A NEW ALGORITHM TO SEARCH FOR SMALL NONZERO $|x^3 - y^2|$ VALUES

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ABSTRACT. In relation to Hall's conjecture, a new algorithm is presented to search for small nonzero $k = |x^3 - y^2|$ values. Seventeen new values of $k < x^{1/2}$ are reported.

1. HALL'S CONJECTURE

Dealing with natural numbers, the difference

$$(1.1) \quad k = x^3 - y^2$$

is zero when $x = t^2$ and $y = t^3$ but, in other cases, it seems difficult to achieve small absolute values. For a given $k \neq 0$, (1.1), known as Mordell's equation, is an elliptic curve and has only finitely many solutions in integers by Siegel's theorem. Therefore, for any nonzero k value, there are only finitely many solutions in x (which is hence bounded). There is a proven lower bound, due to A. Baker [1] and improved by H. M. Stark [14], that places the size of k above the order of $\log^c(x)$ for any $c < 1$. A bound concerning the minimal growth rate of $|k|$ was found early by M. Hall [2, 7] by means of a parametric family of the form

$$(1.2) \quad \begin{aligned} f(t) &= \frac{t}{9}(t^9 + 6t^6 + 15t^3 + 12), \\ g(t) &= \frac{t^{15}}{27} + \frac{t^{12} + 4t^9 + 8t^6}{3} + \frac{5t^3 + 1}{2}, \\ f^3(t) - g^2(t) &= -\frac{3t^6 + 14t^3 + 27}{108}, \end{aligned}$$

with t congruent to 3 mod 6, which supplies infinitely many cases with $|k| < Cx^{3/5}$, where C is a positive constant.

H. Davenport [4], pointed out that the degree of $f^3(t) - g^2(t)$ is always greater than half of the degree of $f(t)$. This fact and experimental results for $x < 700,000$, prompted Marshall Hall [7] to conjecture that $|k|$ cannot be less than $Cx^{1/2}$, for some constant C whose tentative value was fixed to be $C = 1/5$. Later, L. V. Danilov [3] found an infinite family derived from the unbounded solutions of quadratic equations supplying values of $|k| < 217\sqrt{2}x^{1/2}$. N. D. Elkies [5] revised and improved the method, reporting the Fermat-Pell family

$$(1.3) \quad (5^5t^2 + 3000t + 719)^3 - (5^3t^2 - 114t + 26)(5^6t^2 - 5^3 \cdot 123t + 3781)^2 = 27(2t - 1),$$

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for t such that $5^3 t^2 - 114t + 26$ is a perfect square. This family yields infinitely many solutions with $|k| < 0.966 x^{1/2}$. Theoretical probabilistic considerations (see S. Lang [9, p. 163]) and the finding by computer search of smaller values of $|k|$ such as the present record found by Elkies [5] of $|k| \approx 1/46.6 x^{1/2}$ suggest that $1/2$ is not the correct exponent and so the original conjecture was reformulated in a weaker way as follows: *For any exponent $e < 1/2$, a constant $C_e > 0$ exists such that $|x^3 - y^2| > C_e x^e$.* Hall's conjecture is considered to be a particular case of the related and more general ABC conjecture [11, 9] and both seem hard to prove or disprove. S. Mohit and M. R. Murty [10] show that Hall's conjecture implies that there are infinitely many primes such that $a^{p-1} \not\equiv 1 \pmod{p^{16}}$ for any a .

Since, at present, Hall's conjecture is neither proved nor disproved, it is worthy to enumerate the cases when $|k| < x^{1/2}$, which we address as *good examples of Hall's conjecture*, borrowing the notation used in the ABC conjecture. Table 1 gathers the currently known examples (excluding the infinite Danilov-Elkies family), displaying the values of x and $r = \sqrt{x}/k > 1$. The values of y and k are not displayed because y can be calculated as the nearest integer to $x^{3/2}$. Items #2, #3, #12 and #13 correspond to (1.2) for $t = -3, 3, -9$ and 9 respectively. Items #4, #24 and #43 correspond to the three first solutions of (1.3), which, in fact, yield infinitely many examples with $r > 1.035$. J. Gebel, A. Pethö and H.G. Zimmer [6] applied their own algorithm to compute the integral points on elliptic curves in all the cases of (1.1) with $|k| \leq 100,000$. Their algorithm was able to find up to the item #14 in Table 1 and excludes the possibility of the existence of other cases with $|k| \leq 100,000$. N.D. Elkies [5] gives an extensive account of the subject and develops a lattice basis reduction algorithm finding the next items up to #26.

