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A LOCAL MIN-MAX-ORTHOGONAL METHOD FOR FINDING MULTIPLE SOLUTIONS TO NONCOOPERATIVE ELLIPTIC SYSTEMS

XIANJIN CHEN AND JIANXIN ZHOU

Abstract. A local min-max-orthogonal method together with its mathematical justification is developed in this paper to solve noncooperative elliptic systems for multiple solutions in an order. First it is discovered that a noncooperative system has the nature of a zero-sum game. A new local characterization for multiple unstable solutions is then established, under which a stable method for multiple solutions is developed. Numerical experiments for two types of noncooperative systems are carried out to illustrate the new characterization and method. Several important properties for the method are explored or verified. Multiple numerical solutions are found and presented with their profiles and contour plots. As a new bifurcation phenomenon, multiple asymmetric positive solutions to the second type of noncooperative systems are discovered numerically but are still open for mathematical verification.

1. Introduction

Involving two or more components (particles, molecules, species, etc.), nonlinear differential systems (e.g., the nonlinear Schrödinger systems) are known to have many applications. In study of pattern formation, stability/instability, and other evolution dynamics, standing solitary wave or steady state solutions are of great interest to many researchers. More often, those differential systems result in certain semilinear elliptic systems, of which three types have drawn much attention recently [4, 5, 6, 8, 9, 10, 11, 15] due to their wide application background. They are the **cooperative** system

$$\begin{cases} -\Delta u(x) = G_u(x, u(x), v(x)), & x \in \Omega, \\ -\Delta v(x) = G_v(x, u(x), v(x)), & x \in \Omega, \end{cases} \tag{1.1}$$

with energy functional

$$J(u, v) = \int_\Omega \left[\frac{1}{2}(|\nabla u(x)|^2 + |\nabla v(x)|^2) - G(x, u(x), v(x)) \right] dx, \tag{1.2}$$

the **noncooperative** system

$$\begin{cases} -\Delta u(x) = G_u(x, u(x), v(x)), & x \in \Omega, \\ -\Delta v(x) = -G_v(x, u(x), v(x)), & x \in \Omega, \end{cases} \tag{1.3}$$

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with energy functional

$$J(u,v) = \int_\Omega \left[\frac{1}{2}(|\nabla u(x)|^2 - |\nabla v(x)|^2) - G(x,u(x),v(x))\right] dx, \tag{1.4}$$

and the **Hamiltonian type** system

$$\begin{cases} -\Delta u(x) = G_v(x,u(x),v(x)), & x \in \Omega, \\ -\Delta v(x) = G_u(x,u(x),v(x)), & x \in \Omega, \end{cases} \tag{1.5}$$

with energy functional

$$J(u,v) = \int_\Omega [\nabla u(x)\cdot\nabla v(x) - G(x,u(x),v(x))]\, dx, \tag{1.6}$$

where Ω is a bounded open domain in $\mathbb{R}^N (N \geq 1)$, $G : \overline{\Omega}\times\mathbb{R}^2 \to \mathbb{R}$ is of class C^1 in the variables $(u,v) \in \mathbb{R}^2$ with gradient $\nabla G = (G_u, G_v)$ and satisfies some growth conditions. Here, zero Dirichlet or Neumann boundary conditions are assumed. Certain qualitative results for those systems including the existence or multiplicity of their solutions have been established under some suitable assumptions; see [7, 8, 9, 10, 11, 15] and the references therein. As a subsequent paper to [5, 6], we continue developing some computational theory and methods to solve those systems for their multiple solutions in an order.

Example 1.1. The Gross-Pitaevskii system [4, 10, 13]

$$\mathbf{i}\frac{\partial \Phi_j}{\partial t} = \Delta\Phi_j - V_j(x)\Phi_j - \mu_j|\Phi_j|^2\Phi_j - \sum_{i\neq j}\beta_{ij}|\Phi_i|^2\Phi_j,\ j = 1,2,...,m \tag{1.7}$$

has been widely used to describe multi-species Bose-Einstein condensations (BEC) in m different hyperfine spin states on the corresponding condensate wave functions Φ_j, where V_j is the magnetic trapping potential for the jth hyperfine spin state, the constants μ_i and β_{ij} are the intraspecies and interspecies scattering lengths which represent the interactions between "like" and "unlike" particles, respectively; e.g., $\beta_{ij} > 0\,(< 0)$ means repulsive (attractive) interaction between the ith and jth particles. To find its solitary wave solutions of the form $\Phi_j = e^{-\mathbf{i}\lambda_j t}u_j(x)$, one may transform system (1.7) into the elliptic system

$$-\Delta u_j + (V_j(x) + \lambda_j)u_j + \mu_j u_j^3 + \sum_{i\neq j}\beta_{ij}u_i^2 u_j = 0,\ j = 1,...,m. \tag{1.8}$$

Here, λ_j's are some positive constants. When $m = 2$, it is easy to see that system (1.8) is cooperative if $\beta_{ij}\beta_{ji} > 0$, for which some numerical results can be found in [4]; while system (1.8) becomes noncooperative if $\beta_{ij}\beta_{ji} < 0$, for which there is no efficient or reliable numerical method available so far for finding its multiple nontrivial solutions.

Since cooperative systems have already been studied in [5, 6] and many Hamiltonian type systems can actually be converted into noncooperative ones by change of variables, we will focus on noncooperative systems in this work.

Next, to see why our local min-orthogonal method (LMOM) developed in [5, 6] for the cooperative case cannot be applied to the noncooperative case, let us explain an essential difference between these two systems.

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and $\phi \in C^1(H,\mathbb{R})$. A point $u_0 \in H$ is a critical point of ϕ if $\phi'(u_0) = 0$ where ϕ' is the first Fréchet derivative of ϕ. Obviously, any local extremum of ϕ is a critical point. Critical points that are not local extrema of ϕ are called saddle points. The Morse index (MI) of a critical point u_0 of ϕ is the dimension of the maximum negative definite subspace of $\phi''(u_0)$ in H. When ϕ is of the form

$$\phi(\mathbf{u}) = \frac{1}{2}\langle A\mathbf{u}, \mathbf{u}\rangle + b(\mathbf{u}) \tag{1.9}$$

where $A : H \to H$ is an invertible self-adjoint linear operator and b is a nonlinear functional with compact gradient $\nabla b \in C(H, H)$, ϕ is called *strongly indefinite* if both the positive and negative eigenspaces of A are infinite-dimensional; ϕ is called *positive (semi-positive) definite* if the dimension of the negative eigenspace of A is zero (finite).

It is obvious that if ϕ is strongly indefinite, so is $-\phi$. In this case, the Morse index of every critical point of both ϕ and $-\phi$ is infinite and hence provides no help for one to find those critical points [1, 2]. This also implies that a strongly indefinite functional is neither bounded from above nor from below, not even modulo any finite-dimensional subspace [12].

Taking (1.9) into account, one sees that the linear operator, denoted by A_c, in functional (1.2) for the cooperative case and the linear operator, denoted by A_{nc}, in functional (1.4) for the noncooperative case are, respectively,

$$A_c = \begin{bmatrix} -\Delta & 0 \\ 0 & -\Delta \end{bmatrix}, \quad A_{nc} = \begin{bmatrix} -\Delta & 0 \\ 0 & \Delta \end{bmatrix}. \tag{1.10}$$

For both cases, the term $b(\mathbf{u}) = \int_\Omega G(x, u, v)dx$ has a compact gradient if it does not grow too rapidly, e.g., it satisfies some subcritical growth condition [7, 8], see also condition (F_1) in Section 4.1. Hence, functional (1.2) is positive definite and has critical points with a finite Morse index; while functional (1.4) is strongly indefinite and each of its critical points has an infinite Morse index.

By [5, 6], if a critical point u^* found is nondegenerate, there exists a finite-dimensional support L with $\dim(L) = \text{MI}(u^*) - 2$. Since the functional J in (1.4) is strongly indefinite and $\text{MI}(u^*) = \infty$, a finite-dimensional support L for LMOM does not exist; see also Theorem 4.2. Hence, LMOM cannot be applied to solve noncooperative systems. In fact, there is no reliable numerical method available so far for solving such strongly indefinite problems.

