_q([\alpha]+1))}|||x|||_{\alpha,p}.\end{aligned}$$

The evaluation of the exponent $-\eta_q([\alpha])+\{\alpha\}(\eta_q([\alpha])-\eta_q([\alpha]+1))$ yields the desired estimate. If $[\alpha]=\frac{q\pm1}{2}$, then either on $\mathcal{D}(A^{[\alpha]})$ or on $\mathcal{D}(A^{[\alpha]+1})$, we have to include a factor of $\ln(n+1)$ in the estimate according to Theorem 3.2. This shows the result for $\alpha\in(\frac{q-1}{2},\frac{q+3}{2})$.

If $\alpha=1,2,\dots,q$, then we use (4.7) to identify the spaces $X_{\alpha,q+1,p}$. We take $(X_1,X_2)_{\frac{1}{2},p,K}:=(\mathcal{D}(A^{\alpha-1}),\mathcal{D}(A^{\alpha+1}))_{\frac{1}{2},p,K}$ and $(Y_1,Y_2)_{\frac{1}{2},p,K}:=(X,X)_{\frac{1}{2},p,K}=X$

in the Riesz–Thorin inequality (4.10) with equivalence of the respective norms. This yields, for $x \in X_{\alpha,q+1,p}$,

$$\begin{aligned}
&||r^n(\tfrac{t}{n}A)x - T(t)x|| \le C||r^n(\tfrac{t}{n}A)x - T(t)x||_{\frac{1}{2},p.K} \\
&\le C\big(KMt^{\alpha-1-\eta_q(\alpha-1)}\big(\tfrac{t}{n}\big)^{\eta_q(\alpha-1)}\big)^{\frac{1}{2}}\big(KMt^{\alpha+1-\eta_q(\alpha+1)}\big(\tfrac{t}{n}\big)^{\eta_q(\alpha+1)}\big)^{\frac{1}{2}}|||x|||_{\alpha,p} \\
&= \tilde{K}Mt^{\alpha}n^{-\frac{1}{2}(\eta_q(\alpha-1)+\eta_q(\alpha+1))}|||x|||_{\alpha,p}.
\end{aligned}$$

The evaluation of $\frac{1}{2}(\eta_q(\alpha-1)+\eta_q(\alpha+1))$ at $\alpha = 1, 2, \ldots, q$, $\alpha \ne \frac{q}{2}, \frac{q+2}{2}$ if $\alpha$ is even and $\alpha \ne \frac{q\pm1}{2}, \frac{q+3}{2}$ if $\alpha$ is odd, gives the desired result. $\square$

Let us consider the Favard spaces $\mathcal{F}_\alpha$ defined in (3.1) with the norm

$$||x||_\alpha := \begin{cases} |||x|||_{\alpha,\infty} & \text{if } \alpha \notin \mathbb{N}, \\ ||x||_{\mathcal{F}_\alpha} & \text{if } \alpha \in \mathbb{N}, \end{cases}$$

where $|||\cdot|||_{\alpha,\infty}$ and $||\cdot||_{\mathcal{F}_\alpha}$ are defined as in (4.6) and (3.3). We set $\mathcal{F}_0 := X$ and $||x||_0 := ||x||$. For the convenience of the reader we reformulate Theorems 3.2 and 4.1 if $x \in \mathcal{F}_\alpha$.

**Corollary 4.2.** *Let $r$ be a rational approximation method of order $q \ge 1$. If $0 \le \alpha \le q+1$, then there is a constant $K > 0$ such that for $x \in \mathcal{F}_\alpha$ we have*

$$||r^n(\tfrac{t}{n}A)x - T(t)x|| \le KMt^\alpha n^{-\eta_q(\alpha)}||x||_\alpha, \ t \ge 0, \ n \in \mathbb{N},$$

*if $q$ is even or if $q$ is odd but $\alpha \notin (\frac{q-1}{2}, \frac{q+3}{2})$. If $q$ is odd and $\alpha \in (\frac{q-1}{2}, \frac{q+3}{2})$, then*

$$||r^n(\tfrac{t}{n}A)x - T(t)x|| \le KMt^\alpha n^{-\eta_q(\alpha)}[\ln(n+1)]^{1-|\frac{q+1}{2}-\alpha|}||x||_\alpha, \ t \ge 0, \ n \in \mathbb{N}.$$

