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PARITY-REGULAR STEINHAUS GRAPHS

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ABSTRACT. Steinhaus graphs on n vertices are certain simple graphs in bijective correspondence with binary $\{0,1\}$-sequences of length $n-1$. A conjecture of Dymacek in 1979 states that the only nontrivial regular Steinhaus graphs are those corresponding to the periodic binary sequences 110...110 of any length $n-1=3m$. By an exhaustive search the conjecture was known to hold up to 25 vertices. We report here that it remains true up to 117 vertices. This is achieved by considering the weaker notion of *parity-regular* Steinhaus graphs, where all vertex degrees have the same parity. We show that these graphs can be parametrized by an $\mathbb{F}_2$-vector space of dimension approximately $n/3$ and thus constitute an efficiently describable domain where true regular Steinhaus graphs can be searched by computer.

1. INTRODUCTION

Let $s = a_1 \dots a_{n-1}$ be a binary sequence of length $n-1$ with entries a_i in the 2-element field $\mathbb{F}_2 = \{0,1\}$. The Steinhaus *graph* associated with s is the simple graph $G(s)$ on the vertex set $\{0,1,\dots,n-1\}$ whose adjacency matrix $M(s) = (a_{i,j}) \in \mathcal{M}_n(\mathbb{F}_2)$, with indices $0 \le i,j \le n-1$, is defined as follows:

1. $a_{i,i} = 0$ for $0 \le i \le n-1$,
2. $a_{0,i} = a_i$ for $1 \le i \le n-1$,
3. $a_{i,j} = a_{i-1,j-1} + a_{i-1,j}$ for $1 \le i < j \le n-1$,
4. $a_{j,i} = a_{i,j}$ for $0 \le i < j \le n-1$.

Note that the first row of $M(s)$ is the vector $(0, a_1, \dots, a_{n-1})$, and that each subsequent row is determined, in its strict upper triangular part, by its predecessor using rule 3. For example, if $s = a_1 \dots a_4$, then

$$M(s) = \begin{pmatrix} 0 & a_1 & a_2 & a_3 & a_4 \\ a_1 & 0 & a_1+a_2 & a_2+a_3 & a_3+a_4 \\ a_2 & a_1+a_2 & 0 & a_1+a_3 & a_2+a_4 \\ a_3 & a_2+a_3 & a_1+a_3 & 0 & a_1+a_2+a_3+a_4 \\ a_4 & a_3+a_4 & a_2+a_4 & a_1+a_2+a_3+a_4 & 0 \end{pmatrix}.$$

The strict upper triangular part of $M(s)$ is known as the Steinhaus *triangle* associated with s, first defined by Steinhaus in [7]. We say that the graph $G(s)$ is *generated* by the binary sequence s. Steinhaus graphs (in fact, their complements) were introduced by John Molluzzo in [6]. It can easily be shown [3] that all Steinhaus graphs are connected, except those generated by the constant sequences 0...0.

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Here are two easy examples of Steinhaus graphs: if $s = \mathtt{1}$ of length 1, then $G(s) = K_2$, the one-edge graph on 2 vertices; if $s = \mathtt{0}\ldots\mathtt{0}$ of length $n-1$ (also denoted $s = \mathtt{0}^{n-1}$), then $G(s)$ is the zero-edge graph on n vertices. These particular instances are regular graphs.

A general problem on Steinhaus graphs is that of characterizing those satisfying a given graph property. For instance, all bipartite and all planar Steinhaus graphs are now known [2, 4]. As for regular Steinhaus graphs, it is believed that the currently known ones exhaust them all. Besides the trivial instances mentioned above, namely $G(\mathtt{1}) = K_2$ and $G(\mathtt{0}^{n-1})$, there is also a nontrivial infinite family of regular Steinhaus graphs on $n = 3m+1$ vertices, generated by the periodic sequence $s = \mathtt{110}\ldots\mathtt{110}$ of length $3m$ ($m \geq 1$). It is an amusing exercise to check that $G((\mathtt{110})^m)$ is indeed regular of degree $2m$.

Dymacek first conjectured in 1979 that there are no other regular Steinhaus graphs besides those described above [3]. Bailey and Dymacek verified this conjecture for $n \leq 25$ vertices in 1988 (see [1]). In this paper we extend the verification as follows.