To further motivate our new method, let us view the two components u and v as two players in a two-person game and define their objective functions respectively by

$$f(u, v) = \int_\Omega [\frac{1}{2}|\nabla u(x)|^2 - G(x, u(x), v(x))]dx = J(u, v) \pm \alpha(v), \tag{1.11}$$

$$g(u, v) = \int_\Omega [\frac{1}{2}|\nabla v(x)|^2 - (\pm)G(x, u(x), v(x))]dx = \pm\left(J(u, v) + \beta(u)\right), \tag{1.12}$$

where $\alpha(w) = \beta(w) = -\frac{1}{2}\int_\Omega |\nabla w(x)|^2 dx$, the functional J is as in (1.2) and the sign is "+" for the cooperative case (1.1), and the functional J is as in (1.4) and the sign is "$-$" for the noncooperative case (1.3). Then, (u^*, v^*) is a solution to system (1.1) or (1.3) iff (u^*, v^*) solves the system

$$f'_u(u, v) = 0, \qquad g'_v(u, v) = 0. \tag{1.13}$$

Obviously, the term $\pm\alpha(v)$ can be viewed as a constant to the player u and so can the term $\pm\beta(u)$ to the player v. Thus, $J(u, v)$ and $\pm J(u, v)$ are the essential parts of their objective functions, respectively.

For the cooperative system (1.1), the essential parts of the two players' objective functions are the same, i.e., $J(u,v)$. So it is quite natural to call system (1.1) cooperative in the sense of game theory. In this case, the objective functions f and g are positive definite in u and v, respectively, and a solution (u^*,v^*) to system (1.1) can be found through a two-person game

$$\left\{ \begin{array}{lcl} u^* & = & \arg\min_{u\in\mathcal{N}(u^*)} f(u,v^*), \\ v^* & = & \arg\min_{v\in\mathcal{N}(v^*)} g(u^*,v), \end{array} \right. \tag{1.14}$$

where $\mathcal{N}(u^*),\mathcal{N}(v^*)$ are some open neighborhoods of u^*,v^*, respectively. On the other hand, for the noncooperative system (1.3), the essential parts of the two players' objective functions f,g are respectively $J(u,v)$ and $-J(u,v)$, where J is as in (1.4). Hence, a solution (u^*,v^*) to system (1.3) can be found by a two-person zero-sum game

$$\left\{ \begin{array}{l} u^* = \arg\min_{u\in\mathcal{N}(u^*)} f(u,v^*) \\ v^* = \arg\min_{v\in\mathcal{N}(v^*)} g(u^*,v) \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{l} u^* = \arg\min_{u\in\mathcal{N}(u^*)} J(u,v^*) \\ v^* = \arg\max_{v\in\mathcal{N}(v^*)} J(u^*,v) \end{array} \right. \tag{1.15}$$

or equivalently by a local saddle point problem

$$J(u^*,v) \le J(u^*,v^*) \le J(u,v^*), \quad \forall u\in\mathcal{N}(u^*), v\in\mathcal{N}(v^*).$$

Of course, it becomes much more complicated as multiple solutions are concerned. However, the discovery of the nature of a zero-sum game for problem (1.3) leads us to develop a new local saddle point characterization and a new stable numerical method, which hereafter is called a local min-max-orthogonal method (LMMOM), for finding multiple saddle points to certain strongly indefinite functionals. This method is the first one of its kind so far.

This paper is organized as follows. In Section 2, we establish a local min-max-orthogonal characterization for saddle points to strongly indefinite functionals of the form (1.4). In Section 3, we develop a numerical method for saddle points of infinite Morse index. In the final section, we carry out numerical experiments on two types of noncooperative systems to illustrate this new characterization and method. We also verify certain important properties (e.g., existence, differentiability, separation) that are closely related to our method.

2. A local min-max-orthogonal characterization

For $i=1,2$, let H_i be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, let L_i be a closed subspace of H_i and let $H_i = L_i \oplus L_i^\perp$ be its orthogonal decomposition. Denote $H = H_1\times H_2$, $L = L_1\times L_2$. Thus $L_1^\perp \times L_2^\perp = L^\perp$ and $H = L\oplus L^\perp$. Denote $S_B = \{u\in B : \|u\| = 1\}$ for any closed subspace B of $H_i\,(i=1,2)$ or H and let $[L_i,v] = \{tv+w | w\in L_i, t\in\mathbb{R}\}$, $\forall v\in S_{L_i^\perp}, i=1,2$. Assume $J\in C^1(H,\mathbb{R})$ and denote by $\nabla J \equiv (\partial_1 J, \partial_2 J)$ its gradient.

Definition 2.1 ([6]). The set-valued mapping $P: S_{L^\perp}\to 2^H$ is the L-$\perp$ mapping of J if for each $v=(v_1,v_2)\in S_{L^\perp}$

$$P(v) = \Big\{ u\in[L_1,v_1]\times[L_2,v_2] : \partial_1 J(u)\perp[L_1,v_1], \partial_2 J(u)\perp[L_2,v_2] \Big\}.$$

A single-valued mapping $p: S_{L^\perp}\to H$ is called an L-$\perp$ selection of J if $p(v)\in P(v)$, $\forall v\in S_{L^\perp}$. For a given $w\in S_{L^\perp}$, if such p is locally defined in $\mathcal{N}(w)\cap S_{L^\perp}$ where $\mathcal{N}(w)$ is some neighborhood of w, then p is called a local L-$\perp$ selection of J at w; in addition, if $p(v)$ is a local maximum of J in $[L_1,v_1]\times[L_2,v_2]$ for each

$v = (v_1, v_2) \in \mathcal{N}(w) \cap S_{L^\perp}$, then such p is called a local peak selection of J w.r.t. L at w.

The notion of a local peak selection is introduced to find a local mountain pass solution or to design a local min-max method in the literature. Here it is clear that a local peak selection of J w.r.t. L at w is a local L-$\perp$ selection of J at w.

Lemma 2.1. *For every unit vector w in a Hilbert space $(X, \|\cdot\|)$, it follows that*

$$\frac{\|v\|}{\|w \pm v\|} \leq \left\| \frac{w \pm v}{\|w \pm v\|} - w \right\| \leq \frac{2\|v\|}{\|w \pm v\|}, \ \forall v \in X \text{ with } v \perp w. \qquad \blacksquare$$

The following lemma is crucial in establishing a local characterization for multiple saddle points of strongly indefinite functionals of the form (1.4) and a stepsize rule in the LMMOM. Note that the opposite signs/directions in (i)–(ii) and $w^{(j)}(s)$ $(j = 1, 2)$ below reveal a new search process for game-type saddle points in numerical computation.

Lemma 2.2. *Let $J \in C^1(H, \mathbb{R})$, let L be a closed subspace of H and $w = (w_1, w_2) \in S_{L^\perp}$ with $w_1 \neq 0$, $w_2 \neq 0$. Assume that p is a local L-$\perp$ selection of J and continuous at w. Write $p(w) \equiv (p_1(w), p_2(w)) = (t_1 w_1, t_2 w_2) + w_L \in H$, where $w_L \in L$ and $t_1, t_2 \in \mathbb{R}$ with $t_1 t_2 \neq 0$. If $d \equiv (d_1, d_2) = \nabla J(p(w)) \neq 0$, then $\exists s_0 > 0$ such that for each $0 < s \leq s_0$, we have*

$$\textbf{(i)} \quad J(p(w^{(1)}(s))) - J(p(w)) < -\frac{|t_1|}{4}||d_1||\,||w^{(1)}(s) - w|| \leq -\frac{|t_1|}{8} s||d_1||^2 < 0, \quad \textit{if } d_1 \neq 0;$$

$$\textbf{(ii)} \quad J(p(w^{(2)}(s))) - J(p(w)) > \frac{|t_2|}{4}||d_2||\,||w^{(2)}(s) - w|| \geq \frac{|t_2|}{8} s||d_2||^2 > 0, \quad \textit{if } d_2 \neq 0,$$

where $w^{(1)}(s) = \frac{w - sign(t_1)s(d_1,0)}{||w - sign(t_1)s(d_1,0)||} \in S_{L^\perp}$, $w^{(2)}(s) = \frac{w + sign(t_2)s(0,d_2)}{||w + sign(t_2)s(0,d_2)||} \in S_{L^\perp}$.

