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CONSTRUCTING REPRESENTATIONS OF HIGHER DEGREES OF FINITE SIMPLE GROUPS AND COVERS

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Abstract. Let G be a finite group and χ an irreducible character of G. A simple method for constructing a representation affording χ can be used whenever G has a subgroup H such that χ_H has a linear constituent with multiplicity 1. In this paper we show that (with a few exceptions) if G is a simple group or a covering group of a simple group and χ is an irreducible character of G of degree between 32 and 100, then such a subgroup exists.

1. Introduction

Let G be a finite group and χ be an irreducible character of G. We say that a subgroup H is a χ-subgroup if the restriction χ_H of χ to H has at least one linear constituent of multiplicity 1. Not every pair (χ, G) has a χ-subgroup (see [10] and Section 4 below), but χ-subgroups can be found in many cases. The existence of such subgroups is of interest for several reasons such as computing primitive idempotent elements (see [10], [9] and [15]), calculating Schur indices (see [8], [14] and [16]) and computing irreducible matrix representations (see [6], [5] and [4]).

A simple and efficient method for computing matrix representations of finite groups has been described by Dixon [6]. This applies whenever G has a χ-subgroup for an irreducible character χ and works best when the subgroup is small. In practice, the difficulty in using this method lies in finding such a χ-subgroup (if one exists). This could involve searching through the full lattice of the subgroup of G and that becomes impractical for large groups.

In [4] the author developed a general program to compute a matrix representation affording any specified character χ of a group G (see also the GAP package [5]). The program uses a recursive technique which reduces the general problem to the special problem of computing representations affording irreducible characters of degree at most $\chi(1)$ for central covers of simple groups.

Thus we must deal with the problem of computing a representation in the case where χ is irreducible and G is simple or a cover of a simple group. In [4] (see also [3]) the author shows that in this special case we can usually find a suitable χ-subgroup when $\chi(1) < 32$. In a few cases where we failed to find χ-subgroups for characters in this range we found a maximal subgroup M of G such that χ_M is irreducible. In the latter case a representation of M affording χ_M can be constructed (recursively), and we showed how to extend this representation of M to

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a representation of G. Thus our general program can be applied to construct a representation affording a character of degree < 32 for any group.

In the present paper we extend the result of [3] (and hence the capability of the original program) to characters with degrees between 32 and 100. Again we shall show that, in most cases, if χ is an irreducible character of a cover G of a simple group and $32 \leqslant \chi(1) \leqslant 100$, then we can find a χ-subgroup which is easily described in terms of the group structure (see Section 2). In the remaining cases, with two exceptions, we find a maximal subgroup M of G such that χ_M is irreducible (see Section 3). The exceptions occur for the cover $6.A_7$ and two characters of degree 36; in these two cases there are neither χ-subgroups nor irreducible maximal subgroups available, so a different approach will be needed to construct the corresponding matrix representations for these characters (see Section 4).

2. χ-SUBGROUPS

The following theorem describes χ-subgroups for certain irreducible characters χ of alternating groups. For a proof see [4].

Theorem 2.1. *Let χ be an irreducible character of the alternating group A_n.*

(1) *If $n \geqslant 4$ and $\chi(1) = n-1$, then a Sylow 3-subgroup of order 3 of A_4 is a χ-subgroup.*
(2) *If $n \geqslant 6$ and $\chi(1) = (n-1)(n-2)/2$ or $n(n-3)/2$, then a Sylow 3-subgroup of A_6 of order 9 is a χ-subgroup.*
(3) *If $n \geqslant 8$ and $\chi(1) = (n-1)(n-2)(n-3)/6$, $n(n-1)(n-5)/6$ or $n(n-2)(n-4)/3$, then a Sylow 2-subgroup of A_8 of order 64 is a χ-subgroup.*

With the exception of the characters covered in the theorem above, there are only a few cases where an alternating group has an irreducible character of degree between 32 and 100. In these cases a χ-subgroup was computed directly using GAP [7]. These exceptions are listed in Table 1. This table also contains χ-subgroups for the covering groups of alternating groups and some other simple groups and covers listed in the Atlas [1] (see also [2]). For these groups there is no general theorem about their χ-subgroups when $32 \leqslant \chi(1) \leqslant 100$.