Computational Result 1. There are no other regular Steinhaus graphs on $n \leq 117$ vertices besides $G(\mathtt{1})$, $G(\mathtt{0}^{n-1})$ and $G((\mathtt{110})^m)$.

This is achieved by computer-searching regular Steinhaus graphs inside the larger class of parity-regular Steinhaus graphs, which, for n vertices, are proved to depend on at most $\lceil n/3 \rceil$ binary parameters.

2. Parity-regular graphs

We shall say that a locally finite graph is *parity-regular* if its vertex degrees all have the same parity. For instance, regular graphs are parity-regular. We say that a parity-regular graph G is of *odd type* if its vertex degrees are all odd, and of *even type* otherwise. Note that finite parity-regular graphs of even type are also known as *even graphs*, or Eulerian *graphs* if connected.

It is well known that in a finite graph, the number of vertices of odd degree is even. This implies the following.

Lemma 1. *There are no parity-regular graphs of odd type with an odd number of vertices.*

Proof. If V and E denote the vertex set and the edge set, respectively, then $2|E| = \sum_{x\in V} deg(x) \equiv |V| \bmod 2$. □

We now show that parity-regular Steinhaus graphs on n vertices can be parametrized by approximately $n/3$ binary parameters. This fact greatly accelerates the search for true regular Steinhaus graphs.

Theorem 2. *Let $n \geq 2$ be an integer. The set of parity-regular Steinhaus graphs on n vertices is in bijection with a vector space over $\mathbb{F}_2$ of dimension $\lceil \frac{n}{3} \rceil - \varepsilon(n)$, where $\varepsilon(k) = 0$ or 1 and $\varepsilon(k) \equiv k \bmod 2$.*

Proof. Let $s = a_1 \ldots a_{n-1}$ be a binary sequence of length $n-1$, or equivalently let $s \in \mathbb{F}_2^{n-1}$. Let $L_i(s) \in \mathbb{F}_2$ denote the ith row coefficient sum of the associated matrix $M(s)$, for each $i = 0, \dots, n-1$. Thus, we have

$$M(s) \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} L_0(s) \\ \vdots \\ L_{n-1}(s) \end{pmatrix}$$

as matrices with coefficients in $\mathbb{F}_2$. For example, if $n = 5$, then $L_0(s) = a_1 + a_2 + a_3 + a_4$, $L_1(s) = a_4$, $L_2(s) = a_2 + a_3 + a_4$, $L_3(s) = a_4$ and $L_4(s) = a_1$. (Compare with the matrix shown in the Introduction.)

Let $R(n) = \{s \in \mathbb{F}_2^{n-1} : L_i(s) = L_j(s) \ \text{ for } \ 0 \le i, j \le n-1\}$. Then $R(n)$ is a subspace of $\mathbb{F}_2^{n-1}$, as the $L_i(s)$ are linear forms in s. It easily follows from the definitions that *the graph $G(s)$ is parity-regular if and only if $s \in R(n)$*. Thus, the subspace $R(n)$ corresponds bijectively to the set of parity-regular Steinhaus graphs on n vertices.

It is convenient to decompose $R(n)$ as the disjoint union $R_0(n) \cup R_1(n)$, where $R_\lambda(n) = \{s \in \mathbb{F}_2^{n-1} : L_i(s) = \lambda \ \text{ for } \ 0 \le i \le n-1\}$ for $\lambda \in \mathbb{F}_2 = \{0, 1\}$.

The set $R_0(n)$ is a subspace of $R(n)$, corresponding to even[1] Steinhaus graphs on n vertices. The set $R_1(n)$ corresponds to the odd type, and is either empty or an affine translate of $R_0(n)$. It follows from Lemma 1 that $R_1(n) = \emptyset$ if n is odd.

The dimension count of $R(n)$ is made possible by wonderful results in [3]. There, Dymacek shows (Theorem 3.5, page 409) that the set of even Steinhaus graphs on n vertices is of cardinality $2^{\lfloor \frac{n-1}{3} \rfloor}$. This is equivalent to the formula $\dim R_0(n) = \lfloor \frac{n-1}{3} \rfloor$. He further shows (Corollary 3.7, page 411) that for n even, there are also $2^{\lfloor \frac{n-1}{3} \rfloor}$ Steinhaus graphs on n vertices with all vertices of odd degree (i.e., parity-regular of odd type, in our terminology). Thus, for n even, the set $R_1(n)$ is an affine translate of the subspace $R_0(n)$. For n odd, we have already observed that $R_1(n)$ is empty. Hence, for the disjoint union $R(n) = R_0(n) \cup R_1(n)$, we get

$$|R(n)| = \begin{cases} 2^{\lfloor \frac{n-1}{3} \rfloor} & \text{if } n \text{ is odd,} \\ 2^{\lfloor \frac{n-1}{3} \rfloor + 1} & \text{if } n \text{ is even.} \end{cases}$$

