POWER SERIES FOR INVERSE JACOBIAN ELLIPTIC FUNCTIONS

B. C. CARLSON

ABSTRACT. The 12 inverse Jacobian elliptic functions are expanded in power
series by using properties of the symmetric elliptic integral of the first kind.
Suitable notation allows three series to include all 12 cases, three of which have
been given previously. All coefficients are polynomials in the modulus k that
are homogeneous variants of Legendre polynomials. The four series in each of
three subsets have the same coefficients in terms of k.

1. INTRODUCTION

Only partial information about power series for inverse Jacobian elliptic functions
seems to be available if we ask for series in which all coefficients are algebraic
functions of the modulus k. If $z = \text{sn}(u, k)$, the inverse function is $u = \text{arcsn}(z, k)$,
which was expanded in power series by Kelisky [Ke, (4)]:

$$(1.1) \qquad \text{arcsn}(z, k) = \sum_{n=0}^{\infty} k^n P_n \left(\frac{1}{2} \left(k + \frac{1}{k} \right) \right) \frac{z^{2n+1}}{2n+1}, \quad |z| < 1, \quad 0 < k < 1,$$

where P_n is the Legendre polynomial of degree n. Equation (9) in the same paper
gives a similar series for $\text{arcsc}(z, k)$. Lomont and Brillhart [LB, p. 112] cite Kelisky's
paper and verify (1.1). [Wo] shows a series for each of the three functions $\text{arcsp}(z, k)$,
p = c, d, or n, with coefficients containing a $_2F_1$ polynomial. I have not seen series
of the kind specified above for the remaining nine inverse functions. Tölke [Tol,
(884), (886), (888)] gives the first four terms for each of the six functions included
in arcsp and arcps, and three terms (but see the Remarks at the end of this paper)
for each of the six functions included in arcqp, where q, p $\in$ {c, d, n}, q $\neq$ p. The
last two references do not mention Legendre polynomials.

This paper uses known connections between the symmetric elliptic integral R_F
and the inverse Jacobian functions to derive their infinite power series with "homo-
geneous Legendre polynomials" as coefficients. We shall show that the coefficients
for arcsp, arcps, and two cases of arcqp are the same for a given choice of p = c, d,
or n. For example, if p = n, this is true for arcsn, arcns, arccn, and arcdn.

2. Derivation of series

The symmetric incomplete elliptic integral of the first kind, defined by

$$(2.1) \qquad R_F(w_1, w_2, w_3) = \frac{1}{2} \int_0^\infty \prod_{j=1}^3 (t + w_j)^{-1/2} dt, \quad |\mathrm{ph}(w_j)| < \pi,$$

is symmetric and homogeneous of degree $-1/2$ in the w's. With $\{p,q,r\} = \{c,d,n\}$ the 12 inverse Jacobian elliptic functions are related to R_F by

$$(2.2) \qquad \mathrm{arcsp}(z, k) = z R_F(1 + \Delta(q, p)z^2, 1 + \Delta(r, p)z^2, 1),$$

$$(2.3) \qquad \mathrm{arcps}(z, k) = z^{-1} R_F(1 + \Delta(q, p)z^{-2}, 1 + \Delta(r, p)z^{-2}, 1),$$

$$\mathrm{arcqp}(z, k) = \xi R_F(z^2, 1 + \Delta(r, p)\xi^2, 1), \quad \xi = \left(\frac{1 - z^2}{\Delta(p, q)}\right)^{1/2}$$

$$(2.4) \qquad \qquad = \xi R_F(1 + \Delta(q, p)\xi^2, 1 + \Delta(r, p)\xi^2, 1),$$

where

$$(2.5) \qquad \Delta(p, q) = \mathrm{ps}^2(u, k) - \mathrm{qs}^2(u, k) = -\Delta(q, p)$$

and therefore

$$(2.6) \qquad \Delta(d, c) = k'^2 = 1 - k^2, \quad \Delta(n, d) = k^2, \quad \Delta(n, c) = 1.$$