The group SL(2, q) is the unique covering group of the simple group PSL(2, q), except for $q = 9$. In the latter case PSL(2, 9) $\cong A_6$. By [13, Theorem 7.1.1] the group SL(3, q) where $q > 2$, is the unique covering group of the simple group PSL(3, q) except when $q = 4$ (the group PSL(3, 4) has 7 different covering groups, see Table 1). Also PSU(3, q) is a simple group of twisted Lie type ${}^2A_2(q)$ and the group SU(3, q) is the unique covering group of the simple group PSU(3, q) (see [11, Corollary 5.1.3]).

On the other hand, PSL(2, q), PSL(3, q) and PSU(3, q) are the factor groups of SL(2, q), SL(3, q) and SU(3, q) by their centres, so characters of the former groups correspond to those characters of SL(2, q), SL(3, q) and SU(3, q), respectively, whose kernels contain the centre. Thus it is enough to find χ-subgroups for the irreducible characters χ of SL(2, q), SL(3, q) and SU(3, q) (except for PSL(3, 4) and covers). The following theorems describe these χ-subgroups. There is no restriction on the degree of the characters. For a proof of these theorems see [4].

Theorem 2.2. Let $G = \mathrm{SL}(2, q)$ ($G = \mathrm{SL}(3, q)$) *where $q > 4$ ($q > 2$) is a power of a prime p and let H be a Sylow p-subgroup of G. Then H is a χ-subgroup for all irreducible characters χ of G.*

Theorem 2.3. *Let $G = \mathrm{SU}(3, q)$ where $q > 2$ is a power of a prime p and let H be a Sylow p-subgroup of G. Then H is a χ-subgroup for all irreducible characters χ of G such that $\chi(1) \neq q^2 - q$. If $\chi(1) = q^2 - q$, then H contains an abelian subgroup of order q^2 which is a χ-subgroup.*

Let G be a group and χ a character of G. We denote by $\mathrm{Lin}(\chi)$ and $\mathrm{Lin}(G)$ the set of linear constituents of χ and the set of linear characters of G, respectively. Clearly $\mathrm{Lin}(\chi) \subseteq \mathrm{Lin}(G)$. The following simple remark is very useful.

Remark 2.4. Let χ be an irreducible character of G and H a Sylow subgroup of G. Let μ be a character of H such that $\chi(1) = |H| + \mu(1)$ and $\chi_H(h) = \mu(h)$ for all $1 \neq h \in H$. Then $\mathrm{Lin}(\mu) \neq \mathrm{Lin}(H)$ implies H is a χ-subgroup. This is true because the hypotheses show that $\chi_H = \rho + \mu$ where ρ is the regular character of H. Since $\mathrm{Lin}(\mu) \subset \mathrm{Lin}(\rho) = \mathrm{Lin}(H)$, there exists a linear character φ of H such that $\langle \chi_H, \varphi \rangle = 1$.

We now examine each case in the Atlas [1] where G is a simple group or the cover of a simple group and χ is an irreducible character of degree between 32 and 100, and which was not covered by Theorems 2.1, 2.2 and 2.3. Using GAP [7], the remark above and the information available from [1] we searched for a subgroup of G of a simply described form which is a χ-subgroup. We found that in most cases such χ-subgroups exist.

The group libraries in GAP and [2] are used as the sources for generators of the groups listed in Tables 1 and 2. For finding most of the χ-subgroups listed in these tables we used the available standard functions in GAP. For each group G in Table 1 we computed Sylow r-subgroups, say P_r, and some simply relative subgroups to P_r such as the derived subgroup P_r' and the normalizers and centralizers of P_r and P_r' in G. Then we found the smallest possible χ-subgroup among these subgroups. The only exception in Table 1 is an irreducible character of degree 32 of $2.M12$. In this case we found a particular 2-subgroup which is denoted by H^*. These results are summarized in Table 1.