This yields the formula $\dim R(n) = \lfloor \frac{n-1}{3} \rfloor + \varepsilon(n-1)$. It is a simple matter to see that $\lfloor \frac{n-1}{3} \rfloor + \varepsilon(n-1) = \lceil \frac{n}{3} \rceil - \varepsilon(n)$ for all $n \ge 1$, as witnessed by the following table:

n	$6k$	$6k+1$	$6k+2$	$6k+3$	$6k+4$	$6k+5$
$\lfloor \frac{n-1}{3} \rfloor + \varepsilon(n-1)$	$2k$	$2k$	$2k+1$	$2k$	$2k+2$	$2k+1$
$\lceil \frac{n}{3} \rceil - \varepsilon(n)$	$2k$	$2k$	$2k+1$	$2k$	$2k+2$	$2k+1$

As a last remark, note that $\dim R(n)$ satisfies the relation

$$\dim R(n+6) = \dim R(n) + 2$$

for all $n \ge 1$. □

[1] As observed earlier, all Steinhaus graphs are connected except the zero-edge ones. Thus, nontrivial even Steinhaus graphs are actually Eulerian.

3. Parametrizations up to 30 vertices

As an illustration of Theorem 2, here we give the parametrizations of all parity-regular Steinhaus graphs up to 30 vertices. For a given number n of vertices, *the sequence* $s = (x_1, \ldots, x_{n-1}) \in \mathbb{F}_2^{n-1}$ *yields a parity-regular* Steinhaus *graph if and only if* s *is of the form shown below.* Observe that the number of free parameters is everywhere $\lceil \frac{n}{3} \rceil - \varepsilon(n)$, as expected. The cases $n = 7$ and $n = 9$ are illustrated in the next section.

These parametrizations were obtained by a simple *Mathematica* program, in the spirit of the one given in [5]. One key function is `GroebnerBasis`, used to perform Gaussian elimination on the family of linear forms $L_i(s) - L_j(s)$ defining the set $R(n)$ in the proof of Theorem 2. Our program easily produces such parametrizations for n in the hundreds.

$n = 2:\ (x_1)$

$n = 3:\ (0, 0)$

$n = 4:\ (x_1, x_1, x_3)$

$n = 5:\ (0, x_2, x_2, 0)$

$n = 6:\ (x_1, x_1, x_1, x_1, x_5)$

$n = 7:\ (x_1, x_2, x_1 + x_2, x_2, x_2, 0)$

$n = 8:\ (x_1, x_2, x_2, x_1, x_2, x_2, x_7)$

$n = 9:\ (0, x_2, x_3, x_2, x_3, x_2, x_2, 0)$

$n = 10:\ (x_1, x_1, x_3, x_4, x_1, x_1 + x_3 + x_4, x_1, x_1, x_9)$

$n = 11:\ (x_1, x_2, x_1 + x_2, x_4, x_1 + x_2, x_2, x_1 + x_4, x_2, x_2, 0)$

$n = 12:\ (x_1, x_2, x_2, x_1, x_5, x_1 + x_2 + x_5, x_2, x_1, x_2, x_2, x_{11})$

$n = 13:\ (x_1, x_2, x_3, x_2, x_1 + x_3, x_6, x_2 + x_3 + x_6, x_2, x_3, x_2, x_2, 0)$

$n = 14:\ (x_1, x_2, x_3, x_4, x_2, x_1 + x_3 + x_4, x_1 + x_2 + x_3, x_1 + x_2 + x_4, x_2, x_2 + x_3 + x_4, x_2, x_2, x_{13})$

$n = 15:\ (x_1, x_2, x_3, x_4, x_3, x_2, x_1 + x_4, x_4, x_3, x_2, x_2 + x_3 + x_4, x_2, x_2, 0)$

