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SUPERCONVERGENCE OF QUADRATIC FINITE ELEMENTS ON MILDLY STRUCTURED GRIDS

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Abstract. Superconvergence estimates are studied in this paper on quadratic finite element discretizations for second order elliptic boundary value problems on mildly structured triangular meshes. For a large class of practically useful grids, the finite element solution u_h is proven to be superclose to the interpolant u_I and as a result a postprocessing gradient recovery scheme for u_h can be devised. The analysis is based on a number of carefully derived identities. In addition to its own theoretical interests, the result in this paper can be used for deriving asymptotically exact a posteriori error estimators for quadratic finite element methods.

1. Introduction

Superconvergence of finite element methods has been a subject of active research for a few decades; see [4, 8, 10, 12]. In recent years, there has been a revitalization of the research of this subject because of its strong relevance to a posteriori error estimation for finite element grid adaptation.

Using superconvergence to justify certain types of a posteriori error estimates can be traced back to the work of [3]. In the early work of this type, there is a dilemma: adaptive finite element methods often lead to unstructured grids whereas the classic theory of superconvergence has been mostly established on specially structured grids (such as the property of strong regularity). Thus there is a serious theoretical gap between the theory of superconvergence and the theory of finite element a posteriori error estimation.

Such a gap, however, is gradually closing up with a number of recent developments. One notable example of development is the recent work of Bank and Xu [6] who studied superconvergence on some mildly regular finite element grids. The main conclusion of their result is that superconvergence can be indeed established for a large class of grids that can be often found in practical computations. More specifically, for triangular linear elements, they proved that the finite element solution is superclose to the linear interpolant on the grids where most pairs of elements form an approximate parallelogram. Let us call these types of grids, at least temporarily, mildly structured grids. Indeed, triangular grids that are generated by

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popular grid generators are found to be mildly structured. In other words, superconvergence theory can be used to derive new error estimators or to justify the performance of existing estimators. Among the various existing error estimations, the so-called Zienkiewizc-Zhu [14] estimator has been proven to be very useful in many practical applications. Using the result in [6], the Zienkiewizc-Zhu estimator was given a new theoretical justification by Xu and Zhang [13].

The aforementioned results are all for linear finite elements. Extensions of these results to higher order elements are certainly of theoretical and practical interests. The goal of this paper is to extend the results to quadratic finite elements.

The superconvergence analysis of quadratic elements is known to be much more complicated than linear elements. This problem has been studied by many authors. Superconvergence of the gradients was obtained in Zhu [15] and supercenvergence of the function value on the nodes was obtained in Zhu [16] for global strongly-regular grids. Similar results were derived or rederived by Andreev [1] and Andreev and Lazarov [2], Goodsel and Whiteman [9] and [7].

In this paper, we will carefully investigate the superconvergence properties for quadratic triangular elements for mildly structured grids. The basic idea follows from Bank and Xu [6], but the technical details are quite different and more difficult.

Superconvergence analysis for high order finite elements crucially depends on the choice of the interpolation operators. According to an observation first made by Li [11], the more traditional nodal value interpolant was known to be inappropriate for the analysis of cubic finite element methods. While the traditional nodal value interpolant was actually used in the all the existing works mentioned above for quadratic finite elements, [15, 16], we find in this paper it is more convenient to use interpolation operator based on moment conditions on edges (for quadratic and cubic interpolants) and moment condition on the element (for cubic interpolants). These are interpolants that preserve the de Rham exact sequences on the discrete level in the context of differential forms; see Hiptmair [10]. We find these interpolants are useful in the study of superconvergence.

Let us use Π_{Q} and Π_{C} to denote such quadratic and cubic interpolants respectively. One main technical result in this paper is the following expansion on each triangular element τ:

$$\int_\tau \nabla(u-\Pi_{\mathrm{Q}}u)\cdot\nabla v_{\mathrm{Q}} = \sum_{k=1}^{3}\sum_{s=0}^{3}\left(a_k^s(\tau)\int_\tau + b_k^s(\tau)\int_{e_k}\right)\frac{\partial^3(\Pi_{\mathrm{C}}u)}{\partial \boldsymbol{n}_k^s\partial \boldsymbol{t}_k^{3-s}}\frac{\partial^2 v_{\mathrm{Q}}}{\partial \boldsymbol{t}_k^2} \tag{1.1}$$

which holds for any function $u\in H^1(\tau)\cap C^0(\bar{\tau})$ and quadratic polynomial v_{Q}. Here e_k are the edges of τ, $\boldsymbol{n}_k$ and $\boldsymbol{t}_k$ are the normal and tangential direction of e_k respectively; $a_k^s(\tau)$ and $b_k^s(\tau)$ are constants depending on the geometry of τ.

By summing up (1.1) on all elements τ for a given triangulation of Ω, we obtain the following global error expansion for a Poisson problem:

$$\begin{aligned}\int_\Omega \nabla(u_h-\Pi_{\mathrm{Q}}u)\cdot\nabla v_h &= \int_\Omega \nabla(u-\Pi_{\mathrm{Q}}u)\cdot\nabla v_h \\ &= \sum_{e=\tau\cap\tau'\in\mathcal{E}_h}\sum_{s=0}^{3}\left(a_e^s(\tau)\int_\tau - a_e^s(\tau')\int_{\tau'} + [b_e^s(\tau)-b_e^s(\tau')]\int_e\right)\frac{\partial^3(\Pi_{\mathrm{C}}u)}{\partial \boldsymbol{n}_e^s\partial \boldsymbol{t}_e^{3-s}}\frac{\partial^2 v_h}{\partial \boldsymbol{t}_e^2}\end{aligned} \tag{1.2}$$

which holds for any quadratic finite element function v_h. Here u_h is the Galerkin projection (namely finite element approximation) of u.

The expansion (1.1) is neat and easy to use for superconvergence analysis. Superconvergence occurs when the terms under the summations in (1.2) have certain cancellations that would lead to a higher order error estimate. More specifically such cancellations would indeed occur in (1.2) if τ and τ' are two adjacent triangles that form an approximate parallelogram. If Ω is partitioned by a strongly regular triangulation in which every pair of triangles forms an $\mathcal{O}(h^2)$-approximate parallelogram, then it is very easy to deduce from (1.2) that

$$|(\nabla(u_h - \Pi_Q u), \nabla v_h)| \lesssim h^4 |u|_{4,p,\Omega} \|\nabla_h^2 v_h\|_{0,q,\Omega} \lesssim h^3 |u|_{4,p,\Omega} |v_h|_{1,q,\Omega}. \tag{1.3}$$

Here $\nabla_h^2 v_h = \nabla^2 v_h$ on each element.

With such a basic error estimate, the derivation of the superconvergence is standard. Taking $p = q = 2$ and $v_h = u_h - \Pi_Q u$ in (1.3), we obtain

$$|u_h - \Pi_Q u|_{1,\Omega} \lesssim h^3 |u|_{4,\Omega}. \tag{1.4}$$

Taking $p = \infty$, $q = 1$ and v_h to be some appropriate discrete Green's function, we obtain

$$\|u_h - \Pi_Q u\|_{0,\infty,\Omega} \lesssim h^4 |\log h| |u|_{4,\infty,\Omega}. \tag{1.5}$$

Both (1.4) and (1.5) are superconvergence results that were already known in the literature, but they are obtained here using a quite different and more transparent approach. We can use the same approach to obtain a new superconvergence result that is valid on a mildly structured grid in which most pairs of triangles form an approximate parallelogram:

$$|u_h - \Pi_Q u|_{1,\Omega} \lesssim h^{2+\min(\sigma,1)/2} \left(\|u\|_{4,\Omega} + |u|_{3,\infty,\Omega}\right). \tag{1.6}$$

Here $\sigma > 0$ in some sense measures the extent to which the approximate parallelogram property is violated; see Section 2 for details. Roughly speaking, such mildly structured meshes have a smooth transition between large and small elements, and appear locally to be quasi-uniform. This is typical for meshes generated by some adaptive meshing procedures [5].