*Proof.* For $\alpha \notin \mathbb{N}$ the statement follows from Theorem 4.1 using (4.8). If $\alpha = 0, 1, \ldots, q+1$, then Theorem 3.2 implies the desired inequality, noting that $M_x^\alpha \le ||x||_\alpha$, $\alpha = 1, \ldots, q+1$. $\square$

Note that if $A$ has a bounded inverse, then the norm $||\cdot||_\alpha$ is equivalent to the norm $|||x|||_\alpha := \sup_{t\in(0,\infty)} ||t^{-\beta}(T(t)-I)A^k x||$, $k = 0, \ldots, q$, $\alpha = k+\beta$, $\beta \in (0,1]$. As a corollary we obtain the following stability result for the discrete orbits $r^n(\frac{t}{n}A)x$.

**Corollary 4.3.** *If $r$ is a rational approximation method of order $q \ge 1$, then $||r^n(\frac{t}{n}A)x|| \le KM(1+t^{\frac{1}{2}})|||x|||_{\frac{1}{2},p}$, $x \in X_{\frac{1}{2},q+1,p}$, $1 \le p \le \infty$, $t \ge 0$, $n \in \mathbb{N}$.* $\square$

For stable methods, such as the Backward Euler method, solely from the error estimate on $\mathcal{D}(A^{q+1})$ we obtain optimal error estimates on $X_{\alpha,q+1,\infty}$ for all $0 < \alpha \le q+1$ and also on $\mathcal{D}(A^k)$ $(k = 1, 2, \ldots, q)$ including the case $\alpha = \frac{q+1}{2}$.

**Corollary 4.4.** *Assume that $r$ is a rational approximation method of order $q \ge 1$ and that $||r^n(\frac{t}{n}A)|| \le C$ for some $C \ge 1$ and all $n \in \mathbb{N}$. Then there is a constant $K \ge 1$ such that, for all $0 < \alpha \le q+1$,*

$$||r^n(\tfrac{t}{n}A)x - T(t)x|| \le KMt^{\frac{\alpha}{q+1}}\big(\tfrac{t}{n}\big)^{\alpha\frac{q}{q+1}}||x||_{\alpha,q+1,\infty}, \ t \ge 0, \ n \in \mathbb{N}, \ x \in X_{\alpha,q+1,\infty}.$$

*In particular, if $x \in \mathcal{D}(A^k)$ $(k = 1, \ldots, q+1)$, then*

$$||r^n(\tfrac{t}{n}A)x - T(t)x|| \le KMt^{\frac{k}{q+1}}\big(\tfrac{t}{n}\big)^{k\frac{q}{q+1}}||x||_{\mathcal{D}(A^k)}, \ t \ge 0, \ n \in \mathbb{N}. \tag{4.11}$$

*Proof.* If $\alpha = q+1$, then the statement follows from Theorem 3.2 and (4.9). Since $||r^n(\frac{t}{n}A)|| \le C$ for all $n \in \mathbb{N}$, there is constant $K \ge 1$ such that $||r^n(\frac{t}{n}A)x - T(t)x|| \le KM||x||$ for all $x \in X$. By Theorem 3.2,

$$||r^n(\tfrac{t}{n}A)x - T(t)x|| \le KMt\big(\tfrac{t}{n}\big)^q ||x||_{\mathcal{D}(A^{q+1})},\ t \ge 0,\ n \in \mathbb{N},\ x \in \mathcal{D}(A^{q+1}).$$

Now, for $0 < \alpha < q+1$ consider the intermediate spaces

$$(X_1, X_2)_{\frac{\alpha}{q+1},\infty,K} := (X, \mathcal{D}(A^{q+1}))_{\frac{\alpha}{q+1},\infty,K} = X_{\alpha,q+1,\infty}$$

and $(Y_1, Y_2)_{\frac{\alpha}{q+1},\infty,K} := (X, X)_{\frac{\alpha}{q+1},\infty,K} = X$ (with equivalence of the respective norms). By the Riesz–Thorin inequality (4.10), for $0 < \alpha < q+1$, we have

$$\begin{aligned} ||r^n(\tfrac{t}{n}A)x - T(t)x|| &\le (KM)^{1-\frac{\alpha}{q+1}} \big(KMt\big(\tfrac{t}{n}\big)^q\big)^{\frac{\alpha}{q+1}} ||x||_{\alpha,q+1,\infty} \\ &= KMt^{\frac{\alpha}{q+1}} \big(\tfrac{t}{n}\big)^{\alpha\frac{q}{q+1}} ||x||_{\alpha,q+1,\infty},\ t \ge 0,\ n \in \mathbb{N},\ x \in X_{\alpha,q+1,\infty}. \end{aligned}$$