Proof. (i) First, by the definition of p, we have $d_i \perp [L_i, w_i], i = 1, 2$. Then we have

$$w^{(1)}(s) \equiv (w_1^{(1)}(s), w_2^{(1)}(s)) = \left(\frac{w_1 - sign(t_1)sd_1}{\sqrt{1 + s^2\|d_1\|^2}}, \frac{w_2}{\sqrt{1 + s^2\|d_1\|^2}}\right) \to w = (w_1, w_2)$$

as $s \to 0$. Since p is continuous at w, $p(w^{(1)}(s)) \to p(w)$ as $s \to 0$. On the other hand, for each s near zero,

$$p(w^{(1)}(s)) \equiv (p_1(w^{(1)}(s)), p_2(w^{(1)}(s))) = (\tilde{t}_1(s) w_1^{(1)}(s), \tilde{t}_2(s) w_2^{(1)}(s)) + w_L(s)$$

for some scalars $\tilde{t}_1(s), \tilde{t}_2(s)$ and some $w_L(s) \in L$. Thus $\tilde{t}_i(s) \to t_i$ as $s \to 0, i = 1, 2$.

With $t_1 t_2 \neq 0$, we have $sign(\tilde{t}_1(s)) = sign(t_1)$, $|\tilde{t}_1(s)| > |t_1|/2$ when s is small. Since $J \in C^1(H, \mathbb{R})$, by the definition of p, $d_i \perp [L_i, w_i], i = 1, 2$, we have $d_2 \perp w_2^{(1)}(s)$ and $d_2 \perp p_2(w^{(1)}(s))$. Thus

$$\begin{aligned} & J(p(w^{(1)}(s))) - J(p(w)) \\ = \;& \langle \nabla J(p(w)), p(w^{(1)}(s)) - p(w) \rangle + o(\|p(w^{(1)}(s)) - p(w)\|) \\ = \;& \langle d_1, p_1(w^{(1)}(s)) \rangle + o(\|p(w^{(1)}(s)) - p(w)\|). \end{aligned}$$

Then

$$\begin{aligned}
&\langle d_1, p_1(w^{(1)}(s))\rangle \\
= \;& \langle d_1, \tilde{t}_1(s) w_1^{(1)}(s)\rangle = \langle d_1, \tilde{t}_1(s)\frac{w_1 - sign(t_1) s d_1}{\sqrt{1+s^2\|d_1\|^2}}\rangle \quad (\text{since } w_L(s)\in L) \\
= \;& \langle d_1, \frac{-sign(t_1)\tilde{t}_1(s) s d_1}{\sqrt{1+s^2\|d_1\|^2}}\rangle = -\frac{sign(t_1)\tilde{t}_1(s) s}{\sqrt{1+s^2\|d_1\|^2}}\|d_1\|^2 \; (\text{since } d_1 \perp [L_1, w_1]) \\
= \;& -\frac{sign(\tilde{t}_1(s))\tilde{t}_1(s) s}{\sqrt{1+s^2\|d_1\|^2}}\|d_1\|^2 = -\frac{|\tilde{t}_1(s)| s}{\sqrt{1+s^2\|d_1\|^2}}\|d_1\|^2.
\end{aligned}$$

Since $p(w^{(1)}(s)) \to p(w)$ as $s \to 0$, when s is small, we have

$$\begin{aligned}
&J(p(w^{(1)}(s))) - J(p(w)) \\
= \;& \frac{-|\tilde{t}_1(s)| s \|d_1\|^2}{\sqrt{1+s^2\|d_1\|^2}} + o(\|p(w^{(1)}(s)) - p(w)\|) < \frac{-|t_1| s\|d_1\|^2}{2\sqrt{1+s^2\|d_1\|^2}}.
\end{aligned} \tag{2.1}$$

By Lemma 2.1, we have

$$\frac{s\|d_1\|}{2} \le \frac{s\|d_1\|}{\sqrt{1+s^2\|d_1\|^2}} \le \left\| \frac{w - sign(t_1) s (d_1, 0)}{\|w - sign(t_1) s(d_1,0)\|} - w\right\| \le \frac{2s\|d_1\|}{\sqrt{1+s^2\|d_1\|^2}} \tag{2.2}$$

when $s > 0$ is sufficiently small. Combining (2.1) and (2.2) yields

$$\begin{aligned}
&\langle \nabla J(p(w)), p(w^{(1)}(s)) - p(w)\rangle < -\frac{|t_1|}{4}\|d_1\| \frac{2s\|d_1\|}{\sqrt{1+s^2\|d_1\|^2}} \\
\le \;& -\frac{|t_1|}{4}\|d_1\| \left\|\frac{w - sign(t_1)s(d_1,0)}{\|w - sign(t_1)s(d_1,0)\|} - w\right\| = -\frac{|t_1|}{4}\|d_1\|\cdot\|w^{(1)}(s) - w\| \\
\le \;& -\frac{|t_1|}{8} s\|d_1\|^2
\end{aligned}$$

when s is sufficiently small. Taking (2.1) into account, we conclude that $\exists s_0 > 0$ s.t. (i) holds true $\forall 0 < s \le s_0$. Finally, by similar arguments as above, one can prove (ii). □

Now we are ready to establish a local game-type saddle point characterization of multiple saddle points to strongly indefinite functional J in (1.4).

Theorem 2.1. *Let $J \in C^1(H, \mathbb{R})$ be a strongly indefinite functional of form* (1.4), *$\bar{w} = (\bar{w}_1, \bar{w}_2) \in S_{L^\perp}$. Assume that $p(\cdot) = (p_1(\cdot), p_2(\cdot))$ is a local L-$\perp$ selection of J at $\bar{w}$ s.t.*

(i) *p is continuous at $\bar{w}$,*
(ii) *$dist(p_1(\bar{w}), L_1) > 0$ and $dist(p_2(\bar{w}), L_2) > 0$.*

If there exists an open neighborhood $U \times V \subset L^\perp$ of $(\bar{w}_1, \bar{w}_2)$ s.t.

$$J\left(p\left(\tfrac{(\bar{w}_1, w_2)}{\|(\bar{w}_1, w_2)\|}\right)\right) \le J(p(\bar{w}_1, \bar{w}_2)) \le J\left(p\left(\tfrac{(w_1, \bar{w}_2)}{\|(w_1, \bar{w}_2)\|}\right)\right), \quad \forall (w_1, w_2) \in U \times V, \tag{2.3}$$

then $p(\bar{w})$ is a critical (saddle) point of J in H.

Proof. Suppose, by contradiction, $(d_1, d_2) \equiv \nabla J(p(\bar{w})) \ne 0$. We have either (a) $d_1 \ne 0$ or (b) $d_2 \ne 0$. By definition, we may write $p(\bar{w}) = (p_1(\bar{w}), p_2(\bar{w})) = (t_1\bar{w}_1, t_2\bar{w}_2) + w_L$ for some scalars t_1, t_2 and some $w_L \in L$. Then condition (ii) implies that $\bar{w}_1 \neq 0, \bar{w}_2 \neq 0$ and $t_1 t_2 \neq 0$. By Lemma 2.2, there is $s_0 > 0$ such that when $0 < s \leq s_0$, we have

$$\begin{aligned} J(p(w^{(1)}(s))) &< J(p(\bar{w})) - \frac{1}{4}|t_1|\,\|d_1\|\,\|w^{(1)}(s) - \bar{w}\| < J(p(\bar{w})) \quad \text{if } d_1 \neq 0, \\ J(p(w^{(2)}(s))) &> J(p(\bar{w})) + \frac{1}{4}|t_2|\,\|d_2\|\,\|w^{(2)}(s) - \bar{w}\| > J(p(\bar{w})) \quad \text{if } d_2 \neq 0, \end{aligned}$$

where

$$\begin{aligned} w^{(1)}(s) &\equiv (w_1^{(1)}(s), w_2^{(1)}(s)) = \frac{\bar{w} - sign(t_1)s(d_1,0)}{||\bar{w} - sign(t_1)s(d_1,0)||} \\ &= \frac{(\bar{w}_1 - sign(t_1)sd_1, \bar{w}_2)}{\sqrt{1+s^2\|d_1\|^2}} \in S_{L^\perp}, \\ w^{(2)}(s) &\equiv (w_1^{(2)}(s), w_2^{(2)}(s)) = \frac{\bar{w} + sign(t_2)s(0,d_2)}{||\bar{w} + sign(t_2)s(0,d_2)||}, \\ &= \frac{(\bar{w}_1, \bar{w}_2 + sign(t_2)sd_2)}{\sqrt{1+s^2\|d_2\|^2}} \in S_{L^\perp}. \end{aligned}$$