The relaxation from a globally strongly regular grid to such a mildy structured grid is a significant step from a practical point of view. As a result, superconvergence estimates such as (1.6) would then hold for most grids that are actually used in practice. Similar to the work of Xu and Zhang [13], such a type of new superconvergence estimate can be used to theoretically justify that the popular Zienkiewicz-Zhu [14] error estimator is asymptotically exact for most practical grids for second order elliptic problems.

The rest of this paper is organized as follows: Section 2 contains technical identities and estimates that form the basis for the estimate (1.1). In Section 3, some basic error expansions are given for quadratic finite elements, mainly on any given element. In Section 4, the basic error expansions from the previous section are applied to obtain superconvergence results first for uniform grids, then for strongly regular grids and finally for mildly structured grids. In Section 5, some numerical examples are given.

2. Preliminary Lemmas

In this section, we will establish a number of identities related to finite element triangulations and finite element spaces that will be used in our analysis.

2.1. **Elementary geometric identities.** We begin with some geometric identities for a canonical element τ. Let τ have vertices $\boldsymbol{p}_k^t = (x_k, y_k)$, $1 \le k \le 3$, oriented counterclockwise, and corresponding nodal basis functions (barycentric coordinates) $\{\phi_k\}_{k=1}^3$. Let $\{e_k\}_{k=1}^3$ denote the edges of element τ, $\{\theta_k\}_{k=1}^3$ the angles, $\{\boldsymbol{n}_k\}_{k=1}^3$ the unit outward normal vectors, $\{\boldsymbol{t}_k\}_{k=1}^3$ the unit tangent vectors with counterclockwise orientation, $\{\boldsymbol{x}_k\}_{k=1}^2$ the unit vectors of the Cartesian coordinates, $\{\ell_k\}_{k=1}^3$ the edge lengths, and $\{d_k\}_{k=1}^3$ the perpendicular heights (see Figure 1). Let $\tilde{\boldsymbol{p}}$ be the point of intersection for the perpendicular bisectors of the three sides of τ. Let $|s_k|$ denote the distance between $\tilde{\boldsymbol{p}}$ and side k. If τ has no obtuse angles, then the s_k will be nonnegative; otherwise, the distance to the side opposite the obtuse angle will be negative.

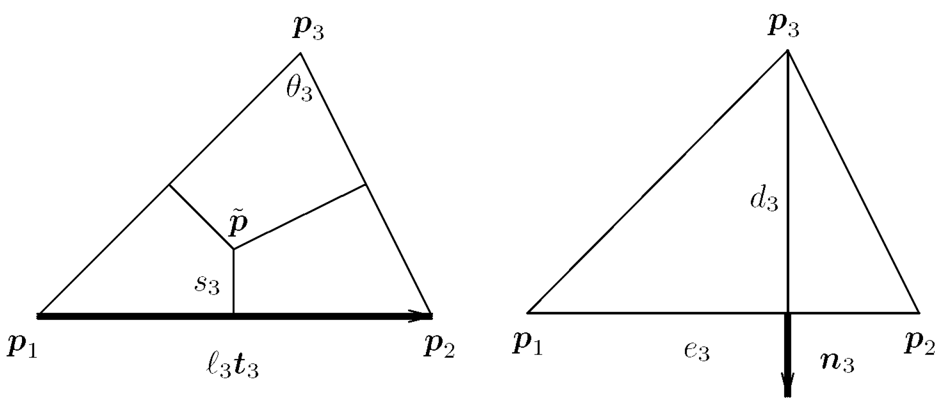

Figure 1. Parameters associated with the triangle τ

There are many relationships among these quantities; in particular we note the following, which hold for $1 \le k \le 3$ and $k \pm 1$ permuted cyclically:

$$\begin{gathered}
\ell_k d_k = \ell_{k+1}\ell_{k-1}\sin\theta_k = 2|\tau|,\\
2\ell_{k+1}\ell_{k-1}\cos\theta_k = \ell_{k+1}^2 + \ell_{k-1}^2 - \ell_k^2,\\
\sin\theta_k = \boldsymbol{n}_{k-1}\cdot\boldsymbol{t}_{k+1} = -\boldsymbol{n}_{k+1}\cdot\boldsymbol{t}_{k-1},\\
\cos\theta_k = -\boldsymbol{t}_{k-1}\cdot\boldsymbol{t}_{k+1} = -\boldsymbol{n}_{k-1}\cdot\boldsymbol{n}_{k+1},\\
\nabla\phi_k = -\boldsymbol{n}_k/d_k.
\end{gathered}$$

Following Bank and Xu [6], let us now deduce several simple identities that will be used later. We first note that

$$\boldsymbol{n}_k = \frac{\boldsymbol{n}_{k+1}\cdot\boldsymbol{n}_k}{\boldsymbol{n}_{k+1}\cdot\boldsymbol{t}_{k-1}}\boldsymbol{t}_{k-1} + \frac{\boldsymbol{n}_{k-1}\cdot\boldsymbol{n}_k}{\boldsymbol{n}_{k-1}\cdot\boldsymbol{t}_{k+1}}\boldsymbol{t}_{k+1} = \frac{\cos\theta_{k-1}}{\sin\theta_k}\boldsymbol{t}_{k-1} - \frac{\cos\theta_{k+1}}{\sin\theta_k}\boldsymbol{t}_{k+1}.$$

Therefore

$$\sin\theta_k\boldsymbol{n}_k = \cos\theta_{k-1}\boldsymbol{t}_{k-1} - \cos\theta_{k+1}\boldsymbol{t}_{k+1}$$

and hence

$$\sin\theta_k \frac{\partial}{\partial \boldsymbol{n}_k} = \cos\theta_{k-1}\frac{\partial}{\partial \boldsymbol{t}_{k-1}} - \cos\theta_{k+1}\frac{\partial}{\partial \boldsymbol{t}_{k+1}}. \tag{2.1}$$

Similarly

$$\frac{\partial}{\partial \boldsymbol{t}_{k+1}} = -\cos\theta_{k-1}\frac{\partial}{\partial \boldsymbol{t}_k} - \sin\theta_{k-1}\frac{\partial}{\partial \boldsymbol{n}_k}, \tag{2.2}$$

$$\frac{\partial}{\partial \boldsymbol{t}_{k-1}} = -\cos\theta_{k+1}\frac{\partial}{\partial \boldsymbol{t}_k} + \sin\theta_{k+1}\frac{\partial}{\partial \boldsymbol{n}_k}. \tag{2.3}$$

Squaring the above two identities leads to:

$$\frac{\partial^2}{\partial \boldsymbol{t}_{k+1}^2} = \cos^2\theta_{k-1}\frac{\partial^2}{\partial \boldsymbol{t}_k^2} + 2\cos\theta_{k-1}\sin\theta_{k-1}\frac{\partial^2}{\partial \boldsymbol{t}_k \partial \boldsymbol{n}_k} + \sin^2\theta_{k-1}\frac{\partial^2}{\partial \boldsymbol{n}_k^2}, \tag{2.4}$$

$$\frac{\partial^2}{\partial \boldsymbol{t}_{k-1}^2} = \cos^2\theta_{k+1}\frac{\partial^2}{\partial \boldsymbol{t}_k^2} - 2\cos\theta_{k+1}\sin\theta_{k+1}\frac{\partial^2}{\partial \boldsymbol{t}_k \partial \boldsymbol{n}_k} + \sin^2\theta_{k+1}\frac{\partial^2}{\partial \boldsymbol{n}_k^2}. \tag{2.5}$$