In this paper we present a new algorithm that adds 17 new items to the table of the known *good examples of Hall's conjecture*. The algorithm has also found 704 cases with $\sqrt{x}/k \in (0, 16]$, which turn out to be uniformly distributed in this range. The data also show that the number of cases for $x < X$ and $k \leq n\sqrt{x}$ may be estimated as $0.80 n \log(X)$. The algorithm was developed following a heuristical approach, noticing the patterns (see Figure 1) of the plots of solutions of (1.1) such that $|k|$ is minimal for each x value, rather than using basis reduction algorithms, common in this kind of problems. The algorithm searches a family of polynomials that contain those minimal solutions. It has the drawback that it does not find all the cases in increasing order and we cannot prove that the algorithm does not miss any solution. Nevertheless, it has been able to find all the *good examples of Hall's conjecture* known before this work, including the first cases of the infinite Danilov-Elkies family (we conjecture that the new algorithm could also find the rest of the cases, with enough time of computation).

2. VALUES OF k FOR x NEAR THE SQUARE OF A RATIONAL

We consider the smallest k satisfying (1.1) as a function of x . Using the notation $[x] = \lfloor x + 1/2 \rfloor$ for the nearest integer value to the value of x , it is given by

$$(2.1) \quad k(x) = x^3 - \left[x^{3/2} \right]^2,$$

where $y = \lfloor x^{3/2} \rfloor$. It is clear that $k(x) = 0$ for $x = t^2$, t an integer.

These expressions for x and y are somehow related with the expressions given by Elkies in equations (38) and (40) of [5], where x and y are approximated by multiples of a square and a cube, respectively. Specifically, he writes $x = 3\alpha^2 + \beta$

and $y = 6\alpha^3 + 3\alpha\beta + \delta$. Then, Elkies argues that the problem can be studied via lattice reduction. We use a different approach, maybe more heuristic and intuitive, which does not employ lattice reductions.

Our method starts by allowing rational values for t and considering values of x for the integers close to t^2 . We define

$$(2.2) \quad t = a/b, \quad \alpha \equiv a^2 \pmod{b^2} \quad \text{with} \quad |\alpha| \leq b^2/2 \quad \text{and} \quad x_0 = \lfloor t^2 \rfloor = \frac{a^2 - \alpha}{b^2}.$$

Function (2.1) for the integer values of x near x_0 , say for $x_0 + i$, where $i = 0, \pm 1, \pm 2, \dots$, takes the form

$$(2.3) \quad k(x_0 + i) = (x_0 + i)^3 - y^2, \quad y = \left\lfloor (x_0 + i)^{3/2} \right\rfloor.$$

Lemma 2.1. *Function (2.1) for integer values of the variable near t^2 , t rational, is equivalent to the equation,*

$$(2.4) \quad k(x_0 + i) = \frac{-2Ca^3 + \frac{3}{4}(b^2i - \alpha)^2a^2 - 3(b^2i - \alpha)Ca - C^2 + (b^2i - \alpha)^3}{b^6},$$

for appropriate values of C and i satisfying

$$(2.5) \quad 2a^3 + 3(b^2i - \alpha)a + 2C \equiv 0 \pmod{2b^3}$$

and such that $|k(x_0 + i)| \leq (x_0 + i)^{3/2} + \frac{1}{4}$.