In (2.2)-(2.4) the third variable of R_F was chosen to be 1 to match (2.16) below. With the help of homogeneity and symmetry, (2.2), (2.3), and (2.4) are easily verified by putting $z = \mathrm{sp}(u, k)$, $z = \mathrm{ps}(u, k)$, and $z = \mathrm{qp}(u, k)$, respectively, to recover in each case the fundamental relation $u = R_F(\mathrm{ps}^2(u, k), \mathrm{qs}^2(u, k), \mathrm{rs}^2(u, k))$ [cdn, (1.8)]. Remark 1 at the end of this section may be helpful in the case of (2.4).

Theorem 1. *Define*

$$(2.7) \qquad R_n(x, y) = R_n(\tfrac{1}{2}, \tfrac{1}{2}; x, y) = \sum_{j=0}^n \frac{(\tfrac{1}{2})_j (\tfrac{1}{2})_{n-j}}{j!(n-j)!} x^j y^{n-j},$$

where $(\tfrac{1}{2})_0 = 1$, $(\tfrac{1}{2})_m = \tfrac{1}{2}\tfrac{3}{2}\tfrac{5}{2} \cdots (m - \tfrac{1}{2})$, $m = 1, 2, 3, \ldots$. *With* $\{p, q, r\} = \{c, d, n\}$,

$$(2.8) \qquad \mathrm{arcsp}(z, k) = \sum_{n=0}^\infty R_n(\Delta(p, q), \Delta(p, r)) \frac{z^{2n+1}}{2n+1},$$

$$(2.9) \qquad \mathrm{arcps}(z, k) = \sum_{n=0}^\infty R_n(\Delta(p, q), \Delta(p, r)) \frac{(1/z)^{2n+1}}{2n+1},$$

$$(2.10) \qquad \mathrm{arcpq}(z, k) = \sum_{n=0}^\infty R_n(\Delta(p, q), \Delta(p, r)) \frac{\xi^{2n+1}}{2n+1}, \quad \xi = \left(\frac{1 - z^2}{\Delta(p, q)}\right)^{1/2},$$

where all variables may be real or complex. Sufficient conditions for absolute convergence of the series (2.8), (2.9), and (2.10) are given by (2.11), (2.12), and (2.13), respectively:

$$(2.11) \qquad \max\{|\Delta(p, q)|, |\Delta(p, r)|\} |z|^2 < 1,$$

$$(2.12) \qquad \max\{|\Delta(p, q)|, |\Delta(p, r)|\} < |z|^2,$$

$$(2.13) \qquad \max\{|\Delta(p, q)|, |\Delta(p, r)|\} |\xi|^2 < 1.$$

For a proof of the stronger statement that each inequality involves the radius of convergence of the corresponding series, see Remark 2 at the end of this section.

Proof. Since the multivariate hypergeometric R-function [Ca, §5.9] includes R_F as the special case [Ca, (9.2-1)]

$$(2.14) \qquad R_F(w_1, w_2, w_3) = R_{-1/2}(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; w_1, w_2, w_3), \quad |\mathrm{ph}(w_j)| < \pi,$$

we can find series expansions of (2.2)-(2.4) from a special case of [Ca, (5.9-4)],

$$(2.15) \quad R_F(1 - v_1, 1 - v_2, 1 - v_3) = \sum_{n=0}^{\infty} \frac{(\tfrac{1}{2})_n}{n!} R_n(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; v_1, v_2, v_3), \quad |v_j| < 1,$$

where the polynomial R_n [Ca, §6.2] is symmetric and homogeneous of degree n in the v's. Putting $v_3 = 0$ and using [Ca, (6.2-6)] with $c_k = \tfrac{3}{2}$ and $c_p = \tfrac{2}{2}$, we have