TABLE 1. The χ-subgroups of simple groups and covers for $32 \leqslant \chi(1) \leqslant 100$

Group	$\chi(1)$	χ-subgroup	Size of χ-subgroup
A_7	35	$N_G(P_5)$	20
$2.A_7$	36	$N_G(P_3)$	72
M_{11}	44, 45	$N_G(P_5)$	20
	55	$N_G(P_{11})$	55
A_8	45, 56, 70	$N_G(P_5)$	60
$2.A_8$	48, 56, 64	$N_G(P_5)$	120
$2.L_3(4)$	36, 64, 70, 90	$N_G(P_3)$	144
$4_1.L_3(4)$	56, 64, 80	$N_G(P_3)$	288
$4_2.L_3(4)$	36	$N_G(P_3)$	288
$U_4(2)$	40, 60, 64, 81	P_2	64
	45	$N_G(P_5)$	20
$2.U_4(2)$	36, 60, 64, 80	P_3	81

(Table 1. continued)

$Sz(8)$	35	$N_G(P_7)$	14
	64, 65, 91	$N_G(P_{13})$	52
$2.Sz(8)$	40, 56, 64	$N_G(P_{13})$	104
M_{12}	45, 54, 55, 66	$N_G(P_5)$	40
	99	$N_G(P_{11})$	55
$2.M_{12}$	32	H^*	64
	44	P_3	27
J_1	56, 76, 77	$N_G(P_7)$	42
A_9	35, 42, 84	$N_G(P_7)$	42
$2.A_9$	48, 56	$N_G(P_7)$	84
M_{22}	45, 55	$N_G(P_5)$	20
	99	$N_G(P_{11})$	55
$2.M_{22}$	56	$N_G(P_5)$	40
$3.M_{22}$	45	$N_G(P_5)$	60
	99	P_2	128
$4.M_{22}$	56	$N_G(P_3)$	288
J_2	63, 70, 90	$N_G(P_7)$	42
	36	P_5	25
$2.J_2$	50, 56, 64	$N_G(P_7)$	84
	84	$N_G(P_3)$	432
$S_4(4)$	34, 51	P_5	25
	50	$N_G(P_3)$	72
	85	P_2	256
$S_6(2)$	35, 56, 70, 84	$N_G(P_7)$	42
$2.S_6(2)$	48, 64	P_3	81
A_{10}	42	P_5	25
	90	P_3	81
$2.A_{10}$	48, 64	P_3	81
$U_4(3)$	35	$N_G(P_5)$	20
	90	$C_G(P_3')$	81
$2.U_4(3)$	56	$N_G(P_5)$	40
	70	$C_G(P_3')$	162
$3_2.U_4(3)$	36, 45	$N_G(P_5)$	60
$6_1.U_4(3)$	84	$C_G(P_3')$	486
$12_1.U_4(3)$	84	$C_G(P_3')$	972
$6_2.U_4(3)$	90	$N_G(P_5)$	120
$12_2.U_4(3)$	36	$N_G(P_5)$	240
$G_2(3)$	64, 78, 91	$N_G(P_7)$	42
$S_4(5)$	40	$N_G(P_{13})$	52
	65, 78	P_2	64
	90	$C_G(P_5')$	125
$2.S_4(5)$	52	$N_G(P_{13})$	104

(Table 1. continued)

$L_4(3)$	39	P_3'	27
	52	$N_G(P_5)$	80
	65	$N_G(P_{13})$	39
	90	$C_G(P_3')$	81
$2.L_4(3)$	40	P_3'	27
M_{23}	45	P_{23}	23
$U_5(2)$	44, 55, 66	$N_G(P_{11})$	55
${}^2F_4(2)'$	78	$N_G(P_{13})$	78
HS	77	$N_G(P_{11})$	55
$2.HS$	56	$N_G(P_7)$	84
J_3	85	P_3'	27
$O_8^+(2)$	35, 50, 84	$N_G(P_7)$	42
$2.O_8^+(2)$	56	$N_G(P_7)$	84
$O_8^-(2)$	34, 84	P_3	81
	51	$N_G(P_{17})$	68
${}^3D_4(2)$	52	P_7	49
M_{24}	45	P_{23}	23
$G_2(4)$	65	$N_G(P_{13})$	78
	78	P_2'	256
He	51	P_3	27
$O_7(3)$	78	P_2'	32
	91	$N_G(P_{13})$	78
$S_6(3)$	78	$N_G(P_7)$	84
	91	$C_G(P_3')$	27
$2.U_6(2)$	56	P_3	729
$S_8(2)$	35	$C_G(P_2')$	16
	51, 85	$N_G(P_{17})$	136
$3.Suz$	66, 78	P_3'	243
$2.F_4(2)$	52	P_7	49