Proof. We note that, by the definition of x_0 ,

$$(x_0 + i)^3 = \frac{a^6 + 3(b^2i - \alpha)a^4 + 3(b^2i - \alpha)^2a^2 + (b^2i - \alpha)^3}{b^6}.$$

Suppose that

$$y^2 = \frac{a^6 + 3(b^2i - \alpha)a^4 + p(a)}{b^6} = \left(\frac{a^3 + Ba + C}{b^3} \right)^2$$

for an appropriate third degree polynomial $p(a)$ and coefficients B and C . Since y must be an integer, the following two conditions must be fulfilled:

- (i) $(a^3 + Ba + C)^2 = a^6 + 3(b^2i - \alpha)a^4 + p(a)$.
- (ii) $b^3 \mid a^3 + Ba + C$.

Expanding the square of the trinomial in (i) we can deduce that $B = 3/2(b^2i - \alpha)$ and $p(a)$ takes the following expression in C :

$$p(a) = 2Ca^3 + \frac{9}{4}(b^2i - \alpha)^2a^2 + 3(b^2i - \alpha)Ca + C^2.$$

The condition (ii) now becomes $a^3 + \frac{3}{2}(b^2i - \alpha)a + C \equiv 0 \pmod{b^3}$. Multiplying by 2, we have (2.5). Substituting the values of $(x_0 + i)^3$ and y^2 in (2.3), we have (2.4), where values C and i must fulfill (2.5). The condition $|k(x_0 + i)| \leq (x_0 + i)^{3/2} + 1/4$ easily follows from the bounds of (2.1). Note that $2C$ is only determined modulo $2b^3$ by (2.5), so that we can take $|2C| < b^3$ and the absolute value of the leading term of (2.4) is always smaller than $a^3/b^3 \approx x_0^{3/2}$; hence the bounds of (2.4) are the bounds of (2.3) for moderate i values. $\square$

The allowed values of C imposed by (2.5) form a small set for a fixed t , due to the coefficient b^2 of i . Figure 1 depicts an example for $t = a/b = 222272/15$. In this case five values are possible, all of them congruent mod $3b^2$ and for each one of the five C values, the points split as integer points of five cubic polynomials.

FIGURE 1. The polynomials for $t = 222272/15$

The number of polynomials and the values of C depend on b , the parity of a and whether 3 divides b or not. The analysis is done in the next section.

3. THE GEOMETRY OF $k(x)$

From (2.5), we obtain

$$3b^2ai \equiv 3a\alpha - 2a^3 - 2C \pmod{2b^3}.$$

Both sides of the equation must be divisible by $g = \gcd(3b^2a, 2b^3)$ to be solvable in i . Note that, since the fraction t is in its lowest terms, $\gcd(a, b) = 1$, then

$$g = c_1c_2b^2, \quad \begin{cases} c_1 = 2 \text{ if } a \text{ is even, } c_1 = 1 \text{ otherwise;} \\ c_2 = 3 \text{ if } 3|b, c_2 = 1 \text{ otherwise.} \end{cases}$$

Cancelling by g , we obtain

$$(3.1) \quad \left(\frac{3a}{c_1c_2} \right) i \equiv \left(\frac{3a\alpha - 2a^3 - 2C}{c_1c_2b^2} \right) \pmod{\frac{2b}{c_1c_2}}.$$

Since the right side of the equation must be an integer, we have that

$$3a\alpha - 2a^3 - 2C \equiv 0 \pmod{c_1c_2b^2}.$$

Note that if $\alpha = a^2 + rb^2$ for some integer r , then

$$3a\alpha \equiv 3a^3 + 3arb^2 \equiv 3a^3 \pmod{c_1c_2b^2}$$

for any of the possible values of c_1 and c_2 . Then,

$$(3.2) \quad (2C) \equiv a^3 \pmod{c_1 c_2 b^2} \begin{cases} (2C) \equiv a^3 \pmod{c_2 b^2}, & \text{when } a \text{ is odd,} \\ C \equiv a^3/2 \pmod{c_2 b^2}, & \text{when } a \text{ is even.} \end{cases}$$

The values of C are of the form $C = C_0 + s c_2 b^2$, for some integer s . Note that whenever a is odd, C may be a rational of the form $C = C'/2$, with C' an integer. For a even, C is always integral.