$$(2.16) \qquad R_F(1 - v_1, 1 - v_2, 1) = \sum_{n=0}^{\infty} \frac{1}{2n+1} R_n(\tfrac{1}{2}, \tfrac{1}{2}; v_1, v_2), \quad |v_1|, |v_2| < 1,$$

and (3.9) below shows that the series converges absolutely. In (2.15) and (2.16) the v's lie in the open unit disk with center 0 in the complex plane. The explicit form of R_n with two variables is a special case of [Ca, (6.2-3)] and is reproduced in (2.7) with the parameters $\tfrac{1}{2}, \tfrac{1}{2}$ suppressed on the left side for brevity. The relation of (2.7) to the Legendre polynomial P_n will be discussed in §3.

Applying (2.16) to the right side of (2.2)-(2.4), we find

$$(2.17) \quad \mathrm{arcsp}(z, k) = z \sum_{n=0}^{\infty} \frac{1}{2n+1} R_n(\Delta(\mathrm{p}, \mathrm{q}) z^2, \Delta(\mathrm{p}, \mathrm{r}) z^2),$$

$$(2.18) \quad \mathrm{arcps}(z, k) = z^{-1} \sum_{n=0}^{\infty} \frac{1}{2n+1} R_n(\Delta(\mathrm{p}, \mathrm{q}) z^{-2}, \Delta(\mathrm{p}, \mathrm{r}) z^{-2}),$$

$$(2.19) \quad \mathrm{arcqp}(z, k) = \xi \sum_{n=0}^{\infty} \frac{1}{2n+1} R_n(\Delta(\mathrm{p}, \mathrm{q}) \xi^2, \Delta(\mathrm{p}, \mathrm{r}) \xi^2), \quad \xi = \left(\frac{1 - z^2}{\Delta(\mathrm{p}, \mathrm{q})} \right)^{1/2}.$$

According to (2.16) these series converge absolutely if each variable of R_n lies inside the open unit disk, as (2.11)-(2.13) indeed require. Equations (2.8)-(2.10) follow from (2.17)-(2.19) by the homogeneity of R_n. $\qquad \square$

Because $\{\mathrm{p,q,r}\} = \{\mathrm{c,d,n}\}$ the choice of p determines q and r except for their order. For $R_n(\Delta(\mathrm{p}, \mathrm{q}), \Delta(\mathrm{p}, \mathrm{r}))$, which is symmetric in its two variables, the order of q and r is immaterial, and the coefficients in (2.8)-(2.10) depend only on the choice of p = c, d, or n. For each choice there are four inverse functions with the same coefficients: one case of arcsp, one case of arcps, and two cases of arcqp (because q appears also in ξ).

First choosing p=n, we find from (2.6) that the variables of R_n are 1 and k^2. Then (2.8) becomes

$$(2.20) \qquad \mathrm{arcsn}(z, k) = \sum_{n=0}^{\infty} R_n(1, k^2) \frac{z^{2n+1}}{2n+1}, \quad \max\{1, |k^2|\} |z|^2 < 1.$$

Using (2.9) and (2.10) we immediately find the series for arcns, arccn, and arcdn by replacing z on the right-hand side of (2.20) by $1/z$, $(1 - z^2)^{1/2}$, and $\left(\frac{1 - z^2}{k^2} \right)^{1/2}$, respectively.

Choosing p=d, we find similarly that (2.8) becomes

$$(2.21) \quad \text{arcsd}(z,k) = \sum_{n=0}^{\infty} R_n(1-k^2,-k^2)\frac{z^{2n+1}}{2n+1}, \quad \max\{|1-k^2|,|k^2|\}\,|z|^2 < 1.$$

The series for arcds, arccd, and arcnd are found by replacing z on the right-hand side of (2.21) by $1/z$, $\left(\frac{1-z^2}{1-k^2}\right)^{1/2}$, and $\left(\frac{z^2-1}{k^2}\right)^{1/2}$, respectively.