Multiplying (2.3) with (2.4) and (2.2), we obtain

(2.6)
$$\frac{\partial^3}{\partial \boldsymbol{t}_{k+1}^2 \partial \boldsymbol{t}_{k-1}} = \ldots\cos^2\theta_{k-1}\frac{\partial^2}{\partial \boldsymbol{t}_k^2} + 2\cos\theta_{k-1}\sin\theta_{k-1}\frac{\partial^2}{\partial \boldsymbol{t}_k \partial \boldsymbol{n}_k} + \sin^2\theta_{k-1}\frac{\partial^2}{\partial \boldsymbol{n}_k^2},$$

(2.7)
$$\frac{\partial^3}{\partial \boldsymbol{t}_{k-1}^2 \partial \boldsymbol{t}_{k+1}} = \ldots\cos^2\theta_{k+1}\frac{\partial^2}{\partial \boldsymbol{t}_k^2} - 2\cos\theta_{k+1}\sin\theta_{k+1}\frac{\partial^2}{\partial \boldsymbol{t}_k \partial \boldsymbol{n}_k} + \sin^2\theta_{k+1}\frac{\partial^2}{\partial \boldsymbol{n}_k^2}.$$

$$\frac{\partial^3}{\partial \boldsymbol{t}_1 \partial \boldsymbol{t}_2 \partial \boldsymbol{t}_3} = \ldots\cos^2\theta_{k+1}\frac{\partial^2}{\partial \boldsymbol{t}_k^2} - 2\cos\theta_{k+1}\sin\theta_{k+1}\frac{\partial^2}{\partial \boldsymbol{t}_k \partial \boldsymbol{n}_k} + \sin^2\theta_{k+1}\frac{\partial^2}{\partial \boldsymbol{n}_k^2}. \tag{2.8}$$

A proper linear combination of the identities (2.4) and (2.5) (namely $\ell_{k+1}^2 * (2.4) - \ell_{k-1}^2 * (2.5)$) yields that

$$4|\tau|\frac{\partial^2}{\partial \boldsymbol{t}_k \partial \boldsymbol{n}_k} = (\ell_{k-1}^2 - \ell_{k+1}^2)\frac{\partial^2}{\partial \boldsymbol{t}_k^2} + \ell_{k+1}^2\frac{\partial^2}{\partial \boldsymbol{t}_{k+1}^2} - \ell_{k-1}^2\frac{\partial^2}{\partial \boldsymbol{t}_{k-1}^2}. \tag{2.9}$$

Here we have used the following elementary identities:

$$\ell_{k+1}^2\sin^2\theta_{k-1} - \ell_{k-1}^2\sin^2\theta_{k+1} = 0,$$
$$\ell_{k+1}^2\cos^2\theta_{k-1} - \ell_{k-1}^2\cos^2\theta_{k+1} = \ell_{k+1}^2 - \ell_{k-1}^2,$$
$$\ell_{k+1}^2 2\cos\theta_{k-1}\sin\theta_{k-1} + \ell_{k-1}^2 2\cos\theta_{k+1}\sin\theta_{k+1} = 4|\tau|.$$

By a direct integration by parts, we have

$$\int_\tau \frac{\partial u}{\partial \boldsymbol{t}_k} = \sum_{j=1}^3 \boldsymbol{n}_j \cdot \boldsymbol{t}_k . \int_{e_j} u = -\sin\theta_{k+1}\int_{e_{k-1}} u + \sin\theta_{k-1}\int_{e_{k+1}} u. \tag{2.10}$$

2.2. **Hierarchical representation of quadratic and cubic polynomials.** Given a smooth function u, we shall make use of linear ($\Pi_{\mathrm{L}} u$), quadratic ($\Pi_{\mathrm{Q}} u$) and cubic ($\Pi_{\mathrm{C}} u$) polynomial interpolations of u. Given each element τ, all three of these interpolations assume the same value as u at the three vertices of τ. Furthermore, the quadratic interpolation $\Pi_{\mathrm{Q}} u$ has the same integral as u on each edge

$$\int_{e_i} (u - \Pi_{\mathrm{Q}} u) = 0, \quad i = 1, 2, 3.$$

The cubic interpolation $\Pi_{\mathrm{C}} u$ has the same first order moment as u on each edge e_i of τ,

$$\int_{e_i} (u - \Pi_{\mathrm{C}} u) p_1 = 0, \quad \forall p_1 \in \mathcal{P}_1, i = 1, 2, 3$$

and has the same integral on the element τ as u,

$$\int_\tau (u - \Pi_{\mathrm{C}} u) = 0.$$

A more common type of interpolant used in superconvergence study is the traditional nodal value interpolant. For a linear element, the above defined interpolation Π_{L} is simply the nodal value interpolant. For a quadratic element, the nodal value interpolant, denoted by I_{Q}, is slightly different. It assumes the same value as u at all vertices of τ and it assumes the same value as u at all the midpoints of element edges.

We have the following simple but very important property for these interpolants.

Lemma 2.1. *The following identity holds:*

$$I_{\mathrm{Q}} \Pi_{\mathrm{C}} = \Pi_{\mathrm{Q}}$$

or equivalently, for every midpoint $a_{ij} = (a_i + a_j)/2$, we have

$$(\Pi_{\mathrm{Q}} u - \Pi_{\mathrm{C}} u)(a_{ij}) = 0.$$

Proof. By definition, we have

$$\begin{aligned} \int_e (u - I_{\mathrm{Q}} \Pi_{\mathrm{C}} u) &= \int_e (u - I_{\mathrm{Q}} u) + \int_e I_{\mathrm{Q}} (u - \Pi_{\mathrm{C}} u) \\ &= \int_e (I - I_{\mathrm{Q}}) u - (I - I_{\mathrm{Q}})(u - \Pi_{\mathrm{C}} u) \\ &= \int_e (I - I_{\mathrm{Q}}) \Pi_{\mathrm{C}} u \\ &= 0 \end{aligned}$$

where the last step is verified by a direct calculation. □

Using the subscripts L, Q and C, we shall use $u_{\mathrm{L}}, u_{\mathrm{Q}}$ and u_{C} to denote any linear, quadratic and cubic function respectively. In the following, the subscripts are equivalent mod 3, for example, $\ell_i = \ell_{i (\mathrm{mod}\, 3)}, \phi_i = \phi_{i (\mathrm{mod}\, 3)}$

Lemma 2.2. *For any quadratic function u_{Q},*

$$(I - \Pi_{\mathrm{L}}) u_{\mathrm{Q}}(x) = -\frac{1}{2} \sum_{i=1}^{3} \ell_i^2 \phi_{i+1} \phi_{i+2} \frac{\partial^2 u_{\mathrm{Q}}}{\partial \boldsymbol{t}_i^2} \tag{2.11}$$

and

$$\Delta u_{\mathrm{Q}} = -\frac{1}{4|\tau|^2} \sum_{i=1}^{3} \ell_i^2 \ell_{i+1} \ell_{i+2} \cos\theta_i \frac{\partial^2 u_{\mathrm{Q}}}{\partial \boldsymbol{t}_i^2}. \tag{2.12}$$

Here ϕ_i $(i = 1, 2, 3)$ are barycentric coordinate functions for τ.