For each value of C , the allowed values of i are of the form $i = j + \omega b'$, $b' = 2b/(c_1 c_2)$, where $\omega \in \mathbb{Z}$, j being a solution of (3.1) for each such C value. We have split up the points $x_0 + i$ into b' congruence classes. For each one, we can associate a unique value of C . For each one of the b' couples (j, C) , we can substitute the value of i in (2.4), getting a cubic polynomial in the variable ω that contains the family of points corresponding to a value j . The effective value of C (out of the b' significant possible values) must be chosen so that $k(x)$ is bounded as imposed by Lemma 2.1.

An example. We apply the theory to the case illustrated in Figure 1, where $a = 222272$ and $b = 15$. Then by (2.2),

$$\alpha = a^2 \pmod{b^2} = 109, \quad x_0 = (a^2 - \alpha)/b^2 = 219577075.$$

Since a is even, $c_1 = 2$ while $c_2 = 3$ because $3|b$. Applying (3.2) we have that

$$C_0 = a^3/2 \pmod{3b^2} = 499; \text{ then } C = C_0 + 3b^2 s = 499 + 675s, \quad s \in \mathbb{Z}.$$

Applying (3.1) we guess that for $C = 499$, $i \equiv 4 \pmod{5}$, then $i = 4 + 5\omega$. That is, for $x = x_0 + 4 + 5\omega$, the values of $k(x)$ correspond to the points labelled as P_4 in the figure.

Theorem 3.1. *The integer points of $k(x) = x^3 - \lfloor x^{3/2} \rfloor^2$, for x near the square of a rational $t = a/b$, are integer points of a set of polynomials*

$$P_j(x) = x^3 - \left(\frac{3}{2}t\right)^2 x^2 + \left(\frac{3}{2}t^4 - \frac{3C}{b^3}t\right)x + \frac{C}{b^3}t^3 - \frac{C^2}{b^6} - \frac{1}{4}t^6$$

for $0 \leq j < b'$, $b' = 2b/\gcd(3a, 2b)$ and a computable number C , provided $|P_j(x)| \leq x^{3/2} + 1/4$ and $x = \lfloor t^2 \rfloor + j + \omega b'$, $\omega \in \mathbb{Z}$.

Proof. We note that, for each C value in (3.1), i is in $Z_{b'}$, with $b' = 2b/(c_1 c_2) = 2b/\gcd(3a, 2b)$. Then, we put $i = j + \omega b'$ and we have b' cases of values (C, j) for each t value. We substitute $j + \omega b'$ for i in (2.4). Then the term $(b^2 i - \alpha)$ becomes $(b^2(j + \omega b') - \alpha) = b^2(b'\omega + j - \frac{\alpha}{b^2})$. We get a polynomial in ω for each j ,

$$P_j(\omega) = (b'\omega + j - \frac{\alpha}{b^2})^3 + \frac{3}{4}t^2(b'\omega + j - \frac{\alpha}{b^2})^2 - \frac{3C}{b^3}t(b'\omega + j - \frac{\alpha}{b^2}) - \frac{2C}{b^3}t^3 - \frac{C^2}{b^6}.$$

Recall that $x = x_0 + j + b'\omega$. Let $z = x - t^2$. Then

$$(3.3) \quad z = x_0 + j + b'\omega - t^2 = j + b'\omega - \frac{\alpha}{b^2},$$

as follows from (2.2). Then

$$(3.4) \quad P_j(z) = z^3 + \frac{3}{4}t^2 z^2 - \frac{3C}{b^3}t z - \frac{2C}{b^3}t^3 - \frac{C^2}{b^6}.$$

We can rewrite the polynomial as a function of x by means of the variable change $z = x - t^2$, obtaining the polynomials $P_j(x)$ in the statement of the theorem.