Finally, choosing p=c, we find that (2.8) becomes

$$(2.22) \qquad \text{arcsc}(z,k) = \sum_{n=0}^{\infty} R_n(k^2-1,-1)\frac{z^{2n+1}}{2n+1}, \quad \max\{|1-k^2|,1\}\,|z|^2 < 1.$$

The series for arccs, arcdc, and arcnc are found by replacing z on the right-hand side of (2.22) by $1/z$, $\left(\frac{z^2-1}{1-k^2}\right)^{1/2}$, and $(z^2-1)^{1/2}$, respectively.

Remark 1. To understand why the coefficients in (2.8) and (2.9) are the same, note that $u = \text{arcps}(z,k)$ implies $z = \text{ps}(u,k)$, $1/z = \text{sp}(u,k)$, and $u = \text{arcsp}(1/z,k)$. Hence $\text{arcps}(z,k) = \text{arcsp}(1/z,k)$. A similar argument applies to (2.8) and (2.10) because

$$\xi^2 = \frac{1-\text{qp}^2}{\Delta(\text{p},\text{q})} = \frac{1-(\text{qs}/\text{ps})^2}{\text{ps}^2-\text{qs}^2} = 1/\text{ps}^2 = \text{sp}^2.$$

Thus the quantity with exponent $2n+1$ in (2.8), (2.9), or (2.10) is in each case sp, which is 0 at $u = 0$, as is the sum u of each series.

Remark 2. The right-hand sides of (2.2)-(2.4) have a common form

$$(2.23) \qquad\qquad w R_F(1-v_1, 1-v_2, 1), \quad |\text{ph}(1-v_j)| < \pi,$$

where $w = z, z^{-1}, \xi$ in (2.2), (2.3), (2.4), respectively, and $v_1 = \Delta(\text{p},\text{q})w^2$, $v_2 = \Delta(\text{p},\text{r})w^2$. The condition for (2.23), taken from (2.1), states that $1 - v_j$ lies in the complex plane cut along the nonpositive real axis, and hence v_j lies in the complex plane cut along the positive real axis from 1 to ∞. In the w^2-plane this entails a cut rotated away from the positive real axis so that it extends with constant phase from $1/\Delta(\text{p},\text{q})$ to ∞ and a similar cut from $1/\Delta(\text{p},\text{r})$ to ∞. Except on these cuts, (2.23) is analytic in w^2. In particular it is analytic inside the circle with center $w^2 = 0$ that passes through the nearer of the two points $1/\Delta(\text{p},\text{q})$ and $1/\Delta(\text{p},\text{r})$, one or both of which may be complex (see (2.6)). That circle is the circle of convergence of the power series (2.8)-(2.10) in w^2 obtained from (2.17)-(2.19) by using the homogeneity of R_n. The radius of the circle is the radius of convergence

$$(2.24) \qquad \rho = \min\{|1/\Delta(\text{p},\text{q})|, |1/\Delta(\text{p},\text{r})|\} = \frac{1}{\max\{|\Delta(\text{p},\text{q})|, |\Delta(\text{p},\text{r})|\}}.$$

Thus the inequalities (2.11)-(2.13), previously described as sufficient, are now seen to be the best possible since they require only that $|w^2|$ be less than ρ.

3. Homogeneous Legendre polynomials

The polynomials R_n have a generating relation that is a special case of [Ca, (6.6-1)]:

$$(3.1) \qquad (1-t(x+y)+t^2xy)^{-1/2} = (1-tx)^{-1/2}(1-ty)^{-1/2} = \sum_{n=0}^{\infty} t^n R_n(x,y).$$