From (4.3) it follows that if $x \in \mathcal{D}(A^k)$ and $k = 1, \dots, q$, then

$$||r^n(\tfrac{t}{n}A)x - T(t)x|| \le KMt^{\frac{k}{q+1}} \big(\tfrac{t}{n}\big)^{k\frac{q}{q+1}} ||x||_{\mathcal{D}(A^k)}. \qquad \square$$

The estimate in (4.11) is an improvement of the inequality in [3, Thm. 4] for the stable case for $k = \frac{q+1}{2}$, as it does no longer contain a factor of $\ln(n+1)$. It also proves and generalizes [9, Thm. 1.7], where the same result is shown for the Backward Euler method on $\mathcal{D}(A)$.

Finally, we briefly discuss the sharpness of Theorem 3.2, Theorem 4.1 and Corollary 4.2. For $f \in L_p(\mathbb{R})$ let

$$\Delta_h^N f := \sum_{i=0}^{N} (-1)^{N-l} \binom{N}{l} f(\cdot + lh),\ h \in \mathbb{R},$$

be the $N$th right difference of $f$, and let

$$\omega_N(t, f, p) := \sup_{h \in (0,t]} (||\Delta_h^N f||_p),\ t \in (0, \infty),$$

denote its $N$th modulus of continuity. For $p \in [1, \infty)$, $N \in \mathbb{N}$, $\alpha \in (0, N)$, and $s \in [1, \infty)$, the homogeneous *Besov spaces* $\mathcal{B}(\alpha, N, s, p)$ are defined as

$$\mathcal{B}(\alpha, N, s, p) := \{f \in L_p(\mathbb{R}) : \int_{-\infty}^{\infty} (|h|^{-\alpha} ||\Delta_h^N f||_p)^s \frac{dh}{|h|} < \infty\}$$

and

$$\mathcal{B}(\alpha, N, \infty, p) := \{f \in L_p(\mathbb{R}) : \sup_{h \in \mathbb{R}} (|h|^{-\alpha} ||\Delta_h^N f||_p) < \infty\},\ \alpha \in [0, N].$$

The Besov spaces $\mathcal{B}(\alpha, N, s, p)$ are Banach spaces under the equivalent norms

$$\begin{aligned} ||f|| &:= ||f||_p + \Big(\int_{-\infty}^{\infty} (|h|^{-\alpha} ||\Delta_h^N f||_p)^s \frac{dh}{|h|}\Big)^{\frac{1}{s}}, \\ ||f||_{\mathcal{B}(\alpha,N,s,p)} &:= ||f||_p + \Big(\int_0^{\infty} (t^{-\alpha} \omega_N(t, f, p))^s \frac{dt}{t}\Big)^{\frac{1}{s}}, \end{aligned}$$

with the usual modification for $s = \infty$ (see [5, Prop. 4.3.5]). Let $W_{N,p}(\mathbb{R})$ be the *Sobolev space* of order $N$ with the norm $||f||_{W_{N,p}(\mathbb{R})} := ||f||_p + ||D^N f||_p$, where $D^N f$ denotes the $N$th generalized derivative of $f$. The Besov spaces are intermediate spaces of $L_p(\mathbb{R})$ and $W_{N,p}(\mathbb{R})$, more precisely, we have that

$$\mathcal{B}(\alpha, N, s, p) = (L^p(\mathbb{R}), W_{N,p}(\mathbb{R}))_{\frac{\alpha}{N},s,K}$$

with equivalent norms (see [5, Thm. 4.3.6]). This means that if we take $X := L_p(\mathbb{R})$ and $Af := f'$ with maximal domain, then by the above and (4.5),

$$\mathcal{B}(\alpha, q+1, s, p) = (L^p(\mathbb{R}), W_{q+1,p}(\mathbb{R}))_{\frac{\alpha}{q+1},s,K} = (X, \mathcal{D}(A^{q+1}))_{\frac{\alpha}{q+1},s,K} = X_{\alpha,q+1,s}.$$