Proof. Since $u_{\mathrm{Q}} - \Pi_{\mathrm{L}} u_{\mathrm{Q}}$ vanishes at all three vertices, we can write in the following form:

$$u_{\mathrm{Q}} - \Pi_{\mathrm{L}} u_{\mathrm{Q}} = \sum_{i=1}^{3} \alpha_i \phi_{i+1} \phi_{i+2}.$$

To determine the coefficients α_i, we take a second order derivative on both sides of the above identity to obtain

$$\frac{\partial^2 u_{\mathrm{Q}}}{\partial t_i^2} = \alpha_i \frac{\partial^2(\phi_{i+1}\phi_{i+2})}{\partial t_i^2} = -2\alpha_i \frac{1}{\ell_i^2}.$$

The first identity (2.11) then follows. The proof of the second identity (2.12) follows from the identity (2.11) together with the following sequence of identities:

$$\Delta u_L = 0, \Delta(\phi_{i+1}\phi_{i+2}) = 2\nabla\phi_{i+1} \cdot \nabla\phi_{i+2},$$

and

$$|\nabla\phi_{i+1}| = -\nabla\phi_{i+1} \cdot n_{i+1} = d_{i+1}^{-1} = \frac{\ell_{i+1}}{2|\tau|},$$

and

$$\nabla\phi_{i+1} \cdot \nabla\phi_{i+2} = |\nabla\phi_{i+1}||\nabla\phi_{i+2}| \cos\theta_i = \frac{\ell_{i+1}\ell_{i+2}}{4|\tau|^2} \cos\theta_i. \quad \square$$

Next we will give a representation of $u_{\mathrm{C}} - u_{\mathrm{Q}}$ in terms of the following cubic "bubble" functions:

$$\psi_0 = \phi_1\phi_2\phi_3 \tag{2.13}$$

and

$$\psi_i = \phi_{i+1}^2\phi_{i+2} - \phi_{i+1}\phi_{i+2}^2. \tag{2.14}$$

The function ψ_0 vanishes on $\partial\tau$, the boundary of the element τ, and ψ_i vanishes at the three nodal points of e_i and also vanishes on the remaining two edges of $\partial\tau$.

The following identities can be obtained by direct calculations:

$$\frac{\partial^3 \psi_i}{\partial \boldsymbol{t}_j^3} = \delta_{ij}\frac{12}{\ell_i^3}, \tag{2.15}$$

$$\frac{\partial^3\psi_0}{\partial\boldsymbol{t}_1\partial\boldsymbol{t}_2\partial\boldsymbol{t}_3} = 0, \qquad \frac{\partial^3\psi_i}{\partial\boldsymbol{t}_1\partial\boldsymbol{t}_2\partial\boldsymbol{t}_3} = \frac{4}{\ell_1\ell_2\ell_3}, \quad i = 1, 2, 3, \tag{2.16}$$

$$\frac{\partial^3\psi_0}{\partial\boldsymbol{t}_i^2\partial\boldsymbol{t}_{i+1}} = -\frac{2}{\ell_i^2\ell_{i+1}}, \qquad \frac{\partial^3\psi_{i-1}}{\partial\boldsymbol{t}_i^2\partial\boldsymbol{t}_{i+1}} = \frac{-2}{\ell_i^2\ell_{i+1}}, \quad i = 1, 2, 3, \tag{2.17}$$

$$\frac{\partial^3\psi_i}{\partial\boldsymbol{t}_i^2\partial\boldsymbol{t}_{i+1}} = \frac{-6}{\ell_i^2\ell_{i+1}}, \qquad \frac{\partial^3\psi_{i+1}}{\partial\boldsymbol{t}_i^2\partial\boldsymbol{t}_{i+1}} = \frac{2}{\ell_i^2\ell_{i+1}}, \quad i = 1, 2, 3. \tag{2.18}$$

Using the formula

$$\int_\tau \phi_1^\alpha\phi_2^\beta\phi_3^\gamma = \frac{2|\tau|}{(\alpha+\beta+\gamma+2)!}$$

we have

$$\int_\tau \psi_i = 0 \qquad \text{for } i = 1, 2, 3$$

and

$$\int_\tau \psi_0 = \frac{1}{60}|\tau|.$$

Lemma 2.3. *For any cubic function* u_{C},
(2.19)
$$u_{\mathrm{C}} - I_{\mathrm{Q}}u_{\mathrm{C}} = \left(\sum_{i=1}^{3} \frac{\ell_i^2\ell_{i+1}}{6}\frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_i^2 \partial \boldsymbol{t}_{i+1}} + \frac{\ell_1\ell_2\ell_3}{4}\frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_1\partial \boldsymbol{t}_2\partial \boldsymbol{t}_3}\right)\psi_0 + \sum_{i=1}^{3}\frac{\ell_i^3}{12}\frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_i^3}\psi_i$$

and

$$\frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_1\partial \boldsymbol{t}_2\partial \boldsymbol{t}_3} = \frac{1}{3\ell_1\ell_2\ell_3}\sum_{i=1}^{3}\ell_i^3\frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_i^3}. \tag{2.20}$$

Proof. By Lemma 2.1, $u_{\mathrm{C}} - I_{\mathrm{Q}}u_{\mathrm{C}}$ is a cubic polynomial vanishing at all three nodes and all three midpoints of edges, so we can write it as

$$u_{\mathrm{C}} - I_{\mathrm{Q}}u_{\mathrm{C}} = \sum_{i=0}^{3}\alpha_i\psi_i \tag{2.21}$$

where the constants α_i are determined as follows. A direct calculation gives the following identities. Using (2.15) and taking the derivative $\frac{\partial^3}{\partial \boldsymbol{t}_i^3}$ on (2.21) we have

$$\alpha_i = \frac{\ell_i^3}{12}\frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_i^3}.$$

Similarly, using (2.16), we have

$$\frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_1\partial \boldsymbol{t}_2\partial \boldsymbol{t}_3} = \frac{4}{\ell_1\ell_2\ell_3}\sum_{i=1}^{3}\alpha_i = \frac{1}{3\ell_1\ell_2\ell_3}\sum_{i=1}^{3}\ell_i^3\frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_i^3}.$$

Using (2.17) and (2.18) we have

$$6\alpha_0 - 6\sum_{i=1}^{3}\alpha_i = \sum_{i=1}^{3}\ell_i^2\ell_{i+1}\frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_i^2\partial \boldsymbol{t}_{i+1}}.$$

So,

$$\alpha_0 = \sum_{i=1}^{3}\alpha_i + \frac{1}{6}\sum_{i=1}^{3}\ell_i^2\ell_{i+1}\frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_i^2\partial \boldsymbol{t}_{i+1}} = \frac{\ell_1\ell_2\ell_3}{4}\frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_1\partial \boldsymbol{t}_2\partial \boldsymbol{t}_3} + \frac{1}{6}\sum_{i=1}^{3}\ell_i^2\ell_{i+1}\frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_i^2\partial \boldsymbol{t}_{i+1}}.$$

Combining these identities completes the proof of the lemma. ☐

3. Basic variational error expansions

In this section, we will derive some basic variational error expansions for the quadratic finite element interpolant Π_{Q}, namely (1.1) and its consequences.

The following is a fundamental identity in our analysis.

Lemma 3.1.

$$\int_\tau \nabla(u_{\mathrm{C}} - I_{\mathrm{Q}}u_{\mathrm{C}})\cdot\nabla v_{\mathrm{Q}} = -\int_\tau\left(\sum_{i=1}^{3}\frac{\ell_i^2\ell_{i+1}}{360}\frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_i^2\partial \boldsymbol{t}_{i+1}} + \frac{\ell_1\ell_2\ell_3}{240}\frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_1\partial \boldsymbol{t}_2\partial \boldsymbol{t}_3}\right)\Delta v_{\mathrm{Q}}$$
$$+ \frac{1}{2880|\tau|}\sum_{i=1}^{3}\int_{e_i}\left(\ell_i^4(\ell_{i+1}^2 - \ell_{i+2}^2)\frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_i^3} + \ell_i\ell_{i+1}^5\frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_{i+1}^3} - \ell_i\ell_{i+2}^5\frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_{i+2}^3}\right)\frac{\partial^2 v_{\mathrm{Q}}}{\partial \boldsymbol{t}_i^2}.$$