In either case, it violates (2.3) when s is small. □

Note that the game-type saddle point characterization in (2.3) has extended the notion of a zero-sum game as well as the saddle point definition in game theory. If introducing a solution set $\mathcal{M} = \left\{p(w) : w \in S_{L^\perp}\right\}$, which naturally generalizes the notion of the Nehari manifold (wherein $L = \{0\}$), we may call $p(\bar{w})$ a saddle point (actually a game-type saddle point) of J on $\mathcal{M}$. With this in mind, instead of finding saddle points of J in H, we actually look for saddle points of J on $\mathcal{M}$. Here, the function $p(\cdot)$ is introduced in order to find multiple nontrivial solutions.

There are some variations of Lemma 2.2 based on which different stepsize rules can be derived. The following is one of such variations.

Lemma 2.3. *Under the assumptions in Lemma* 2.2, *it follows that*

(i) $J(p(w^{(1)}(s_1))) - J(p(w)) < -\frac{|t_1|}{4}||d_1|| \cdot ||w^{(1)}(s_1) - w|| \leq -\frac{|t_1|}{8}s_1||d_1||^2 < 0,$ *if* $d_1 \neq 0$;

(ii) $J(p(w)) - J(p(w^{(2)}(s_2))) < -\frac{|t_2|}{4}||d_2|| \cdot ||w^{(2)}(s_2) - w|| \leq -\frac{|t_2|}{8}s_2||d_2||^2 < 0,$ *if* $d_2 \neq 0$;

$\forall 0 < s_1 \leq \bar{s}_1, 0 < s_2 \leq \bar{s}_2$ *for some* $\bar{s}_1, \bar{s}_2 > 0$, *where*

$$w^{(1)}(s_1) = \frac{w - sign(t_1)s_1(d_1,0)}{||w - sign(t_1)s_1(d_1,0)||} \in S_{L^\perp},$$
$$w^{(2)}(s_2) = \frac{w + sign(t_2)s_2(0,d_2)}{||w + sign(t_2)s_2(0,d_2)||} \in S_{L^\perp}.$$

Corollary 2.1. *With the notation and assumptions in Lemma* 2.2, *if we let* $j = \arg\max_{k\in\{1,2\}} \|d_k\|$, *then there exists* $\bar{s} > 0$ *such that*

$$J(p(w^{(2)}(s))) - J(p(w^{(1)}(s))) > \frac{|t_j|}{8}s||d_j||^2 \geq \frac{|t_j|}{16}s||d||^2, \quad \forall 0 < s \leq \bar{s}.$$

Proof. Follows from Lemma 2.2 and the fact $2||d_j||^2 \geq \|d\|^2 = \|d_1\|^2 + \|d_2\|^2$. □

3. A local min-max-orthogonal algorithm

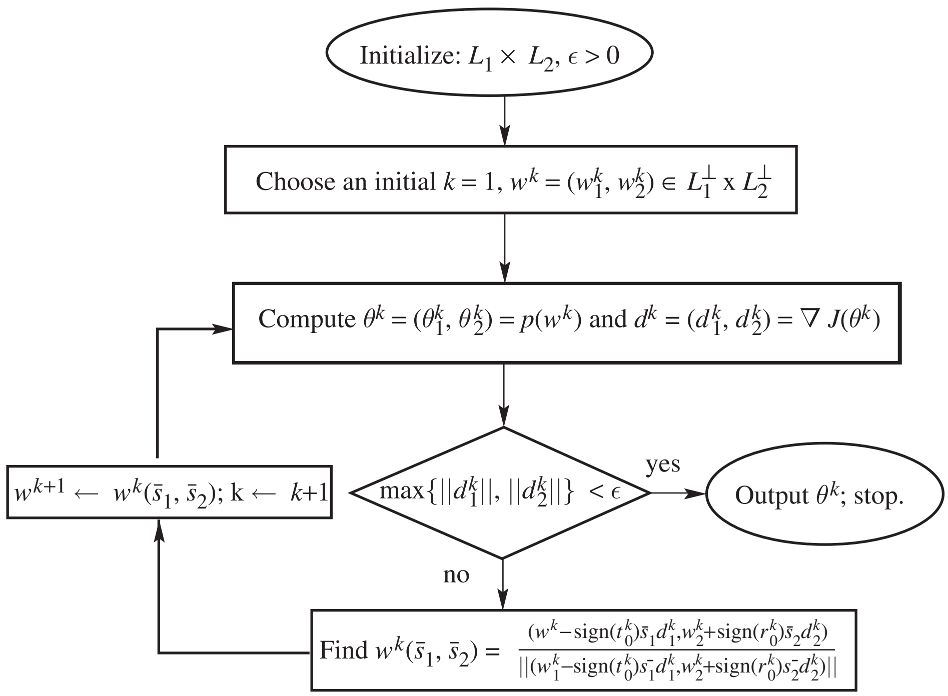

Figure 1. Flow chart of the local min-max-orthogonal algorithm.

Based on the local game-type saddle point characterization in Theorem 2.1 and the stepsize rule in Lemma 2.3, we present a new local min-max-orthogonal method (LMMOM):

Algorithm 3.1. *Local Min-Max-Orthogonal Method (LMMOM)*

Step 0: *Set* $L = L_1 \times L_2 = span\{u_1, ..., u_m\} \times span\{v_1, ..., v_n\}$*, a tolerance* $\varepsilon > 0$*, and choose a control parameter* λ *s.t.* $0 < \lambda < 1$*. Set* $k = 1$*.*

Step 1: *Choose an initial direction* $w^k = (w_1^k, w_2^k) \in S_{L_1^\perp \times L_2^\perp}$ *with* $w_1^k \neq 0$*,* $w_2^k \neq 0$*. Compute*

$$p_1(w^k) = \textstyle\sum_{i=1}^m t_i^k u_i + t_0^k w_1^k \in [L_1, w_1^k] \backslash L_1 \quad (i.e.,\ t_0^k \neq 0),$$
$$p_2(w^k) = \textstyle\sum_{i=1}^n r_i^k v_i + r_0^k w_2^k \in [L_2, w_2^k] \backslash L_2 \quad (i.e.,\ r_0^k \neq 0),$$

where $p(w^k) = (p_1(w^k), p_2(w^k))$ *is an* L-$\perp$ *selection of* J *at* w^k*,* t_i^k *and* r_j^k $(i = 0, \dots, m, j = 0, \dots, n)$ *are solved locally from the* $(m+n+2)$ *equations*

$$\langle \partial_1 J(p(w^k)), w_1^k \rangle = 0, \quad \langle \partial_1 J(p(w^k)), u_i \rangle = 0, \quad i = 1, \dots, m,$$
$$\langle \partial_2 J(p(w^k)), w_2^k \rangle = 0, \quad \langle \partial_2 J(p(w^k)), v_j \rangle = 0, \quad j = 1, \dots, n$$

in $(m + n + 2)$ *variables.*

Step 2: *Set* $\theta^k = p(w^k)$ *and compute the gradient* $d^k = (d_1^k, d_2^k) = \nabla J(\theta^k)$*.*

Step 3: *If* $\max\{||d_1^k||, ||d_2^k||\} \leq \varepsilon$*, Output* θ^k*, Stop; otherwise, Goto Step 4.*

Step 4 (Update the search direction by the stepsize rule): *Find*

$$w^{k+1} \equiv (w_1^{k+1}, w_2^{k+1}) = \phi(s_1, s_2) \equiv \frac{(w_1^k(s_1), w_2^k(s_2))}{\|(w_1^k(s_1), w_2^k(s_2))\|} \tag{3.1}$$

where $w_1^k(s_1) = w_1^k - sign(t_0^k)s_1 d_1^k,\ w_2^k(s_2) = w_2^k + sign(r_0^k)s_2 d_2^k$, *and* $\bar{s}_1, \bar{s}_2$ *are determined by the following* **stepsize rule**.