For $q \in \mathbb{N}$ fixed, define

$$\eta_{q,p}(\alpha) := \begin{cases} \alpha - |\frac{1}{2} - \frac{1}{p}| & \text{if } 0 \le \alpha < (q+1)|\frac{1}{2} - \frac{1}{p}|, \\ \alpha \frac{q}{q+1} & \text{if } (q+1)|\frac{1}{2} - \frac{1}{p}| \le \alpha \le q+1. \end{cases} \tag{4.12}$$

Note that $\eta_{q,1}(\alpha) = \eta_q(\alpha)$ if $q$ is odd and also if $q$ is even and $\alpha \notin (\frac{q}{2}, \frac{q+2}{2})$. If $X := L_p(\mathbb{R})$ and $Af := f'$ with maximal domain, then for a rational approximation method $r$ of order $q \ge 1$ with $|r(is)| = 1$ $(s \in \mathbb{R})$ we have

$$c_t n^{-\eta_{q,p}(\alpha)} \le \sup\{||r^n(\tfrac{t}{n}A)f - T(t)f||_{L_p(\mathbb{R})};\ ||f||_{\mathcal{B}(\alpha,q+1,\infty,p)} \le 1\} \tag{4.13}$$

if $0 < \alpha \le q+1$ and

$$c_t n^{-\eta_{q,p}(k)} \le \sup\{||r^n(\tfrac{t}{n}A)f - T(t)f||_{L_p(\mathbb{R})};\ ||D^k f|| \le 1\} \tag{4.14}$$

if $k = 0, 1, 2, \ldots, q+1$ as [2, (5.6)] together with [2, Lemmas 2.9, 2.11] show. The choice $p = 1$ in (4.13) yields that the general convergence estimate on $X_{\alpha,q+1,\infty}$ that holds for all Banach spaces $X$ and generators of bounded semigroups cannot be better than

$$||r^n(\tfrac{t}{n}A)x - T(t)x|| \le C_t n^{-\eta_{q,1}(\alpha)} |||x|||_{\alpha,\infty},\ x \in X_{\alpha,q+1,\infty}. \tag{4.15}$$

Therefore, our estimates on $X_{\alpha,q+1,\infty}$ in Theorem 4.1 and, correspondingly, on $\mathcal{F}_\alpha$ if $\alpha \notin \mathbb{N}$ in Corollary 4.2, are sharp for $0 < \alpha < q+1$, provided that $\alpha \notin [\frac{q}{2}, \frac{q+2}{2}]$ if $q$ is even and $\alpha \notin [\frac{q-1}{2}, \frac{q+3}{2}]$ if $q$ is odd. We remark that if $q$ is odd and $\alpha \in (\frac{q-1}{2}, \frac{q+1}{2}) \cup (\frac{q+1}{2}, \frac{q+3}{2})$, then Theorem 4.1 and Corollary 4.2 are almost sharp in the sense that the estimates contain an extra factor of $(\ln(n+1))^\beta$ only. Similarly, (4.14) shows the sharpness of Theorem 3.2 on $\mathcal{D}(A^k)$ for $k = 0, 1, \ldots, q+1$, $k \ne \frac{q+1}{2}$, and also the sharpness of Theorem 3.2 and Corrollary 4.2 on integer order Favard classes since $\mathcal{D}(A^k) \subset \mathcal{F}_k$ and $M_x^k = ||A^k x||$ if $x \in \mathcal{D}(A^k)$. Finally, we remark that if $|r(is)| < 1$, $s \ne 0$, and some more detailed information is known about the behavior of $s \mapsto r(is)$, our results can be improved by an order up to $\frac{1}{2}$ for $\alpha < \frac{q+1}{2}$; for details see [3].

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Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803 and Department of Analysis, Mathematics Institute, University of Miskolc, Miskolc-Egyetemváros, Hungary, H-3515

*Current address*: Department of Mathematics and Statistics, University of Otago, P.O. Box 56, Dunedin, New Zealand

*E-mail address*: `kmisi@math.lsu.edu`