Proof. Using Green's theorem, we have

$$\int_\tau \nabla(u_{\mathrm{C}} - I_{\mathrm{Q}}u_{\mathrm{C}}) \cdot \nabla v_{\mathrm{Q}} = -\int_\tau (u_{\mathrm{C}} - I_{\mathrm{Q}}u_{\mathrm{C}})\Delta v_{\mathrm{Q}} + \sum_{k=1}^{3} \int_{e_k} (u_{\mathrm{C}} - I_{\mathrm{Q}}u_{\mathrm{C}}) \frac{\partial v_{\mathrm{Q}}}{\partial \boldsymbol{n}_k}. \tag{3.1}$$

The first term on the right hand of (3.1) equals the first term on the right hand side in Lemma 2.3. To complete the proof, let us consider the integrals on the edges of the element. Note that ψ_0 and $\psi_i, i \neq k$ vanish on e_k. Using Lemma 2.3 and integration by parts, we obtain

$$\begin{aligned} J_k &= \int_{e_k} (u_{\mathrm{C}} - I_{\mathrm{Q}}u_{\mathrm{C}}) \frac{\partial v_{\mathrm{Q}}}{\partial \boldsymbol{n}_k} = \int_{e_k} \frac{\ell_k^3}{12} \frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_k^3} \psi_k \frac{\partial v_{\mathrm{Q}}}{\partial \boldsymbol{n}_k} \\ &= \frac{\ell_k^4}{24} \int_{e_k} \frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_k^3} \frac{\partial}{\partial \boldsymbol{t}_k} [\phi_{k+1}^2 \phi_{k+2}^2] \frac{\partial v_{\mathrm{Q}}}{\partial \boldsymbol{n}_k} \\ &= -\frac{\ell_k^4}{24} \int_{e_k} \frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_k^3} \phi_{k+1}^2 \phi_{k+2}^2 \frac{\partial^2 v_{\mathrm{Q}}}{\partial \boldsymbol{n}_k \partial \boldsymbol{t}_k} = -\frac{\ell_k^4}{720} \int_{e_k} \frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_k^3} \frac{\partial^2 v_{\mathrm{Q}}}{\partial \boldsymbol{n}_k \partial \boldsymbol{t}_k}. \end{aligned}$$

Using (2.9), we have

$$\begin{aligned} J_k &= \frac{\ell_k^4}{2880|\tau|} \left(\int_{e_k} (\ell_{k+1}^2 - \ell_{k+2}^2) \frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_k^3} \frac{\partial^2 v_{\mathrm{Q}}}{\partial \boldsymbol{t}_k^2} \right. \\ &\qquad \left. - \int_{e_k} \ell_{k+1}^2 \frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_k^3} \frac{\partial^2 v_{\mathrm{Q}}}{\partial \boldsymbol{t}_{k+1}^2} + \int_{e_k} \ell_{k+2}^2 \frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_k^3} \frac{\partial^2 v_{\mathrm{Q}}}{\partial \boldsymbol{t}_{k+2}^2} \right) \\ &= \frac{\ell_k^4}{2880|\tau|} \left(\int_{e_k} (\ell_{k+1}^2 - \ell_{k+2}^2) \frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_k^3} \frac{\partial^2 v_{\mathrm{Q}}}{\partial \boldsymbol{t}_k^2} \right. \\ &\qquad \left. - \int_{e_{k+1}} \ell_k \ell_{k+1} \frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_k^3} \frac{\partial^2 v_{\mathrm{Q}}}{\partial \boldsymbol{t}_{k+1}^2} + \int_{e_{k+2}} \ell_{k+2} \ell_k \frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{t}_k^3} \frac{\partial^2 v_{\mathrm{Q}}}{\partial \boldsymbol{t}_{k+2}^2} \right). \end{aligned}$$

Taking the summation over k and reorganizing the terms complete the proof. □

Using identities (2.1)-(2.12), we can further transform the above identities into a form that is useful to our forthcoming analysis. The results are summarized in the following lemma.

Lemma 3.2.

$$\int_\tau \nabla(u_{\mathrm{C}} - I_{\mathrm{Q}}u_{\mathrm{C}}) \cdot \nabla v_{\mathrm{Q}} = \sum_{k=1}^{3} \sum_{s=0}^{3} \left(a_k^s(\tau) \int_\tau - b_k^s(\tau) \int_{e_k} \right) \frac{\partial^3 u_{\mathrm{C}}}{\partial \boldsymbol{n}_k^s \partial \boldsymbol{t}_k^{3-s}} \frac{\partial^2 v_{\mathrm{Q}}}{\partial \boldsymbol{t}_k^2} \tag{3.2}$$

where, with $M_k = \sin 2\theta_k / (\sin 2\theta_1 + \sin 2\theta_2 + \sin 2\theta_3)$,

$$\begin{aligned} a_k^0(\tau) = M_k \Bigg(& \frac{l_{k-1}^2 l_k}{180} \cos^2 \theta_{k+1} + \frac{l_1 l_2 l_3}{120} \cos\theta_{k-1} \cos\theta_{k+1} \\ & - \frac{l_k^2 l_{k+1}}{180} \cos\theta_{k-1} - \frac{l_{k+1}^2 l_{k-1}}{180} \cos^2\theta_{k-1} \cos\theta_{k+1} \Bigg), \end{aligned} \tag{3.3}$$

$$(3.4)\quad a_k^1(\tau) = M_k \left(-\frac{l_{k-1}^2 l_k}{90} \sin\theta_{k+1}\cos\theta_{k+1} - \frac{l_1 l_2 l_3}{120}\cos\theta_{k-1}\sin\theta_{k+1} \right. \\ - \frac{l_{k+1}^2 l_{k-1}}{90}\cos\theta_{k-1}\sin\theta_{k-1}\cos\theta_{k+1} + \frac{l_1 l_2 l_3}{120}\cos\theta_{k+1}\sin\theta_{k-1} \\ \left. -\frac{l_k^2 l_{k+1}}{180}\sin\theta_{k-1} + \frac{l_{k+1}^2 l_{k-1}}{180}\cos^2\theta_{k-1}\sin\theta_{k+1} \right),$$

$$(3.5)\quad a_k^2(\tau) = M_k \left(\frac{l_{k-1}^2 l_k}{180}\sin^2\theta_{k+1} - \frac{l_1 l_2 l_3}{120}\sin\theta_{k-1}\sin\theta_{k+1} \right. \\ \left. + \frac{l_{k+1}^2 l_{k-1}}{90}\cos\theta_{k-1}\sin\theta_{k-1}\sin\theta_{k+1} - \frac{l_{k+1}^2 l_{k-1}}{180}\sin^2\theta_{k-1}\cos\theta_{k+1} \right),$$

$$(3.6)\quad a_k^3(\tau) = M_k \frac{l_{k+1}^2 l_{k-1}}{180}\sin^2\theta_{k-1}\sin\theta_{k+1}$$

and

$$(3.7)\qquad b_k^0(\tau) = \frac{1}{1440}\left(\frac{\ell_k^4(\ell_{k+1}^2 - \ell_{k-1}^2)}{2|\tau|} + \frac{\ell_{k-1}^4\cos^3\theta_{k+1}}{\sin\theta_{k+1}} - \frac{\ell_{k+1}^4\cos^3\theta_{k-1}}{\sin\theta_{k-1}} \right),$$

$$(3.8)\qquad b_k^1(\tau) = \frac{1}{1440}(-3\ell_{k-1}^4\cos^2\theta_{k+1} - 3\ell_{k+1}^4\cos^2\theta_{k-1}),$$

$$(3.9)\qquad b_k^2(\tau) = \frac{1}{1440}(3\ell_{k-1}^4\cos\theta_{k+1}\sin\theta_{k+1} - 3\ell_{k+1}^4\cos\theta_{k-1}\sin\theta_{k-1}),$$

$$(3.10)\qquad b_k^3(\tau) = \frac{1}{1440}(-\ell_{k+1}^4\sin^2\theta_{k-1} - \ell_{k-1}^4\sin^2\theta_{k+1}).$$

The superconvergence results that we will obtain depend on the magnitudes of $a_k^s(\tau)$ and $b_k^s(\tau)$ and how they vary from one element to its neighbor.