In terms of the arithmetic and geometric means

$$(3.2) \qquad \alpha = \frac{x+y}{2}, \quad \gamma = \sqrt{xy},$$

the generating function is $(1 - 2\alpha t + \gamma^2 t^2)^{-1/2}$, and by putting $t = s/\gamma$ we get

$$(3.3) \qquad \left(1 - 2\frac{\alpha}{\gamma} s + s^2\right)^{-1/2} = \sum_{n=0}^{\infty} \frac{s^n}{\gamma^n} R_n(x,y).$$

Comparison with [Ca, (6.7-3)] shows that R_n is related to the Legendre polynomial P_n by

$$(3.4) \qquad R_n(x,y) = \gamma^n P_n\left(\frac{\alpha}{\gamma}\right)$$

and reduces to $P_n(\alpha)$ if $xy = 1$. This close relationship suggests thinking of R_n as a homogeneous Legendre polynomial in the variables α and γ (see (3.5) below). An example of (3.4) is seen in arcsn, where (2.8) contains the coefficient $R_n(\Delta(\mathrm{n,c}), \Delta(n,\mathrm{d})) = R_n(1, k^2)$. Then $\alpha = (1+k^2)/2$, $\gamma = k$, and the right-hand side of (3.4) is $k^n P_n\left(\frac{1+k^2}{2k}\right)$, the coefficient in (1.1). (Incidentally, in [Ke, (2)] a function $R_n(x)$ is defined that can be shown to be $R_n(1, x^2)$ in the notation used here.)

Why, then, use R_n instead of P_n in Theorem 1? If x and y are real with opposite signs, all quantities in (2.7) are real, but γ and the variable of P_n are pure imaginary. This unnecessary complication occurs, for example, in arcsd if $0 < k^2 < 1$; it is dealt with in [Ke, (5),(6)] by introducing a second polynomial S_n that differs from P_n by changing minus signs to plus signs. The polynomial $R_n(x, y)$ suffices for both signs of xy, and its symmetry was important in showing that the four series in each of three subsets have the same coefficients.

To evaluate R_n as a polynomial in x and y, we can use (2.7), but if the coefficients c_{nj} in $P_n(\alpha) = \sum_{j=0}^{[n/2]} c_{nj}\alpha^{n-2j}$ are at hand, we can use (3.4) to find a more concise expression as a homogeneous Legendre polynomial $h_n(\alpha, \gamma)$ such that $h_n(\alpha, 1) = P_n(\alpha)$:

$$(3.5) \qquad R_n(x, y) = h_n(\alpha, \gamma) = \sum_{j=0}^{[n/2]} c_{nj}\alpha^{n-2j}\gamma^{2j}.$$

Thus we have

$$R_0(x,y) = h_0(\alpha, \gamma) = 1,$$
$$R_1(x,y) = h_1(\alpha, \gamma) = \alpha = \tfrac{1}{2}(x+y),$$
$$(3.6) \qquad R_2(x,y) = h_2(\alpha, \gamma) = \tfrac{1}{2}(3\alpha^2 - \gamma^2) = \tfrac{1}{8}\left(3x^2 + 2xy + 3y^2\right),$$
$$R_3(x,y) = h_3(\alpha, \gamma) = \tfrac{1}{2}(5\alpha^3 - 3\alpha\gamma^2) = \tfrac{1}{16}\left(5x^3 + 3x^2 y + 3xy^2 + 5y^3\right),$$
$$R_4(x,y) = h_4(\alpha, \gamma) = \tfrac{1}{8}(35\alpha^4 - 30\alpha^2\gamma^2 + 3\gamma^4)$$
$$= \tfrac{1}{128}\left(35x^4 + 20x^3 y + 18x^2 y^2 + 20xy^3 + 35y^4\right).$$

A recurrence relation that is a special case of [Ca, (5.9-25)] may be more useful for automatic computation,

$$(3.7) \qquad (n+1)h_{n+1} = (2n+1)\alpha\, h_n - n\gamma^2\, h_{n-1},$$

showing by induction that h_n is divisible by α for odd n.