(i) *First, initialize the stepsizes* $\bar{s}_1 = \bar{s}_2 = 0$.

(ii) *If* $||d_1^k|| > \varepsilon$, *then*

$$\bar{s}_1 = \max_{i\in\mathbb{N}} \Big\{ \frac{\lambda}{2^i} \Big| 2^i > ||d_1^k||, J(p(\phi(\frac{\lambda}{2^i}, 0))) - J(p(w^k)) < -\frac{|t_0^k|}{4}||d_1^k|| \cdot ||\phi(\frac{\lambda}{2^i}, 0) - w^k|| \Big\};$$

If $||d_2^k|| > \varepsilon$, *then*

$$\bar{s}_2 = \max_{i\in\mathbb{N}} \Big\{ \frac{\lambda}{2^i} \Big| 2^i > ||d_2^k||, J(p(w^k)) - J(p(\phi(0, \frac{\lambda}{2^i}))) < -\frac{|r_0^k|}{4}||d_2^k|| \cdot ||\phi(0, \frac{\lambda}{2^i}) - w^k|| \Big\}.$$

Here $(t_0^k, t_1^k, \ldots, t_m^k, r_0^k, r_1^k, \ldots, r_n^k)$ *is used as an initial guess to evaluate* $p(\phi(\frac{\lambda}{2^i}, 0))$ *and/or* $p(\phi(0, \frac{\lambda}{2^i}))$ *in the same way as in Step 1.*

Step 5: *Compute* $p(w^{k+1})$ *with the same initial guess as in Step 4. Set* $k \leftarrow k+1$. *Goto Step 2.* ∎

Remark 3.1. (a) A flow chart of Algorithm 3.1 is shown in Figure 1 wherein the stepsizes $\bar{s}_1, \bar{s}_2$ are determined by Step 4 of Algorithm 3.1 and satisfy

$$\bar{s}_1 \begin{cases} > 0, & \text{if } \|d_1^k\| > \varepsilon, \\ = 0, & \text{if } \|d_1^k\| \le \varepsilon, \end{cases} \qquad \bar{s}_2 \begin{cases} > 0, & \text{if } \|d_2^k\| > \varepsilon, \\ = 0, & \text{if } \|d_2^k\| \le \varepsilon. \end{cases} \tag{3.2}$$

Thus Algorithm 3.1 produces two byproducts, $\{w^{k,1}\}$ and $\{w^{k,2}\}$, given by

$$w^{k,1} = \begin{cases} \phi(\bar{s}_1, 0), & \text{if } \|d_1^k\| > \varepsilon, \\ w^k, & \text{if } \|d_1^k\| \le \varepsilon, \end{cases} \qquad w^{k,2} = \begin{cases} \phi(0, \bar{s}_2), & \text{if } \|d_2^k\| > \varepsilon, \\ w^k, & \text{if } \|d_2^k\| \le \varepsilon, \end{cases} \tag{3.3}$$

where ϕ is defined in (3.1). Then from Step 4 of Algorithm 3.1, one can see that

$$J(p(w^{k,2})) \ge J(p(w^k)) \ge J(p(w^{k,1})),\ \forall k.$$

(b) The algorithm usually starts with $L = \{0\} \times \{0\}$ with which a first solution $W_1 = (u_1, v_1)$ is found. Then we may set $L = \text{span}\{u_1\} \times \text{span}\{v_1\}$ to find a new solution $W_2 = (u_2, v_2)$. As L is gradually expanded by newly found solutions W_k, more solutions can be found in a partial order defined by the dimension of L.

A symmetry, if available, can also be used to reduce L and make the algorithm more efficient. The algorithm can also be followed by Newton's method with Armijo's stepsize rule to speed up local convergence. See [14] for more details.

4. Applications to noncooperative systems

In this section, we apply our method (i.e., the LMMOM) to solve two types of noncooperative systems for multiple solutions and verify some of their important properties.

4.1. **Noncooperative systems of definite type.** Consider noncooperative elliptic systems of the form [7, 8, 9, 11, 15]

$$\text{(4.1)} \quad \begin{cases} -\Delta u = \lambda u - \delta v + G_u(x;u,v) & x \in \Omega, \\ -\Delta v = \delta u + \gamma v - G_v(x;u,v) & x \in \Omega, \end{cases} \qquad u = v = 0,\ x \in \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$, $\gamma \leq \lambda, \delta > 0$. The nonlinear term $G(x;U) \in C^1(\Omega \times \mathbb{R}^2;\mathbb{R})$ (in the variables $U = (u,v) \in \mathbb{R}^2$) satisfies the following conditions [7, 8]

(F_1) $|\nabla G(x,U)| \leq c(1 + |U|^{\xi-1})$, $\forall U \in \mathbb{R}^2$, a.e. $x \in \Omega$, for some $c > 0$ and $2 \leq \xi < \frac{2N}{N-2}$ if $N \geq 3$ or $2 \leq \xi < +\infty$ if $N = 1, 2$; (subcritical),

(F_2) $\liminf\limits_{|U|\to\infty} \dfrac{U \cdot \nabla G(x;U) - 2G(x;U)}{|U|^\mu} \geq a > 0$ uniformly a.e. $x \in \Omega$ with $\mu > N(\xi - 2)/2$ if $N \geq 3$ or $\mu > \xi - 2$ if $N = 1, 2$; (nonquadratic),

(F_3) $G(x;U) \geq 0$, $\forall U \in \mathbb{R}^2$, $\lim\limits_{|U|\to 0} \dfrac{G(x;U)}{|U|^2} = 0$ uniformly a.e. $x \in \Omega$.

If we let $H = L^2(\Omega) \times L^2(\Omega)$, denoting $\nabla G = (G_u, G_v)$ and

$$A = \begin{bmatrix} \lambda & -\delta \\ \delta & \gamma \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad -\vec{\Delta} = \begin{bmatrix} -\Delta & 0 \\ 0 & -\Delta \end{bmatrix},$$

then (4.1) becomes

$$\mathcal{L}U = \nabla G(x;U)$$

where $\mathcal{L} : D(\mathcal{L}) \subset H \to H$ is a self-adjoint operator given by $\mathcal{L}U = R(-\vec{\Delta} - A)U$ and

$$D(\mathcal{L}) = W^{2,2}(\Omega, \mathbb{R}^2) \cap W_0^{1,2}(\Omega, \mathbb{R}^2).$$

Problem (4.1) was particularly studied in [7, 8]. As pointed out in [8], the following *asymptotic noncrossing* conditions

(F_4^+) $\lambda_{k-1} < \liminf\limits_{|U|\to\infty} \dfrac{2G(x;U)}{|U|^2} \leq \limsup\limits_{|U|\to\infty} \dfrac{2G(x;U)}{|U|^2} \leq \lambda_k$ unif. a.e. $x \in \Omega$,

(F_4^-) $\lambda_{k-1} \leq \liminf\limits_{|U|\to\infty} \dfrac{2G(x;U)}{|U|^2} \leq \limsup\limits_{|U|\to\infty} \dfrac{2G(x;U)}{|U|^2} < \lambda_k$ unif. a.e. $x \in \Omega$,

or *crossing* conditions

(F_5) $G(x;U) \geq \dfrac{1}{2}\lambda_{k-1}|U|^2$ a.e. $x \in \Omega$, $\forall U \in \mathbb{R}^2$,

(F_6) $\limsup\limits_{|U|\to 0} \dfrac{2G(x;U)}{|U|^2} \leq \alpha < \lambda_k < \beta \leq \liminf\limits_{|U|\to\infty} \dfrac{2G(x;U)}{|U|^2}$ unif. a.e.$x \in \Omega$,

where $\lambda_{k-1} < \lambda_k$ are two consecutive eigenvalues of the operator $\mathcal{L}$, were used to assure the existence of nonzero solutions to (4.1). In some sense, the assumption $G(x;U) \geq 0, \forall U \in \mathbb{R}^2$ in (F_3) is a necessity for conditions $(F_4^{\pm})$ or (F_5)–(F_6). Meanwhile, other authors [3, 9, 15] proved that such an assumption may be weakened by, e.g., $G(x;0,v) \geq 0$, for a.e. $x \in \Omega, v \in \mathbb{R}$, under which the existence results can still be obtained.