As in Lemma 2.1, we must impose that $k(x)$ is bounded by $x^{3/2} + 1/4$. We can compute C by (3.2). $\square$

We analyze the polynomial for x near t^2 , i.e. for $z \approx 0$. Note that, for small z , the cubic term is small compared with the quadratic one. Then it is locally approximated by a parabola with the vertex near $z = 0$. The value of the polynomial at the vertex is therefore near

$$P_j(z = 0) \approx -\frac{2C}{b^3}t^3.$$

Since the values of C for some t are equally spaced (they are of the form $C = C_0 + sc_2b^2$ with s integral), the vertices are also approximately equally spaced. Note that, for $C > 0$, the value of the polynomial is negative in the vertex, but the polynomial grows until crossing $P_j(z) = 0$. Recall that for integral ω , the polynomials give $k(x)$. In this way, it is possible to find a small $k(x)$, selecting for each fixed a/b the polynomial with lowest positive C value (the polynomial P_4 in the example of Figure 1) and calculating the value ω where the polynomial has its smallest absolute value. Nevertheless, better results can be achieved by calculating the optimal value of a for each value b , for which we can select an optimal polynomial. First, we can impose C to be positive and small (say $C = 1/2, 1, 3/2, \dots$). We can also impose that the smallest value of the polynomial is for $\omega = 0$, so the value of z in (3.3) is small and we can hope for a small value of $P_j(z)$. Even more, if $j = 0$, then $z = \alpha/b^2 \leq 1/2$, which is the smallest value possible for z . We can approximate the value of P_0 , for $\omega = 0$, by taking only the terms in t^3 and t^2 in (3.4), getting

$$(3.5) \quad P_0(\omega = 0) \approx \frac{3}{4}t^2\frac{\alpha^2}{b^4} - \frac{2C}{b^3}t^3.$$

Suppose that $a < O(b^{5/2})$, $\alpha < O(b^{5/4})$ and $C = O(1)$. Then $x \approx a^2/b^2 < O(b^3)$ and both terms of the approximation are smaller than $O(x^{1/2})$. Even if a and α are above those bounds, the polynomial may have a small value if both terms are balanced.

4. THE ALGORITHM

The algorithm runs over $b = 2, 3, 4, \dots$, and for small values of C whose bounds will be discussed later. For each pair (b, C) , we must compute the values of a (if any value exists) in (2.5) corresponding to the polynomial P_0 (that is, for $j = 0$) whose smallest absolute value is for $\omega = 0$. So, (2.5) becomes

$$(4.1) \quad 2a^3 - 3a\alpha + 2C \equiv 0 \pmod{2b^3}.$$

First, we compute the value of a modulo b^2 and then we lift to modulo $2b^3$. From (3.2), we have that

$$(4.2) \quad a \equiv (2C)^{1/3} \pmod{b^2}.$$

Note that the equation is solvable only when $\gcd(2C, b) = 1$ because a and b are co-prime. The cube root may be computed, for example, using the algorithms described in [12]. Before lifting the solution of (4.2) to modulo $2b^3$ in (2.5), we must analyze the possible values of the variables whose constraints can be summarized as

$$\gcd(2C, b) = 1, \quad b \begin{cases} \text{even, } C \text{ rational,} \\ \text{odd, } \begin{cases} C \text{ rational,} \\ C \text{ integer.} \end{cases} \end{cases}$$

In fact, if b is even, a must be odd because t is in its lower terms. b being even and a odd, α must be odd, by the definition of α in (2.2), which implies that $(2C)$ in (4.1) must be odd. Therefore, $C = C'/2$ for some odd integer C' . If b is odd, a may be even or odd, then C may be rational or integer.

Let a_0 be a solution of (4.2). Then $a = a_0 + kb^2$ for some integer k that must fulfill (4.1). So, we have

$$2(a_0 + kb^2)^3 - 3\alpha(a_0 + kb^2) + 2C \equiv 0 \pmod{2b^3}.$$

Expanding and neglecting the vanishing terms, we have

$$2a_0^3 + 6a_0^2kb^2 - 3\alpha a_0 - 3\alpha kb^2 + 2C \equiv 0 \pmod{2b^3}.$$

Then

$$3b^2(2a_0^2 - \alpha)k \equiv -(2a_0^3 - 3\alpha a_0 + 2C) \pmod{2b^3}.$$

Let

$$d = \gcd(3(2a_0^2 - \alpha), 2b).$$

Then

$$(4.3) \quad k \equiv - \left(\frac{3(2a_0^2 - \alpha)}{d} \right)^{-1} \left(\frac{2a_0^3 - 3\alpha a_0 + 2C}{db^2} \right) \pmod{\frac{2b}{d}}.$$