Lemma 3.3. *The coefficients $a_k^s(\tau)$ given in* (3.3)-(3.6) *and $b_k^s(\tau)$* (3.7)-(3.10) *have the following properties:*

(1)

$$(3.11)\qquad a_k^s(\tau) = \mathcal{O}(h^3), \quad b_k^s(\tau) = \mathcal{O}(h^4).$$

(2) *Assume that τ and τ' are two adjacent elements that share a common edge $e = e_k$. If τ and τ' form an exact parallelogram, then*

$$(3.12)\qquad a_e^s(\tau) = a_e^s(\tau'), \quad b_e^s(\tau) = b_e^s(\tau'), \quad (s = 0, 1, 2, 3).$$

(3) *If τ and τ' form an $\mathcal{O}(h^2)$ approximate parallelogram (namely the lengths of any two opposite edges differ only by $O(h^2)$), then*

$$(3.13)\qquad a_e^s(\tau) - a_e^s(\tau') = \mathcal{O}(h^4), \quad b_e^s(\tau) - b_e^s(\tau') = \mathcal{O}(h^5), \quad (s = 0, 1, 2, 3).$$

Proof. Note that $M_k = \mathcal{O}(1)$, so the first result is obvious and the second result is also fairly clear. The third result can be obtained by a chain difference, for example,

$$\begin{aligned} \ell_{k-1}^2\ell_k\sin^2\theta_{k+1} \quad - \quad & \ell_{k-1}'^2\ell_k'\sin^2\theta_{k+1}' \\ = \quad & (\ell_{k-1}^2 - \ell_{k-1}'^2)\ell_k\sin^2\theta_{k+1} + \ell_{k-1}'^2(\ell_k - \ell_k')\sin^2\theta_{k+1} \\ + \quad & \ell_{k-1}'^2\ell_k'(\sin^2\theta_{k+1} - \sin^2\theta_{k+1}') = \mathcal{O}(h^4). \quad \square \end{aligned}$$

From Lemma 3.1, using a standard scaling argument, an application of the Bramble-Hilbert Lemma in the reference element gives the following.

Lemma 3.4. *Let $u \in W^{4,p}(\tau)$, then it holds that*

$$\begin{aligned}\int_\tau \nabla(u-\Pi_{\mathrm{Q}}u)\cdot\nabla v_{\mathrm{Q}} &= \sum_{k=1}^{3}\sum_{s=0}^{3}\left(a_k^s(\tau)\int_\tau + b_k^s(\tau)\int_{e_k}\right)\frac{\partial^3 \Pi_{\mathrm{C}}u}{\partial \boldsymbol{n}_k^s \partial \boldsymbol{t}_k^{3-s}}\frac{\partial^2 v_{\mathrm{Q}}}{\partial \boldsymbol{t}_k^2}\\ &= \sum_{k=1}^{3}\sum_{s=0}^{3}\left(a_k^s(\tau)\int_\tau + b_k^s(\tau)\int_{e_k}\right)\frac{\partial^3 u}{\partial \boldsymbol{n}_k^s \partial \boldsymbol{t}_k^{3-s}}\frac{\partial^2 v_{\mathrm{Q}}}{\partial \boldsymbol{t}_k^2}\\ &\quad + \mathcal{O}(h^{4-\alpha})|u|_{4,p,\tau}|v_{\mathrm{Q}}|_{2-\alpha,q,\tau}, \quad \alpha = 0,1.\end{aligned}$$

Proof. We first note, by definition of $\Pi_{\mathrm{C}}u$, that

$$\int_\tau \nabla(u-\Pi_{\mathrm{C}}u)\cdot\nabla v_{\mathrm{Q}} = \int_{\partial\tau}(u-\Pi_{\mathrm{C}}u)\frac{\partial v_{\mathrm{Q}}}{\partial n} - \int_\tau (u-\Pi_{\mathrm{C}}u)\Delta v_{\mathrm{Q}} = 0.$$

Hence, by Lemmas 2.1 and 3.2, we have

$$\begin{aligned}\int_\tau \nabla(u-\Pi_{\mathrm{Q}}u)\cdot\nabla v_{\mathrm{Q}} &= \int_\tau \nabla(\Pi_{\mathrm{C}}u - I_{\mathrm{Q}}\Pi_{\mathrm{C}}u)\cdot\nabla v_{\mathrm{Q}} + \int_\tau \nabla(u-\Pi_{\mathrm{C}}u)\cdot\nabla v_{\mathrm{Q}}\\ &= \int_\tau \nabla(\Pi_{\mathrm{C}}u - I_{\mathrm{Q}}\Pi_{\mathrm{C}}u)\cdot\nabla v_{\mathrm{Q}}\\ &= \sum_{k=1}^{3}\sum_{s=0}^{3}\left(a_k^s(\tau)\int_\tau + b_k^s(\tau)\int_{e_k}\right)\frac{\partial^3 \Pi_{\mathrm{C}}u}{\partial \boldsymbol{n}_k^s \partial \boldsymbol{t}_k^{3-s}}\frac{\partial^2 v_{\mathrm{Q}}}{\partial \boldsymbol{t}_k^2}\\ &= \sum_{k=1}^{3}\sum_{s=0}^{3}\left(a_k^s(\tau)\int_\tau + b_k^s(\tau)\int_{e_k}\right)\frac{\partial^3 u}{\partial \boldsymbol{n}_k^s \partial \boldsymbol{t}_k^{3-s}}\frac{\partial^2 v_{\mathrm{Q}}}{\partial \boldsymbol{t}_k^2} + R\end{aligned}$$

where

$$R = \sum_{k=1}^{3}\sum_{s=0}^{3}\left(a_k^s(\tau)\int_\tau + b_k^s(\tau)\int_{e_k}\right)\frac{\partial^3 (u-\Pi_{\mathrm{C}}u)}{\partial \boldsymbol{n}_k^s \partial \boldsymbol{t}_k^{3-s}}\frac{\partial^2 v_{\mathrm{Q}}}{\partial \boldsymbol{t}_k^2} = \mathcal{O}(h^{4-\alpha})|u|_{4,p,\tau}|v_{\mathrm{Q}}|_{2-\alpha,q,\tau}.$$

In the above estimate, we have used the following inequality:

$$\left|\int_e f\right| \lesssim h^{-1}\int_\tau |f| + \int_\tau |\nabla f|. \quad \square \tag{3.14}$$

For a given triangulation $\mathcal{T}_h$ and the corresponding quadratic finite element space $\mathcal{V}_h \subset H_0^1(\Omega)$, we can obtain an error expansion as stated in the following lemma.

Lemma 3.5. *Let $u \in W^{4,p}(\Omega)$. Then for any $v_h \in \mathcal{V}_h$,*

$$\begin{aligned}&(\nabla(u-\Pi_{\mathrm{Q}}u), \nabla v_h)\\ &= \sum_{e=\tau\cap\tau'\in\mathcal{E}_h}\sum_{s=0}^{3}\left(a_e^s(\tau)\int_\tau - a_e^s(\tau')\int_{\tau'} + [b_e^s(\tau) - b_e^s(\tau')]\int_e\right)\frac{\partial^3 u}{\partial \boldsymbol{n}_e^s \partial \boldsymbol{t}_e^{3-s}}\frac{\partial^2 v_h}{\partial \boldsymbol{t}_e^2}\\ &\quad + \mathcal{O}(h^{4-\alpha})|u|_{4,p,\Omega}|v_h|'_{2-\alpha,q,\Omega}.\end{aligned}$$

Here $|v_h|'_{2-\alpha,q,\Omega} = (\sum_\tau |v_h|^q_{2-\alpha,q,\tau})^{\frac{1}{q}}$ and the first sum $\sum_{e=\tau\cap\tau'\in\mathcal{E}_h}$ is taken over each interior edge e which is the intersection of two adjacent triangles τ and τ', and $\boldsymbol{n}_e$ and $\boldsymbol{t}_e$ are the normal and tangential directions of e respectively. $a_e^s(\tau)$ and $b_e^s(\tau)$ are as given in (3.3)-(3.6) *and* (3.7)-(3.10).