$n = 16:\ (x_1, x_2, x_3, x_4, x_5, x_3 + x_4 + x_5, x_2, x_1, x_2 + x_3 + x_5, x_2 + x_4 + x_5, x_2, x_2 + x_3 + x_4, x_2, x_2, x_{15})$

$n = 17:\ (0, x_2, x_3, x_4, x_5, x_6, x_3 + x_4 + x_6, x_2, x_5, x_2 + x_4 + x_6, x_2 + x_3 + x_6, x_2, x_2 + x_3 + x_4, x_2, x_2, 0)$

$n = 18:\ (x_1, x_1, x_3, x_4, x_5, x_6, x_4 + x_5 + x_6, x_3 + x_5 + x_6, x_1, x_1 + x_5 + x_6, x_1 + x_4 + x_6, x_1 + x_3 + x_6, x_1, x_1 + x_3 + x_4, x_1, x_1, x_{17})$

$n = 19:\ (x_1, x_2, x_1 + x_2, x_4, x_5, x_6, x_7, x_4, x_1 + x_2, x_2, x_1 + x_5 + x_6 + x_7, x_2 + x_4 + x_6, x_1 + x_6, x_2, x_1 + x_4, x_2, x_2, 0)$

$n = 20:\ (x_1, x_2, x_2, x_1, x_5, x_6, x_7, x_8, x_1 + x_6 + x_7, x_2 + x_6 + x_7, x_2, x_1 + x_5 + x_6 + x_7 + x_8, x_1 + x_2 + x_6, x_6, x_2, x_1, x_2, x_2, x_{19})$

$n = 21:\ (x_1, x_2, x_3, x_2, x_1 + x_3, x_6, x_7, x_8, x_1 + x_3, x_2 + x_6 + x_8, x_3 + x_6 + x_8, x_2, x_1 + x_6 + x_7 + x_8, x_6, x_2 + x_3 + x_6, x_2, x_3, x_2, x_2, 0)$

$n = 22:\ (x_1, x_2, x_3, x_4, x_2, x_1 + x_3 + x_4, x_7, x_8, x_9, x_1 + x_2 + x_3 + x_4 + x_9, x_1 + x_3 + x_9, x_1 + x_4 + x_9, x_2, x_1 + x_7 + x_8, x_1 + x_2 + x_3, x_1 + x_2 + x_4, x_2, x_2 + x_3 + x_4, x_2, x_2, x_{21})$

$n = 23:\ (x_1, x_2, x_3, x_4, x_3, x_2, x_1 + x_4, x_8, x_9, x_{10}, x_1 + x_2 + x_3 + x_4 + x_9 + x_{10}, x_2 + x_4 + x_{10}, x_2 + x_3 + x_{10}, x_2, x_1 + x_8, x_4, x_3, x_2, x_2 + x_3 + x_4, x_2, x_2, 0)$

$n = 24:\ (x_1, x_2, x_3, x_4, x_5, x_3 + x_4 + x_5, x_2, x_1, x_9, x_{10}, x_2 + x_3 + x_4 + x_5 + x_{10}, x_1 + x_5 + x_9, x_2 + x_4 + x_{10}, x_2 + x_3 + x_{10}, x_2, x_1, x_2 + x_3 + x_5, x_2 + x_4 + x_5, x_2, x_2 + x_3 + x_4, x_2, x_2, x_{23})$

$n = 25:$ $(x_1, x_2, x_3, x_4, x_5, x_6, x_3 + x_4 + x_6, x_2, x_1 + x_5, x_{10}, x_{11}, x_2 + x_6 + x_{10}, x_3 + x_4 + x_5 + x_6 + x_{11}, x_2 + x_4 + x_{10}, x_2 + x_3 + x_{10}, x_2, x_5, x_2 + x_4 + x_6, x_2 + x_3 + x_6, x_2, x_2 + x_3 + x_4, x_2, x_2, 0)$

$n = 26:$ $(x_1, x_2, x_3, x_4, x_5, x_6, x_4 + x_5 + x_6, x_3 + x_5 + x_6, x_2, x_1 + x_5 + x_6, x_{11}, x_{12}, x_1 + x_2 + x_5, x_1 + x_2 + x_3 + x_4 + x_6 + x_{11} + x_{12}, x_1 + x_2 + x_4 + x_5 + x_6, x_1 + x_2 + x_3 + x_5 + x_6, x_2, x_2 + x_5 + x_6, x_2 + x_4 + x_6, x_2 + x_3 + x_6, x_2, x_2 + x_3 + x_4, x_2, x_2, x_{25})$