Remark 4.1. Due to $G(x;U) \geq 0$ in (F_3), system (4.1) is called a noncooperative system of definite type. In Section 4.2, we will consider an indefinite type noncooperative system where $G(x;U)$ changes sign.

Example 4.1. As a typical problem studied in [3, 7, 8, 9, 15], we choose $N = 2$ (i.e., $\Omega \subset \mathbb{R}^2$) and $G(x;u,v) \equiv G(u,v) = \frac{1}{p+1}|u|^{p+1} + \frac{1}{q+1}|v|^{q+1}$ with $p,q > 1$. Then (4.1) becomes

$$\text{(4.2)} \qquad \begin{cases} -\Delta u = \lambda u - \delta v + |u|^{p-1}u & x \in \Omega, \\ -\Delta v = \delta u + \gamma v - |v|^{q-1}v & x \in \Omega, \end{cases} \qquad u = v = 0, \; x \in \partial\Omega.$$

For this particular example, one sees that conditions (F_1)–(F_3) are satisfied. Let $H = H_0^1(\Omega) \times H_0^1(\Omega)$ and $\|\cdot\|$ be its norm, i.e., $\|(u,v)\|^2 = \int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx$, $\forall (u,v) \in H$. Then, weak solutions of (4.2) are critical points of the following C^2-functional on H:

$$\text{(4.3)} \quad J(u,v) = \frac{1}{2}\int_\Omega \Big[(|\nabla u|^2 - |\nabla v|^2) - (\lambda u^2 - 2\delta uv - \gamma v^2) - 2(\frac{|u|^{p+1}}{p+1} + \frac{|v|^{q+1}}{q+1})\Big]dx.$$

Now we define the solution set

$$\text{(4.4)} \qquad \widetilde{\mathcal{M}} = \Big\{(u,v) \in H : \partial J/\partial u \perp u, \partial J/\partial v \perp v\Big\}$$

where $\nabla J = (\frac{\partial J}{\partial u}, \frac{\partial J}{\partial v})$ is the gradient of J. Clearly, $\widetilde{\mathcal{M}}$ contains all critical points of J. Next, we verify or state some basic properties of J in (4.3) which are closely related to the LMMOM and our numerical computations. For simplicity, from now on, we denote $\int_\Omega$ by $\int$.

Proposition 4.1. *Any critical point of J in* (4.3) *has an infinite Morse index.* ▮

Proposition 4.2. *For J in* (4.3) *and $\forall (u,v) \in \widetilde{\mathcal{M}}$, it follows that*

$$J(u,v) = \int \Big[(\frac{1}{2} - \frac{1}{p+1})|u|^{p+1} + (\frac{1}{2} - \frac{1}{q+1})|v|^{q+1}\Big]dx \geq 0.$$

Consequently, $(0,0) \in \widetilde{\mathcal{M}}$ *is the least energy saddle point of J with* $J(0,0) = 0$.

Proof. For every point $(u,v) \in \widetilde{\mathcal{M}}$, the conditions $\frac{\partial J}{\partial u} \perp u$ and $\frac{\partial J}{\partial v} \perp v$ lead to

$$\text{(4.5)} \qquad \begin{aligned} \int |\nabla u|^2 dx &= \int \left[\lambda u^2 - \delta uv + |u|^{p+1}\right] dx, \\ \int |\nabla v|^2 dx &= \int \left[\gamma v^2 + \delta uv - |v|^{q+1}\right] dx. \end{aligned}$$

Plugging them into (4.3) and since $p,q > 1$, we obtain

$$J(u,v) = \int \Big[(\frac{1}{2} - \frac{1}{p+1})|u|^{p+1} + (\frac{1}{2} - \frac{1}{q+1})|v|^{q+1}\Big]dx \geq 0. \qquad \square$$

If denoting by σ_1 the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$, then we have

Proposition 4.3. *For any critical point $(\bar{u},\bar{v}) \neq (0,0)$ of J, it follows that*

(i) $\bar{u} \neq 0, \bar{v} \neq 0$ *and*
(ii) *if* $\gamma < \sigma_1$, *then* $\int \bar{u}\bar{v}dx > 0$.

Proof. (i) is trivial. For (ii), by the second equation in (4.5), we have

$$\delta\int \bar{u}\bar{v}dx = \int |\nabla \bar{v}|^2 dx - \gamma\int \bar{v}^2 dx + \int |\bar{v}|^{q+1}dx.$$

Then (ii) follows via the Poincaré inequality. □

Property (ii) in Proposition 4.3 can help us select an initial guess (u, v) for our method. The next lemma further confirms the existence and differentiability of an L-$\perp$ selection $\tilde{p}$ of J in (4.3) when $L = \{0\} \times \{0\}$.

Lemma 4.1. *Assume $\gamma \le \lambda < \sigma_1$. For every unit vector $(\bar{u}, \bar{v})$ with $\int \bar{u}\bar{v}dx \neq 0$, there exists a differentiable local peak selection $\tilde{p}$ of J w.r.t. $L = \{0\} \times \{0\}$ around $(\bar{u}, \bar{v})$ such that $\tilde{p}(\bar{u}, \bar{v}) = (\bar{t}\bar{u}, \bar{s}\bar{v})$ and $\frac{\bar{t}}{\bar{s}}\int \bar{u}\bar{v}dx > 0$ for some $\bar{t}, \bar{s}$.*

Proof. By definition, an L-$\perp$ selection $p(\bar{u}, \bar{v}) = (t\bar{u}, s\bar{v})$ is solved from the nonlinear system

$$\frac{\partial J}{\partial t}(t\bar{u}, s\bar{v}) = t(\int[|\nabla \bar{u}|^2 - \lambda\bar{u}^2]dx) + \delta s\int \bar{u}\bar{v}dx - |t|^{p-1}t\int |\bar{u}|^{p+1}dx = 0, \tag{4.6}$$

$$\frac{\partial J}{\partial s}(t\bar{u}, s\bar{v}) = s(\int[\gamma\bar{v}^2 - |\nabla \bar{v}|^2]dx) + \delta t\int \bar{u}\bar{v}dx - |s|^{q-1}s\int |\bar{v}|^{q+1}dx = 0 \tag{4.7}$$

for a nonzero solution (t, s) (i.e., $ts \neq 0$), where J is defined in (4.3). Denote

$$a_0 = \delta\int \bar{u}\bar{v}dx,\ a_1 = \int[|\nabla \bar{u}|^2 - \lambda\bar{u}^2]dx, \tag{4.8}$$

$$a_2 = \int |\bar{u}|^{p+1}dx,\ b_1 = \int[|\nabla \bar{v}|^2 - \gamma\bar{v}^2]dx,\ b_2 = \int |\bar{v}|^{q+1}dx. \tag{4.9}$$

By our assumptions, we have $a_1, a_2, b_1, b_2 > 0$. Then, (4.7) gives

$$t = \frac{b_1 s + b_2|s|^{q-1}s}{a_0}. \tag{4.10}$$

Since we seek nonzero solutions of (4.6)–(4.7), plugging (4.10) into (4.6) yields

$$(b_1 + b_2|s|^{q-1})\frac{a_1}{a_0} + a_0 - \left|\frac{b_1 s + b_2|s|^{q-1}s}{a_0}\right|^{p-1}\frac{(b_1 + b_2|s|^{q-1})}{a_0}a_2 = 0. \tag{4.11}$$