Among the remaining cases, not covered in Table 1, a direct search showed that there were χ-subgroups of a more complicated form. These are summarized in Table 2, and the χ-subgroups are described in more detail as follows.

Table 2. The χ-subgroups of exceptional characters

Group	$\chi(1)$	Size of χ-subgroup
$6.L_3(4)$	36, 42, 60	1008
	90	60
$12_1.L_3(4)$	48	720
$12_2.L_3(4)$	36	288
	48, 60, 84	720
$6.M_{22}$	66	330
Fi_{22}	78	32

If $G = 6.L_3(4)$, then G has a maximal subgroup H of order 1008 such that $H/Z(G) \cong L_2(7)$. The subgroup H is a χ-subgroup for irreducible characters χ of G of degrees 36, 42 and 60. For $\chi(1) = 90$ the group G has a maximal subgroup M such that $M/Z(G) \cong A_6$. Then M contains a normal subgroup H of order 60 such that $H \cong A_5$ and is a χ-subgroup.

If $G = 12_1.L_3(4)$, then G has a maximal subgroup M such that $M/Z(G) \cong A_6$. The subgroup M contains a maximal subgroup H of order 720 such that $H/Z(G) \cong A_5$ and is a χ-subgroup for $\chi(1) = 48$.

If $G = 12_2.L_3(4)$, then G has a maximal subgroup M such that $M/Z(G) \cong L_2(7)$. The subgroup M contains a maximal subgroup H of order 288 such that $H/Z(G) \cong S_4$ and is a χ-subgroup for $\chi(1) = 36$. Also the group G has a maximal subgroup M such that $M/Z(G) \cong A_6$. Then M contains a maximal subgroup H of order 720 such that $H/Z(G) \cong A_5$ and is a χ-subgroup for $\chi(1) \in \{48, 60, 84\}$.

If $G = 6.M_{22}$, then G has a maximal subgroup M such that $M/Z(G) \cong L_2(11)$. The subgroup M contains a maximal subgroup H of order 330 which is a χ-subgroup for both of the characters of degree 66.

Finally if $G = Fi_{22}$ and χ is an irreducible character of G of degree 78, then G contains a maximal subgroup $M \cong O_7(3)$ such that χ_M is irreducible. Now using Table 1 we see that the derived subgroup of order 32 of a Sylow 2-subgroup of M is a χ-subgroup.

3. Maximal subgroups

There are three groups in [1] with characters of degree $\leqslant 100$ for which we do not know whether there are χ-subgroups in these cases. These were the covering groups $2.A_{12}$, $2.A_{13}$ and $6.A_7$. We shall consider $6.A_7$ in the next section. For the first two groups and each of their characters χ of degree $\leqslant 100$ we shall find a maximal subgroup M such that χ_M is irreducible (see the Introduction). To do this we use the technique described in [3].

If $G = A_{12}$, then the covering group $2.G$ has an irreducible character χ of degree 32, and G has a maximal subgroup $M \cong (A_6 \times A_6) : 2^2$ of index 462 such that $\chi_{\tilde{M}}$ is irreducible. If we take H a subgroup of G isomorphic to A_6 and K the normalizer of H in G, then M is the normalizer of K in G (i.e., $M = N_G(N_G(H))$).

For $G = A_{13}$, the covering group $2.G$ has two irreducible characters χ of degree 32. In this case G has a maximal subgroup M of index 13 such that $M \cong A_{12}$ and $\chi_{\tilde{M}}$ is irreducible.

4. Exceptional characters of $6.A_7$

Finally we show that if $\tilde{G} = 6.A_7$ and χ is one of the two faithful irreducible characters of $\tilde{G}$ of degree 36, then $\chi_{\tilde{M}}$ is not irreducible for any maximal subgroup $\tilde{M}$ of $\tilde{G}$, and $\tilde{G}$ has no χ-subgroups.