$n = 27:$ $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_4, x_3, x_2, x_1 + x_5 + x_6 + x_7, x_{12}, x_1 + x_3 + x_5 + x_6, x_6, x_1 + x_3 + x_4 + x_6 + x_7 + x_{12}, x_4, x_3, x_2, x_2 + x_3 + x_5 + x_6 + x_7, x_2 + x_4 + x_6, x_2 + x_3 + x_6, x_2, x_2 + x_3 + x_4, x_2, x_2, 0)$

$n = 28:$ $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_4 + x_6 + x_7, x_3 + x_6 + x_7, x_2, x_1 + x_5 + x_6 + x_7 + x_8, x_{13}, x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_{13}, x_2 + x_3 + x_7, x_1 + x_4 + x_8, x_2 + x_3 + x_4 + x_6 + x_7, x_2 + x_6 + x_7, x_2, x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8, x_2 + x_4 + x_6, x_2 + x_3 + x_6, x_2, x_2 + x_3 + x_4, x_2, x_2, x_{27})$

$n = 29:$ $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_5, x_4 + x_6 + x_8, x_3 + x_6 + x_8, x_2, x_1 + x_6 + x_7 + x_8, x_{14}, x_2 + x_4 + x_6 + x_7 + x_{14}, x_2 + x_4 + x_8, x_5, x_2 + x_6 + x_8, x_2 + x_3 + x_4 + x_6 + x_8, x_2, x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8, x_2 + x_4 + x_6, x_2 + x_3 + x_6, x_2, x_2 + x_3 + x_4, x_2, x_2, 0)$

$n = 30:$ $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_5 + x_6 + x_9, x_4 + x_6 + x_9, x_3 + x_6 + x_9, x_2, x_1 + x_7 + x_8, x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7, x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_8, x_2 + x_5 + x_9, x_2 + x_6 + x_9, x_2 + x_4 + x_5 + x_6 + x_9, x_2 + x_3 + x_5 + x_6 + x_9, x_2, x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8, x_2 + x_4 + x_6, x_2 + x_3 + x_6, x_2, x_2 + x_3 + x_4, x_2, x_2, x_{29})$

4. The cases $n = 7$ and $n = 9$

We illustrate the above parametrizations with two examples, the cases $n = 7$ and $n = 9$. According to the preceding section, the complete list of parity-regular Steinhaus graphs $G(s)$ on 7 vertices is given by all sequences $s \in \mathbb{F}_2^6$ of the form $s = (x_1, x_2, x_1 + x_2, x_2, x_2, 0)$ with $x_1, x_2 \in \mathbb{F}_2$. Thus, there are exactly four such graphs. Among them, two are truly regular, namely for $x_1 = x_2 = 0$, yielding $s = (0, 0, 0, 0, 0, 0)$, and for $x_1 = x_2 = 1$, yielding $s = (1, 1, 0, 1, 1, 0)$.

For $n = 9$, the complete list of parity-regular Steinhaus graphs on 9 vertices is given by all sequences $s \in \mathbb{F}_2^8$ of the form $s = (0, x_2, x_3, x_2, x_3, x_2, x_2, 0)$ with $x_2, x_3 \in \mathbb{F}_2$. Again, there are four such graphs. But here, only the zero-edge one is truly regular. The other three graphs are pictured in Figure 1. (There, the vertices are denoted $1, 2, \ldots, 9$ rather than $0, 1, \ldots, 8$).

5. The main result

Using Theorem 2 and our explicit parametrizations of parity-regular Steinhaus graphs, we have been able to greatly extend the verification of the conjecture of Dymacek, by searching regular Steinhaus graphs inside the larger class of parity-regular ones. Our result is as follows:

There are no new regular Steinhaus graphs on $n \leq 117$ *vertices other than those already known, i.e.* $G(\mathtt{1})$, $G(\mathtt{0}^{n-1})$ *and* $G((\mathtt{110})^{\lfloor n/3 \rfloor})$.

Recall from Section 2 that for any given n, the search space $R(n)$ is of size $2^{\lceil n/3 \rceil - \varepsilon(n)}$. Up to $n \leq 81$ vertices, our search was performed in a few days on a standard PC running a simple uncompiled *Mathematica 5.0* program.