Define, for each $s \in [0, \infty)$,

$$\psi(s) = [b_1 + b_2|s|^{q-1}]\frac{a_1}{a_0} + a_0 - \left|\frac{b_1 s + b_2|s|^{q-1}s}{a_0}\right|^{p-1}\frac{[b_1 + b_2|s|^{q-1}]}{a_0}a_2.$$

Clearly, ψ is continuous with $\psi(0) = \frac{b_1 a_1}{a_0} + a_0, \psi(s) \approx -|s|^{pq-1}\frac{b_2^p a_2}{|a_0|^{p-1}a_0}$ (when s is sufficiently large). We then see that $\psi(0)\psi(\infty) < 0$ because a_1, a_2, b_1, b_2 are all positive. Thus by the mean value theorem, there exists $\bar{s} > 0$ such that $\psi(\bar{s}) = 0$. Plugging $\bar{s}$ into (4.10) gives $\bar{t} = \frac{(b_1 + b_2|\bar{s}|^{q-1})\bar{s}}{a_0} \neq 0$ since $b_1 + b_2|\bar{s}|^{q-1} > 0$. Thus,

$$\frac{\bar{t}}{\bar{s}}\delta\int \bar{u}\bar{v}dx = \frac{\bar{t}}{\bar{s}}a_0 = b_1 + b_2|\bar{s}|^{q-1} > 0, \quad \text{or} \quad \frac{\bar{t}}{\bar{s}}\int \bar{u}\bar{v}dx > 0 \ \text{since } \delta > 0. \tag{4.12}$$

Next, we show that $\tilde{p}(\bar{u},\bar{v}) = (\bar{t}\bar{u}, \bar{s}\bar{v})$ is a local maximum of J in the subspace $\mathrm{span}\{\bar{u}\} \times \mathrm{span}\{\bar{v}\}$; i.e., we verify that the Hessian matrix

$$\begin{aligned} Q &= \left[\begin{array}{cc} \frac{\partial^2 J(t\bar{u},s\bar{v})}{\partial t^2} & \frac{\partial^2 J(t\bar{u},s\bar{v})}{\partial t \partial s} \\ \frac{\partial^2 J(t\bar{u},s\bar{v})}{\partial s \partial t} & \frac{\partial^2 J(t\bar{u},s\bar{v})}{\partial s^2} \end{array} \right] \Bigg|_{(t,s)=(\bar{t},\bar{s})} \\ &= \left[\begin{array}{cc} a_1 - a_2 p |\bar{t}|^{p-1} & a_0 \\ a_0 & -b_1 - b_2 q |\bar{s}|^{q-1} \end{array} \right] \end{aligned} \tag{4.13}$$

is negative definite. Since $(\bar{t},\bar{s})$ solves (4.6)–(4.7), we have

$$a_1 = -\frac{\bar{s}}{\bar{t}} a_0 + a_2 |\bar{t}|^{p-1}, \; b_1 = \frac{\bar{t}}{\bar{s}} a_0 - b_2 |\bar{s}|^{q-1}. \tag{4.14}$$

Substituting (4.14) into (4.13) gives

$$Q = \left[\begin{array}{cc} -\frac{\bar{s}}{\bar{t}} a_0 - a_2(p-1)|\bar{t}|^{p-1} & a_0 \\ a_0 & -\frac{\bar{t}}{\bar{s}} a_0 - b_2(q-1)|\bar{s}|^{q-1} \end{array} \right]. \tag{4.15}$$

Since $a_2, b_2 > 0, p, q > 1$, (4.12) implies that the diagonal elements of Q are negative and the determinant $|Q| > a_2 b_2 (p-1)(q-1)|\bar{t}|^{p-1}|\bar{s}|^{q-1} > 0$. Thus Q is negative definite. Consequently, $\tilde{p}(\bar{u},\bar{v}) = (\bar{t}\bar{u}, \bar{s}\bar{v})$ is a local maximum of J in $\mathrm{span}\{\bar{u}\} \times \mathrm{span}\{\bar{v}\}$.

Finally, we show that such $\tilde{p}$ can be extended locally as a differentiable local peak selection of J around $(\bar{u},\bar{v})$. Consider the equations

$$\left\{ \begin{array}{l} F_1(u,v,t,s) \equiv \frac{\partial J}{\partial t}(tu,sv) = 0, \\ F_2(u,v,t,s) \equiv \frac{\partial J}{\partial s}(tu,sv) = 0, \end{array} \right. \tag{4.16}$$

and define a matrix function

$$\mathcal{Q}(u,v,t,s) \equiv \frac{\partial(F_1,F_2)}{\partial(t,s)} = \left[\begin{array}{cc} \frac{\partial^2}{\partial t^2} J(tu,sv) & \frac{\partial^2}{\partial t \partial s} J(tu,sv) \\ \frac{\partial^2}{\partial s \partial t} J(tu,sv) & \frac{\partial^2}{\partial s^2} J(tu,sv) \end{array} \right]. \tag{4.17}$$

Obviously, $(\bar{u},\bar{v},\bar{t},\bar{s})$ solves (4.16) and $\mathcal{Q}(u,v,t,s)\Big|_{(u,v,t,s)=(\bar{u},\bar{v},\bar{t},\bar{s})} = Q$. Since $|Q| > 0$, by the implicit function theorem, there exists an open neighborhood $\mathcal{N}(\bar{u},\bar{v})$ of $(\bar{u},\bar{v})$ such that for every $(u,v) \in \mathcal{N}(\bar{u},\bar{v}) \cap S_{L^\perp}$, (4.16) can be uniquely solved for $(t(u,v), s(u,v))$, where $t(u,v), s(u,v)$ are differentiable functions of (u,v) with $(t(\bar{u},\bar{v}), s(\bar{u},\bar{v})) = (\bar{t},\bar{s})$. Hence a differentiable local L-$\perp$ selection $\tilde{p}$ with $\tilde{p}(\bar{u},\bar{v}) = (\bar{t}\bar{u},\bar{s}\bar{v})$ is well defined in $\mathcal{N}(\bar{u},\bar{v}) \cap S_{L^\perp}$. With $J \in C^2$, it follows that $\mathcal{Q}(t(u,v),s(u,v)) \equiv \mathcal{Q}(u,v,t,s)$ is continuous in $\mathcal{N}(\bar{u},\bar{v}) \cap S_{L^\perp}$. Since Q is strictly negative definite and $\mathcal{Q}(t(\bar{u},\bar{v}),s(\bar{u},\bar{v})) = Q$, one can conclude that $\mathcal{Q}(t(u,v),s(u,v))$ is strictly negative definite, $\forall (u,v) \in \mathcal{N}(\bar{u},\bar{v}) \cap S_{L^\perp}$. Therefore, such $\tilde{p}$ is also a local peak selection of J w.r.t. L. The lemma is thus proved. $\square$

For a general $L = L_1 \times L_2 \subset H$, define the solution set

$$\mathcal{M} = \Big\{ \tilde{p}(u,v) \neq (0,0) : (u,v) \in S_{L^\perp} \Big\}.$$

In particular, for $L = \{0\} \times \{0\}$, denote the solution set

$$\mathcal{M}_0 = \Big\{ \tilde{p}(u,v) \neq (0,0) : \|(u,v)\| = 1 \Big\}.$$

Clearly, $\mathcal{M} \subseteq \mathcal{M}_0 \subseteq \widetilde{\mathcal{M}}, \forall L \subset H$, where $\widetilde{\mathcal{M}}$ is defined in (4.4). Here the trivial solution $(0,0)$ is excluded from the solution set $\mathcal{M}$ or $\mathcal{M}_0$. Next, we verify property (ii) in Theorem 2.1 which insures that solutions found by our method are nontrivial.

Theorem 4.1. *Assume $\lambda, \gamma < \sigma_1$. Then there exists a constant $\alpha > 0$ such that*

$$dist(\mathcal{M}_0, (0,0)) \geq \alpha > 0. \tag{4.18}$$

Consequently, $(0,0) \notin \overline{\mathcal{M}}_0$.