The proof of this basic result can be obtained directly by summing up the local error expansion given in Lemma 3.4 together with the following simple observations:

(1) $\frac{\partial^2 v_h}{\partial \boldsymbol{t}_e^2}$ assumes the same value on the two adjacent elements, τ and τ', that share e as an edge and it is zero when e is on the boundary of Ω, namely $e \subset \partial\Omega$.

(2) For each $s = 0, 1, 2, 3$, $\frac{\partial^3 u}{\partial \boldsymbol{n}_e^s \partial \boldsymbol{t}_e^{3-s}}$ has the opposite sign on τ and τ', namely

$$\frac{\partial^3 u}{\partial \boldsymbol{n}_{\tau,e}^s \partial \boldsymbol{t}_{\tau,e}^{3-s}} = -\frac{\partial^3 u}{\partial \boldsymbol{n}_{\tau',e}^s \partial \boldsymbol{t}_{\tau',e}^{3-s}}$$

where $\boldsymbol{n}_{\tau,e}$ is the external normal direction of e of τ and $\boldsymbol{t}_{\tau,e}$ is the counterclockwise tangential direction of e of τ.

4. Superconvergence results

With the preliminary results obtained in previous sections, we are now ready to present various superconvergence results for quadratic finite elements. We will first recover various known results and finally will present our new result on mildly unstructured grids.

4.1. **Globally uniform grid.** Let us first take a look at a simple case where the triangulation $\mathcal{T}_h$ is uniform, namely every two adjacent triangles τ and τ' form a perfect parallelogram.

Lemma 4.1. *If the triangulation $\mathcal{T}_h$ is uniform, then*

$$(\nabla(u - \Pi_{\mathrm{Q}} u), \nabla v_h) = \sum_{e=\tau\cap\tau'\in\mathcal{E}_h} \sum_{s=0}^{3} a_e^s \left(\int_\tau - \int_{\tau'}\right) \frac{\partial^3 u}{\partial \boldsymbol{n}_e^s \partial \boldsymbol{t}_e^{3-s}} \frac{\partial^2 v_h}{\partial \boldsymbol{t}_e^2}$$
$$+ \mathcal{O}(h^{4-s})|u|_{4,p,\Omega}|v_h|'_{2-s,q,\Omega}. \tag{4.1}$$

In the above lemma, $a_e^s = a_e^s(\tau)$ is independent of τ.

Theorem 4.2. *If the triangulation $\mathcal{T}_h$ is uniform, then, for $p, q \ge 1$ with $1/p + 1/q = 1$, we have*

$$|(\nabla(u - \Pi_{\mathrm{Q}} u), \nabla v_h)| \lesssim h^{4-\alpha}|u|_{4,p,\Omega}|v_h|'_{2-\alpha,q,\Omega}, \quad v_h \in V_h \tag{4.2}$$

and

$$|u_h - \Pi_{\mathrm{Q}} u|_{1,\Omega} \lesssim h^3|u|_{4,\Omega}, \tag{4.3}$$

$$|u_h - \Pi_{\mathrm{Q}} u|_{0,\infty,\Omega} \lesssim h^4|\log h||u|_{4,\infty,\Omega}. \tag{4.4}$$

Proof. By Lemma 4.1, it suffices to prove that

$$\left(\int_\tau - \int_{\tau'}\right) \frac{\partial^3 u}{\partial \boldsymbol{n}_e^s \partial \boldsymbol{t}_e^{3-s}} = \mathcal{O}(h^{3-2/p})|u|_{4,p,\tau\cup\tau'}$$

which can be obtained by a standard scaling argument using the Bramble-Hilbert Lemma.

The superconvergence estimate (4.3) can be obtained by taking $v_h = u_h - \Pi_{\mathrm{Q}} u$ and $\alpha = 1$ in (4.2) and the estimate (4.4) can be obtained by taking $v_h = G_h^z$

(the finite element approximation of the Green's function), $\alpha = 0$ and $p = \infty$ in (4.5) and using the following bound (see Zhu and Lin [17]):

$$|G_h^z|'_{2,1} \lesssim |\log h|. \qquad \square$$

4.2. **Strongly regular grid.** Classic superconvergence results are often obtained for the so-called *strongly regular* grids. A strongly regular grid is such that any two adjacent triangles form an $O(h^2)$ approximate parallelogram. With our basic error expansion in Lemma 3.5, the derivation of such results is rather straightforward.

Simply speaking, results for globally strongly regular grids are totally analogous to those for globally uniform grid presented in the previous section.

Theorem 4.3. *If the triangulation $\mathcal{T}_h$ is strongly regular, then*

$$|(\nabla(u - \Pi_{\mathrm{Q}} u), \nabla v_h)| \lesssim h^{4-\alpha}|u|_{4,p,\Omega}|v_h|'_{2-\alpha,q,\Omega}, \quad v_h \in V_h \tag{4.5}$$

and

$$|u_h - \Pi_{\mathrm{Q}} u|_{1,\Omega} \lesssim h^3 |u|_{4,p,\Omega}, \tag{4.6}$$

$$|u_h - \Pi_{\mathrm{Q}} u|_{0,\infty,\Omega} \lesssim h^4 |\log h| |u|_{4,\infty,\Omega}. \tag{4.7}$$

Proof. Thanks to Lemma 3.5, we only need to verify the following two estimates:

$$\begin{aligned} &\left| \left(a_e^s(\tau) \int_\tau - a_e^s(\tau') \int_{\tau'} \right) \frac{\partial^3 u}{\partial \boldsymbol{n}_e^s \partial \boldsymbol{t}_e^{3-s}} \frac{\partial^2 v_h}{\partial \boldsymbol{t}_e^2} \right| \\ &\lesssim h^{4-\alpha} |u|_{4,p,\tau\cup\tau'} (|v_h|_{2-\alpha,q,\tau} + |v_h|_{2-\alpha,q,\tau'}) \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} &\left| [b_e^s(\tau) - b_e^s(\tau')] \int_e \frac{\partial^3 u}{\partial \boldsymbol{n}_e^s \partial \boldsymbol{t}_e^{3-s}} \frac{\partial^2 v_h}{\partial \boldsymbol{t}_e^2} \right| \\ &\lesssim h^{4-\alpha} \|u\|_{4,p,\tau\cup\tau'} (|v_h|_{2-\alpha,q,\tau} + |v_h|_{2-\alpha,q,\tau'}). \end{aligned} \tag{4.9}$$

Both of these estimates can be obtained by standard arguments and by using Lemma 3.3. $\square$

4.3. **Mildly structured grids.** The main result in this paper is on superconvergence for mildly structured grids. In practical computations, triangulations obtained by most grid generators cannot be completely unstructured. In most practical grids, most pairs of triangles form an approximate parallelogram although the entire grid is not strongly regular. We will show that a similar (but weaker) superconvergence result also holds on this type of grid.