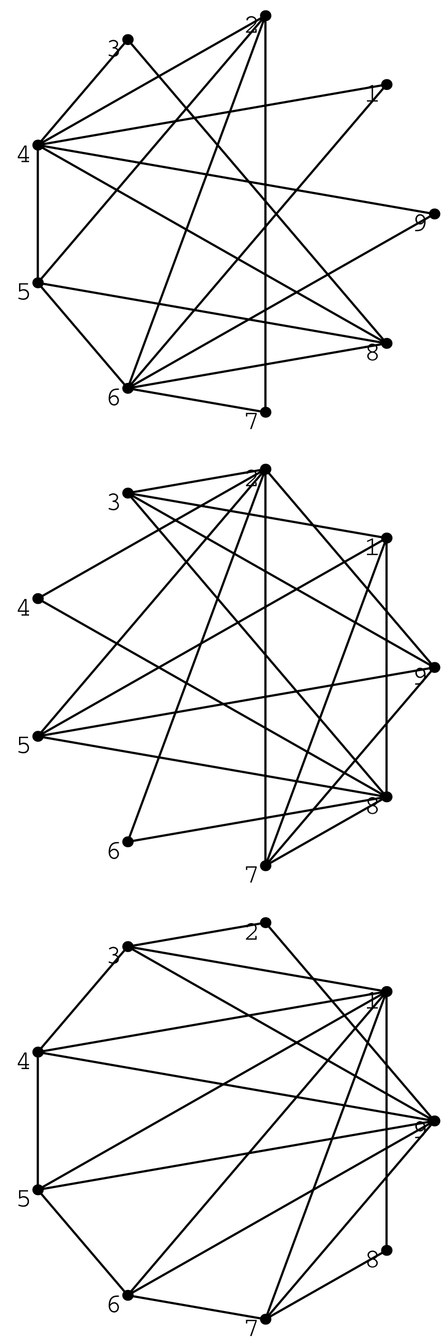

FIGURE 1. All nontrivial parity-regular Steinhaus graphs on 9 vertices

However, to treat the range $82 \leq n \leq 117$, we had to switch to a carefully written and highly optimized program in C. This computation was run on a 16-processor Bull NovaScale machine in less than two weeks.

6. Related problems

Two interesting problems arose during our investigations on parity-regular Steinhaus graphs. As for regularity, they both concern the vertex degree distribution of such graphs. For convenience, we shall sometimes write "PRS graph" for "parity-regular Steinhaus graph" in this section.

Question 1. Are there parity-regular Steinhaus graphs on n vertices with exactly *two* distinct vertex degrees?

For even n, the following binary sequences yield solutions to Question 1:

- If $s = 0^{n-2}1$ or 1^{n-1}, the set of vertex degrees of $G(s)$ is $\{1, n-1\}$.
- If $s = 1^{n-2}0$, the set of vertex degrees of $G(s)$ is $\{2, n-2\}$.

The hypothesis that n is even ensures that the above graphs are indeed parity-regular. There are other systematic solutions, depending on the class of n mod 6:

- If $n \equiv 1 \bmod 6$ and $s = (101000)^{(n-1)/6}$, then the degree set of $G(s)$ is $\{k, 2k\}$ where $k = \frac{n-1}{3}$.
- If $n \equiv 2$ or $5 \bmod 6$ and $s = 0(110)^{(n-2)/3}$, then $G(s)$ has degree set equal to $\{2k, 2k+2\}$ where $k = \frac{n-2}{3}$.

Surprisingly, in the remaining case $n \equiv 3 \bmod 6$, there seem to be *no* solutions, except for $n = 3$ and $n = 15$. We have verified this up to $n \leq 45$ vertices.

Conjecture 1. *For every $n \equiv 3 \bmod 6$ with $n \geq 21$, there are no parity-regular Steinhaus graphs on n vertices having exactly two distinct vertex degrees.*

In the same type of question, we can show that, *for every $n \geq 9$, there are PRS graphs on n vertices with exactly three distinct degrees.* Solutions $s \in \mathbb{F}_2^{n-1}$ of the form $0(1)^{2k}0$, $(001)^{2k+1}$, $1(001)^{2k}$, $(0110)^{2k}110$ and $0(0110)^{2k+1}0^4$ suffice to cover all $n \geq 9$.