Note that the possible values of d may be $d = 1$, $d = 3$ if $3|b$, $d = 2$ if α is even or $d = 6$ if both conditions hold. Let k_0 be a solution of the above equation. Then $k = k_0 + n2b/d$ for some integer n . Substituting the value of k in $a = a_0 + kb^2$, we have finally

$$a = a_0 + k_0b^2 + n\frac{2b^3}{d}.$$

Since we are looking for small values of P_0 in (3.5), that is, for $\omega = 0$, we want that $a \approx (3\alpha^2)/(8C)$. Then

$$n = \left\lfloor \frac{d}{2b^3} \left(\frac{3\alpha^2}{8C} - a_0 - k_0b^2 \right) \right\rfloor.$$

Once the most appropriate values a for each couple (b, C) are computed, we can compute $x_0 = (a^2 - \alpha)/b^2$ expecting to find a small value of $k(x)$.

5. RESULTS AND CONCLUSIONS

The algorithm was programed as a PARI script and translated to C with the utility GP2C [13]. The executable was run on various computers during several periods, the total time of computation being equivalent to about 1000 days in an AMD-64 3400+. The algorithm was able to find all the so far known items and the remaining 17 new items displayed in Table 1, with $r = \sqrt{x}/k \geq 1$. Three items were found by Johan Bosman using the software of the authors. We include the third term of the Danilov-Elkies family, item #43, because we can compute the value b , by taking an appropriate convergent of $\sqrt{x} \approx a/b$, for which the algorithm would find this item. We conjecture that the algorithm would be able to find all the terms in this infinite family, checking the values of C from $1/2$ to $b^{1/3}$, if it would be run with enough computational time.

TABLE 1. Good examples of Hall's conjecture.

H: Found by M. Hall [7]. **GPZ**: Found by J. Gebel, A. Pethö and H. G. Zimmer [6] with the elliptic curve method.

E: Found by N.D. Elkies with a lattice basis reduction algorithm.

JHS: Found by the authors of this paper.

JB: Found by Johan Bosman with the software of the authors.

(1.2): Items #2, #3, #12 and #13 correspond to the polynomial family for $t = -3, 3, -9$ and 9 respectively in equation (1.2).

(1.3): The three first terms of the infinite Fermat-Pell family from equation (1.3).

!: Record of N.D. Elkies.

*: Obtained from #20 scaling (x, y, k) to $(2^2x, 2^3y, 2^6k)$.