Note the special cases

$$(3.8) \qquad\qquad R_n(x, 0) = \frac{(\frac{1}{2})_n}{n!}\, x^n, \quad R_n(x, x) = x^n.$$

The first equation is obvious from (2.7). The second follows from the nature of R_n as a Dirichlet average of x^n [Ca, (6.1-1)] or alternatively from a special case of Vandermonde's theorem [Ca, (2.3-1)], which also serves to establish the bound

$$(3.9) \qquad\qquad |R_n(x, y)| \leq [\max\{|x|, |y|\}]^n.$$

This bound provides an elementary route to (2.11)-(2.13) instead of using the convergence condition for [Ca, (5.9-4)].

The two parts of (3.8) serve to check the reductions of (2.20)-(2.22) to power series for inverse circular functions if $k = 0$ or inverse hyperbolic functions if $k = 1$ [AS, 16.6, 4.4, 4.6].

4. Leading terms

The coefficients for the leading terms of (2.20)-(2.22) can be shown as explicit polynomials in k^2 by using (3.6) or (3.7):

$$(4.1) \qquad \begin{aligned} \operatorname{arcsn}(z, k) &= z + \tfrac{1}{6}(1 + k^2)z^3 + \tfrac{1}{40}(3 + 2k^2 + 3k^4)z^5 \\ &\quad + \tfrac{1}{112}(5 + 3k^2 + 3k^4 + 5k^6)z^7 + \ldots, \end{aligned}$$

$$(4.2) \qquad \begin{aligned} \operatorname{arcsd}(z, k) &= z + \tfrac{1}{6}(1 - 2k^2)z^3 + \tfrac{1}{40}(3 - 8k^2 + 8k^4)z^5 \\ &\quad + \tfrac{1}{112}(5 - 18k^2 + 24k^4 - 16k^6)z^7 + \ldots, \end{aligned}$$

$$(4.3) \qquad \begin{aligned} \operatorname{arcsc}(z, k) &= z + \tfrac{1}{6}(k^2 - 2)z^3 + \tfrac{1}{40}(3k^4 - 8k^2 + 8)z^5 \\ &\quad + \tfrac{1}{112}(5k^6 - 18k^4 + 24k^2 - 16)z^7 + \ldots. \end{aligned}$$

Corresponding terms for the other nine functions are obtained by replacing z as prescribed after each of (2.20)-(2.22).

Remarks. Recurrence relations equivalent to the special case of (3.7) with $\alpha = (1 + k^2)/2$ and $\gamma = k$ were used by [Wr, (1.4), Table I] and [LB, (8.4), Table 8.1] to list $R_n(k^2, 1)$ for $0 \leq n \leq 7$ and $0 \leq n \leq 9$, respectively. [LB] shows the connection with arcsn and the divisibility mentioned after (3.7).

The leading terms given by [Tol, (888)] for the six cases of arcqp were mentioned in the Introduction. They involve powers of $(1 - z)^{1/2}$ (in present notation) instead of $(1 - z^2)^{1/2}$. They are obtained from the leading terms of $1 - z = 1 - \operatorname{qp}(u, k)$ as power series in u^2 [Tol, (816)-(818)]. Reversion of series to get the leading terms of u^2 as power series in $1 - z$ is followed by taking the square root to get leading terms of $u = \operatorname{arcqp}(z, k)$ as series of odd powers of $(1 - z)^{1/2}$. Each of these two operations would make it increasingly difficult to get higher terms of the final series even if higher terms of the initial series for $1 - z$ were readily available.

The equations for arccn given by [Ke, (6)] and [Wo] follow from the power series for arcsd and the relation $\operatorname{arccn}(z, k) = K(k) - \operatorname{arcsd}(z/k')$ [La, (3.2.18)].

Acknowledgment

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Ames Laboratory and Department of Mathematics, Iowa State University, Ames, Iowa 50011-3020

E-mail address: bcarlson@scl.ameslab.gov