Our second problem concerns the reconstructibility of PRS graphs from their degree distributions. We denote by d_i the degree of vertex i.

Question 2. Is it true that a parity-regular Steinhaus graph on n vertices is determined by its degree sequence $(d_0, d_1, \ldots, d_{n-1})$?

For example, there are exactly four PRS graphs on $n = 6$ vertices. Their respective degree sequences are $(0,0,0,0,0,0)$, $(1,1,1,1,1,5)$, $(5,1,1,1,1,1)$ and $(4,2,2,2,2,4)$. These four sequences being distinct, Question 2 has a positive answer for $n = 6$.

While the answer to Question 2 turns out to be negative in general, counterexamples seem to be very rare. To facilitate their discussion, we define a *collision* to be a pair of distinct binary sequences $s_1, s_2 \in \mathbb{F}_2^{n-1}$ such that

- the associated Steinhaus graphs $G(s_1), G(s_2)$ are parity-regular, and
- the degree sequences of $G(s_1), G(s_2)$ are identical.

The smallest collision occurs at $n = 26$ vertices. It is given by the pair $s_1, s_2 \in \mathbb{F}_2^{25}$, where

$$\begin{aligned} s_1 &= (0,0,1,0,1,0,1,0,0,1,1,1,1,1,1,0,0,1,0,1,0,1,0,0,1), \\ s_2 &= (1,1,0,1,0,1,0,1,1,0,0,0,0,0,0,1,1,0,1,0,1,0,1,1,1). \end{aligned}$$

Observe the relation $s_1 + s_2 = (1,1,\dots,1,0) \in \mathbb{F}_2^{25}$. The common degree sequence of their associated Steinhaus graphs $G(s_1), G(s_2)$ is

$$(13,15,9,9,13,13,17,11,9,19,9,9,11,11,9,9,19,9,11,17,13,\ 13,9,9,15,13).$$

(As required, all vertex degrees have the same parity.)

This is the sole collision for $n \leq 26$. We found that, up to 50 vertices, the only other collisions occur at $n = 34, 38, 42, 46$ and 50. Moreover, each one satisfies the relation $s_1+s_2 = (1,1,\dots,1,0) \in \mathbb{F}_2^{n-1}$. Below we give the *complete list of collisions* s_1, s_2 *up to* $n \leq 50$ *vertices.* As these pairs all satisfy $s_2 = s_1 + (1,1,\dots,1,0)$, we display s_1 only, in the form of a binary string.

$n = 26:$	0010101001111110010101001
$n = 34:$	000011101110110000110111011100001
$n = 38:$	0000001011110110111101101111010000001
$n = 42:$	00001100001111101110011101111100001100001
	00100110010011101011110101110010011001001
	00101010011100100111111001001110010101001
	00101010011111000101101000111110010101001
	00101010011111001010010100111110010101001
	00101010011111101000000101111110010101001
$n = 46:$	000000101111010011110110111100101111010000001
	000011100011010011101111011100101100011100001
	001001100110010010111111110100100110011001001
	001010101010010001111111111000100101010101001
$n = 50:$	0000110011001100001111111111111000011001100100001
	0000111011101100001101011010110000110111011100001
	0000111011101100001110011001110000110111011100001
	0010011010010110010011111111100100110100101100100 1
	0010011010110010011001111110011001001101011001001
	0010011010110010011010111101011001001101011001001
	0010100001011110010111100111101001111010000101001
	0010100001111010011101100110111001011110000101001
	0010100001111010011110100101111001011110000101001
	0010100010101110010111011011101001110101000101001
	0010101001011010011111000011111001011010010101001
	0010101001111110010101000010101001111110010101001
	0010101001111110010110000001101001111110010101001
	0010101010001110010101111110101001110001010101001
	0010101010001110010110111101101001110001010101001
	0010101010101010011100111100111001010101010101001

This leads us to the following conjecture.

Conjecture 2. *If* $s_1, s_2 \in \mathbb{F}_2^{n-1}$ *is a collision, then* $n \equiv 2 \bmod 4$ *and* $s_1 + s_2 = (1,1,\dots,1,0)$.

In other words, we conjecture that whenever $n \not\equiv 2 \bmod 4$, a parity-regular Steinhaus graph on n vertices is completely determined by its vertex degree sequence. If true, it would be extremely interesting to know how to reconstruct the sequence s from the degree sequence of $G(s)$.

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