If $\tilde{M}_i$ is a maximal subgroup of $\tilde{G}$, then $\tilde{M}_i/Z(\tilde{G})$ is isomorphic to one of the groups $(A_4 \times 3) : 2$, $\mathrm{PSL}(2,7)$, S_5 and A_6 for $i = 1, 2, 3$ and 4, respectively (see [1]). It is straightforward to check that $\chi_{\tilde{M}_i}$ is not irreducible for any maximal subgroup $\tilde{M}_i$ of $\tilde{G}$.

A direct method to show that $\tilde{G}$ has no χ-subgroups, is to search among all non-conjugate subgroups of $\tilde{G}$ to see that the restriction of χ to all of these subgroups has no linear constituents of multiplicity 1.

Another approach is the restriction of χ to maximal subgroups, as follows. Since $\chi_{\tilde{M}_1} = 3\theta_1 + 3\theta_2$ where θ_1 and θ_2 are irreducible characters of $\tilde{M}_1$ of degree 6, $\tilde{M}_1$ does not contain any χ-subgroups. Furthermore, $\chi_{\tilde{M}_i}$ has no linear constituents for $i = 2, 3, 4$. Now we restrict χ to the maximal subgroups $\tilde{M}_i$ for $i = 2, 3, 4$.

The group $\tilde{M}_3$ has 4 non-conjugate maximal subgroups. Let $\tilde{M}_{34}$ be a maximal subgroup of $\tilde{M}_3$ of index 3. Then $\tilde{M}_{34} \cong \mathrm{SL}(2, 7)$ and it is the only maximal subgroup of $\tilde{M}_3$ such that $\chi_{\tilde{M}_{34}}$ has some constituents of multiplicity 1. Since these constituents are not linear, $\tilde{M}_{34}$ is not a χ-subgroup but, if $\tilde{M}_3$ contains a χ-subgroup, it must be a subgroup of $\tilde{M}_{34}$. Straightforward computations show that the multiplicities of the constituents of the restriction of χ to all maximal subgroups of $\tilde{M}_{34}$ are larger than 1. This implies that $\tilde{M}_{34}$ and, in turn, $\tilde{M}_3$ does not contain any χ-subgroups.

A similar argument holds for $\tilde{M}_2$ and $\tilde{M}_4$. The group $\tilde{M}_2$ has two maximal subgroups $\tilde{M}_{24}$ and $\tilde{M}_{25}$ of indices 2 and 3, respectively, such that $\tilde{M}_{24}/Z(\tilde{G}) \cong S_5$ and $\tilde{M}_{25}/Z(\tilde{G}) \cong A_5$. Also $\tilde{M}_4$ contains two maximal subgroups $\tilde{M}_{44}$ and $\tilde{M}_{45}$ of index 6 such that $\tilde{M}_{44}/Z(\tilde{G}) \cong \tilde{M}_{45}/Z(\tilde{G}) \cong A_5$. These are the only maximal subgroups of $\tilde{M}_2$ and $\tilde{M}_4$ such that the restriction of χ has some constituents of multiplicity 1. Since none of these constituents are linear, if $\tilde{M}_2$ or $\tilde{M}_4$ contains a χ-subgroup, it needs to be a subgroup of one of these maximal subgroups. The groups $\tilde{M}_{24}$, $\tilde{M}_{25}$, $\tilde{M}_{44}$ and $\tilde{M}_{45}$ contain only one maximal subgroup $\tilde{H} \cong 2.A_5$ such that $\chi_{\tilde{H}}$ has some constituents of multiplicity 1. It is easy to see that the multiplicities of the constituents of the restriction of χ to all of the maximal subgroups of $\tilde{H}$ are bigger than 1. Therefore all the maximal subgroups of $\tilde{M}_2$ and $\tilde{M}_4$ and, in turn, $\tilde{M}_2$ and $\tilde{M}_4$ have no χ-subgroups. This proves that $\tilde{G}$ has no χ-subgroups.

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