Theorem 4.4. *Assume that the triangulation $\mathcal{T}_h = \mathcal{T}_{1,h} \cup \mathcal{T}_{2,h}$ satisfies the following properties: $\mathcal{T}_{1,h}$ is strongly regular and, for $\mathcal{T}_{2,h}$,*

$$|\Omega_{2,h}| = O(h^\sigma), \quad \bar{\Omega}_{2,h} \equiv \bigcup_{\tau \in \mathcal{T}_{2,h}} \bar{\tau}.$$

Then

$$(\nabla(u - \Pi_{\mathrm{Q}} u), \nabla v_h) = \mathcal{O}(h^{2+\min(\sigma/p,1/2)}) \, (|u|_{4,p,\Omega} + |u|_{3,\infty,\Omega}) \, |v_h|_{1,q,\Omega}. \tag{4.10}$$

Proof. By Lemma 3.5, we can write

$$(\nabla(u - \Pi_{\mathrm{Q}} u), \nabla v_h) = I_1 + I_2 + I_3 \tag{4.11}$$

where

$$\begin{aligned}
I_1 &= \sum_{e\in\mathcal{E}_1}\sum_{s=0}^{3}\left(a_e^s(\tau)\int_\tau - a_e^s(\tau')\int_{\tau'} + [b_e^s(\tau) - b_e^s(\tau')]\int_e\right)\frac{\partial^3 u}{\partial \boldsymbol{n}_e^s \partial \boldsymbol{t}_e^{3-s}}\frac{\partial^2 v_h}{\partial \boldsymbol{t}_e^2},\\
I_2 &= \sum_{e\in\mathcal{E}_2}\sum_{s=0}^{3}\left(a_e^s(\tau)\int_\tau - a_e^s(\tau')\int_{\tau'} + [b_e^s(\tau) - b_e^s(\tau')]\int_e\right)\frac{\partial^3 u}{\partial \boldsymbol{n}_e^s \partial \boldsymbol{t}_e^{3-s}}\frac{\partial^2 v_h}{\partial \boldsymbol{t}_e^2},\\
I_3 &= \mathcal{O}(h^3)|u|_{4,p,\Omega}|v_h|_{1,q,\Omega}.
\end{aligned}$$

Here $\mathcal{E}_1$ is the set of all edges $e = \tau \cap \tau'$ such that both τ and τ' belong to $\mathcal{T}_{1,h}$ and are $\mathcal{E}_2$ the rest of the edges.

Obviously I_1 can be estimated in the same way as in the proof of Theorem 4.3. To estimate I_2, we use the Hölder inequality. □

5. Numerical experiments

In this section, we will report some numerical experiments that support our theoretical estimates. We consider the Dirichlet problem for the Poisson equation posed on a unique square and a pentagon domain with the exact solution chosen to be e^{x+y}. We consider three different meshes as shown in Figure 2.

Table 1 records results for a uniform grid for the unit square domain. The results clearly indicate that $|u_h - \Pi_{\mathrm{Q}} u|_{1,\Omega} = \mathcal{O}(h^3)$ and and $|u_h - \Pi_{\mathrm{Q}} u|_{0,\infty,\Omega} = \mathcal{O}(h^4)$. Table 2 and Table 3 record results for a pentagon domain discretized with two sequences of grids that are obtained by a commercial mesh generator; the first is a sequence of piecewise uniform grids, and the other is a sequence of unstructured grids. In both cases, there are no superconvergence phenomenon observed for $|u_h - \Pi_{\mathrm{Q}} u|_{0,\infty,\Omega}$, which is expected; but we do observe superconvergence phenomenon for $|u_h - \Pi_{\mathrm{Q}} u|_{1,\Omega}$. Such a superconvergence is rather weak for the unstructured grids, but its presence is clear and significant.

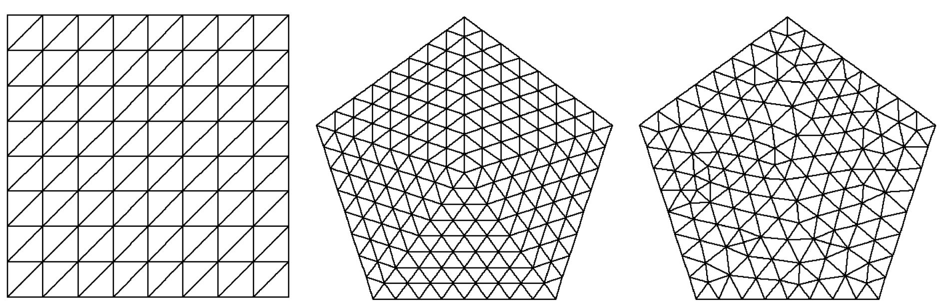

Figure 2. Three different triangulations

TABLE 1. Results for square domain, uniform refinement.

N	$\|\nabla u - \nabla u_h\|_0$		$\|\nabla \Pi_Q u - \nabla u_h\|_0$		$\|\Pi_Q u - u_h\|_{0,\infty}$	
	Error	*Order*	*Error*	*Order*	*Error*	*Order*
8	$1.93E-01$	1.98	$1.08E-02$	2.82	$2.57E-03$	3.71
32	$4.91E-02$	1.99	$1.52E-03$	2.91	$1.96E-04$	3.85
128	$1.23E-02$	2.00	$2.01E-04$	2.95	$1.36E-05$	3.93
512	$3.08E-03$	2.00	$2.58E-05$	2.98	$8.97E-07$	3.99
2048	$7.71E-04$	2.00	$3.27E-06$	2.97	$5.64E-08$	3.92
8192	$1.93E-04$		$4.17E-07$		$3.72E-09$	

TABLE 2. Results for structured mesh on polygon.

N	$\|\nabla u - \nabla u_h\|_0$		$\|\nabla \Pi_Q u - \nabla u_h\|_0$		$\|\Pi_Q u - u_h\|_{0,\infty}$	
	Error	*Order*	*Error*	*Order*	*Error*	*Order*
20	$6.65E-02$	2.00	$7.01E-03$	3.00	$1.77E-03$	3.09
80	$1.67E-02$	2.00	$1.18E-03$	2.83	$2.07E-04$	2.92
320	$4.17E-03$	2.00	$1.99E-04$	2.92	$2.73E-05$	2.92
1280	$1.04E-03$	2.00	$3.42E-05$	2.96	$3.61E-06$	2.96
5120	$2.61E-04$	2.00	$5.96E-06$	2.97	$4.65E-07$	2.97
20480	$6.52E-05$		$1.05E-06$		$5.95E-08$	

TABLE 3. Results for mildly structured mesh on polygon.

N	$\|\nabla u - \nabla u_h\|_0$		$\|\nabla \Pi_Q u - \nabla u_h\|_0$		$\|\Pi_Q u - u_h\|_{0,\infty}$	
	Error	*Order*	*Error*	*Order*	*Error*	*Order*
16	$1.01E-01$	2.19	$1.37E-02$	2.67	$3.79E-03$	3.44
64	$2.22E-02$	2.07	$2.14E-03$	2.25	$3.49E-04$	2.20
264	$5.29E-03$	2.34	$4.50E-04$	2.50	$7.62E-05$	3.62
962	$1.47E-03$	2.22	$1.14E-04$	2.30	$1.04E-05$	2.69
4537	$3.16E-04$	2.08	$2.32E-05$	2.34	$1.62E-06$	2.79
9372	$1.54E-04$	2.32	$1.03E-05$	2.59	$6.16E-07$	2.92
20779	$6.89E-05$		$4.19E-06$		$2.24E-07$	

6. Concluding remarks

The main result we obtain in this paper is that superconvergence estimates are still valid on a large class of practical grids in which most pair of triangles form approximate parallelograms. We expect this result to be useful for a posterior error estimation and adaptive finite element methods. We will report these applications in a future work.

We would like to make some further remarks on the techniques developed in this paper to establish our result. Classic techniques for analyzing higher order elements are complicated and difficult to make rigorous. Our new technique, based on the identity (1.1) is simple and transparent. In the identity (1.1) and its derivation, we have used the moment-based interpolants Π_{Q} and Π_{C}. We expect our new techniques to also apply to general higher order elements, which is a topic of future works.

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