#	x	r	b	C	Comments
1	2	1.41	-	-	-
2	5234	4.26	26	1/2	H, GPZ, (1.2)
3	8158	3.76	28	1/2	H, GPZ, (1.2)
4	93844	1.03	53	1	H, GPZ, (1.3)
5	367806	2.93	117	1/2	H, GPZ
6	421351	1.05	26	1/2	H, GPZ
7	720114	3.77	42	1/2	H, GPZ
8	939787	3.16	115	2	H, GPZ
9	28187351	4.87	159	5	H, GPZ
10	110781386	1.23	95	1/2	H, GPZ
11	154319269	1.08	228	1/2	H, GPZ
12	384242766	1.34	728	1/2	H, GPZ, (1.2)
13	390620082	1.33	730	1/2	H, GPZ, (1.2)
14	3790689201	2.20	1155	4	GPZ
15	65589428378	2.19	5235	17/2	E
16	952764389446	1.15	1448	5/2	E
17	12438517260105	1.27	13415	6	E
18	35495694227489	1.15	97266	1/2	E
19	53197086958290	1.66	13777	1	E
20	5853886516781223	46.60	137035	9	E, !
21	12813608766102806	1.30	6291	35/2	E
22	23415546067124892	1.46	1315447	32	E, *
23	38115991067861271	6.50	321346	1/2	E
24	322001299796379844	1.04	1313479	11	E, (1.3)
25	471477085999389882	1.38	3281374	95/2	E
26	810574762403977064	4.66	5346121	49/2	E
27	9870884617163518770	1.90	4928788	109/2	JHS
28	42532374580189966073	3.47	583876	9/2	JHS
29	51698891432429706382	1.75	19061951	29	JHS
30	44648329463517920535	1.79	11744301	13	JHS
31	231411667627225650649	3.71	11694866	347/2	JHS
32	601724682280310364065	1.88	7496613	13	JHS
33	4996798823245299750533	2.17	76010518	67/2	JHS
34	5592930378182848874404	1.38	93203798	139/2	JHS
35	14038790674256691230847	1.27	61769318	53/2	JHS
36	77148032713960680268604	10.18	184388019	4	JB
37	180179004295105849668818	5.65	292889921	45/2	JB
38	372193377967238474960883	1.33	2554989	4	JHS
39	664947779818324205678136	16.53	678534061	39/2	JHS
40	2028871373185892500636155	1.14	490670918	55/2	JB
41	37223900078734215181946587	1.87	530793746	1457/2	JHS
42	3690445383173227306376634720	1.51	685266726	1/2	JHS
43	1114592308630995805123571151844	1.04	52019836686	1475/2	(1.3)
44	6078673043126084065007902175846955	1.03	8144029787	3	JHS

The value of the integer b runs from 2 to a bound B while for each value of b , values of C from $1/2$ to b^u were checked. Experimentally, it was found that a good interval for C corresponds to $u = 1/3$. With a shorter interval, for example $u = 1/4$, the algorithm runs faster but cannot find some items in the table. In the first case, for $1/2 \leq C \leq b^{1/3}$, the bound reached for b was $B = 6 \cdot 10^8$, while for $1/2 \leq C \leq b^{1/4}$ the bound was $B = 5 \cdot 10^9$. Items #42 and #43 were found constraining the search to lower values of C . Although the bounds used for C seem reasonable, it cannot be proved that the algorithm can find all the existing *good examples of Hall's conjecture*.

The exact complexity of the algorithm must be expressed only as a function of the bound of b . For each b value, we must factorize b and, for each of the b^u values of C , we must perform a cube root and other polynomial time operations. Suppose that the factorization of b takes $O(b^v)$ operations. Then it takes $O(B^{1+\max\{u,v\}} \log^{O(1)}(B))$ operations to explore the $b \leq B$ values. Since a simple trial division with a table of primes was used, the exponent v must be under 0.35 in about half of the instances (see [8]). It is not possible to express the complexity as a fixed function of the bound of x ; nevertheless, an estimation can be done. The value of a is $O(b^\theta)$ for a variable exponent θ that is 2.5 in average (see the last paragraph in Section 3). Then, the corresponding value of x is $x \approx a^2/b^2 = b^{2\theta-2}$. Since $\max\{u, v\} = 0.35$, the complexity takes the form $O(X^{\frac{1.35}{2\theta-2}} \log^{O(1)}(X))$ for $x < X$. If we consider the average value $\theta = 2.5$ we obtain an equivalent complexity of $O(X^{0.45})$, which coincides with the experimental average complexity.

Another search was run, seeking for cases in which $k \leq 16\sqrt{x}$ for $b < 12,500,000$. The algorithm discovered 704 cases whose analysis shows that the distribution of the values of $k/\sqrt{x} \in (0, 16]$ is very close to a uniform distribution for this range of k (the mean is 7.996 and the Kolmogorov-Smirnov test gives a confidence of 98.60% to the hypothesis of a uniform distribution). This gives extra support to the possibility that the size of $k/\sqrt{x}$ may be arbitrarily small, and so the corresponding exponent in Hall's conjecture must be at most $1/2 - \epsilon$. The data also support that the number of cases for $x < X$ and $k \leq n\sqrt{x}$ may be estimated as $0.80 n \log(X)$, somewhat less than the expected value $n \log(X)$.

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