Proof. We start the proof by defining

$$\mathcal{M}' = \Big\{\tilde{p}(u,v) \equiv (tu, sv) : tu \neq 0, \|(u,v)\| = 1\Big\}.$$

Clearly, $\mathcal{M}' \subseteq \mathcal{M}_0$. To prove $\mathcal{M}_0 = \mathcal{M}'$, we verify that $t\bar{u} = 0$ implies $s\bar{v} = 0$ for every L-$\perp$ selection $\bar{p}(\bar{u},\bar{v}) = (t\bar{u}, s\bar{v})$ of J w.r.t. $L = \{0\} \times \{0\}$. For each unit vector $(\bar{u},\bar{v}) \in H$, assume $\bar{p}(\bar{u},\bar{v})$ is an L-$\perp$ selection of J with $L = \{0\} \times \{0\}$. By (4.7), $t\bar{u} = 0$ gives

$$s\Big(\int (|\nabla\bar{v}|^2 - \gamma\bar{v}^2)dx + |s|^{q-1}\int |\bar{v}|^{q+1}dx\Big) = 0 \tag{4.19}$$

from which we have either $s = 0$ or $\int(|\nabla\bar{v}|^2 - \gamma\bar{v}^2)dx + |s|^{q-1}\int|\bar{v}|^{q+1}dx = 0$. By the Poincaré inequality, $\gamma < \sigma_1$ implies $\int(|\nabla\bar{v}|^2 - \gamma\bar{v}^2)dx > 0$, $\forall \bar{v} \neq 0$. Thus $\int(|\nabla\bar{v}|^2 - \gamma\bar{v}^2)dx + |s|^{q-1}\int|\bar{v}|^{q+1}dx = 0$ if and only if $\bar{v} = 0$. Hence $t\bar{u} = 0$ implies $s\bar{v} = 0$. Thus $\tilde{p}(u,v) \in \mathcal{M}'$ for every $\tilde{p}(u,v) \in \mathcal{M}_0$, i.e., $\mathcal{M}_0 \subseteq \mathcal{M}'$. So, $\mathcal{M}_0 = \mathcal{M}'$.

Next, for each $\tilde{p}(\bar{u},\bar{v}) = (t\bar{u}, s\bar{v}) \in \mathcal{M}_0 = \mathcal{M}'$, (4.6) gives

$$|t|^{p-1}\int|\bar{u}|^{p+1}dx = \int[|\nabla\bar{u}|^2 - \lambda\bar{u}^2]dx + \frac{s}{t}\delta\int\bar{u}\bar{v}dx. \tag{4.20}$$

Note that $\frac{s}{t}\delta\int\bar{u}\bar{v}dx \geq 0$ due to (4.12), wherein $\bar{t}, \bar{s}$ are replaced by t, s, respectively. Hence

$$\begin{aligned} c_0|t|^{p-1}\left[\int|\nabla\bar{u}|^2dx\right]^{\frac{p+1}{2}} &\geq |t|^{p-1}\int|\bar{u}|^{p+1}dx \\ &\geq \int(|\nabla\bar{u}|^2 - \lambda\bar{u}^2)dx \geq (1 - \frac{\lambda}{\sigma_1})\int|\nabla\bar{u}|^2dx \end{aligned} \tag{4.21}$$

or equivalently

$$|t|\left(\int|\nabla\bar{u}|^2dx\right)^{1/2} \geq \left(\frac{1}{c_0}(1 - \frac{\lambda}{\sigma_1})\right)^{\frac{1}{p-1}} > 0 \tag{4.22}$$

for some constant $c_0 > 0$ independent of $\bar{u}$ via the Poincaré and Sobolev inequalities.

Setting $\alpha = (\frac{1}{c_0} - \frac{\lambda}{c_0\sigma_1})^{\frac{1}{p-1}}$ gives

$$dist(\tilde{p}(\bar{u},\bar{v}), (0,0)) \geq |t|\left(\int|\nabla\bar{u}|^2dx\right)^{1/2} \geq \alpha > 0,\ \forall\tilde{p}(\bar{u},\bar{v}) \in \mathcal{M}_0,$$

from which it follows that $\operatorname{dist}(\mathcal{M}_0, (0,0)) \geq \alpha > 0$. ∎

As proved in [5, 6], if $\bar{w}$ is a local minimum of $J(\tilde{p}(\cdot))$ on $S_{L^\perp}$, then $\mathrm{MI}(\tilde{p}(\bar{w})) \le \dim(L)+2$. LMOM is designed to find such local minima. When J is strongly indefinite, each critical point $\tilde{p}(\bar{w})$ has an infinite Morse index. Thus, $\bar{w}$ must be a saddle point of $J(\tilde{p}(\cdot))$ on $S_{L^\perp}$ and hence cannot be found by LMOM. Instead, the new method LMMOM is designed to find such type of saddle points. This assertion can also be stated as follows:

Theorem 4.2. *Let $L = L_1 \times L_2 \subset H$ with $\dim(L) < \infty$ and let p be a differentiable L-$\perp$ selection of J in (4.3) at $\bar{w} = (\bar{u}, \bar{v}) \in S_{L^\perp}$ such that $p_i(\bar{w}) \notin L_i$, $i = 1, 2$, where $p(\bar{w}) = (p_1(\bar{w}), p_2(\bar{w}))$. If, in addition, $\nabla J(p(\bar{w})) = 0$, then $\bar{w}$ is a saddle point of $J(p(\cdot))$ on $S_{L^\perp}$. Consequently, $p(\bar{w})$ is a saddle point of J on $\mathcal{M}$.* ▮

4.1.1. *Numerical experiments.* In this section, we apply the LMMOM to find multiple solutions to problem (4.2). We choose $p = q = 3$, $\lambda = \gamma = -0.5, \delta = 5$ and two different domains: a square $\Omega_1 = (-1, 1) \times (-1, 1) \subset \mathbb{R}^2$ and a disk $\Omega_2 = \{x \in \mathbb{R}^2 : |x| < 1.4\}$.

For each $(u, v) \in H$, the gradient $d \equiv (d_1, d_2) = \nabla J(u, v)$ of J in (4.3) can be found as follows. Since for every $\phi = (\phi_1, \phi_2) \in H$ we have

$$\begin{aligned} \langle d, \phi\rangle_H &= \int \nabla d \cdot \nabla \phi dx = -\int (\Delta d_1 \phi_1 + \Delta d_2 \phi_2)\, dx \equiv \frac{d}{dt}\Big|_{t=0} J((u, v) + t\phi) \\ &= \int \left[(-\Delta u - \lambda u + \delta v - |u|^{p-1}u)\phi_1 + (\Delta v + \delta u + \gamma v - |v|^{q-1}v)\phi_2\right] dx. \end{aligned} \tag{4.23}$$

Thus, d satisfies the following two linear elliptic equations

$$\begin{cases} \Delta d_1 = \Delta u + \lambda u - \delta v + |u|^{p-1}u & x \in \Omega, \\ \Delta d_2 = -(\Delta v + \delta u + \gamma v - |v|^{q-1}v) & x \in \Omega, \end{cases} \quad d_1 = d_2 = 0,\ x \in \partial\Omega, \tag{4.24}$$

which can be solved by a finite-element or boundary-element solver, e.g., the MATLAB subroutine ASSEMPDE as used in our numerical experiments.

In our experiments, 32768 (resp. 18432) triangle elements are used on Ω_1 (resp. Ω_2). In both cases, the tolerance $\varepsilon = 10^{-4}$. Figures 2-3 (resp. Figures 4-5) display both the profiles (left) and contour (right) plots of the first few solutions to system (4.2) on Ω_1 (resp. Ω_2). For both positive solutions depicted in Figure 2(a) and 4(a), $L = \{0\} \times \{0\}$. All the sign-changing solutions in the figures are found by using symmetries (i.e., applying the Haar projection, see also [5]) while setting $L = \{0\} \times \{0\}$. The sign-changing solutions in Figures 2-3 may also be found by using nontrivial L's, e.g., $L = \mathrm{span}\{u_1\} \times \mathrm{span}\{v_1\}$ can be used to find the sign-changing solution shown in Figure 2(b), where (u_1, v_1) is the first solution found on Ω_1; see also Figure 2(a). Figure 6 (resp. Figure 7) shows the convergence of the energy gap $|J(p(w^{k,2})) - J(p(w^{k,1}))|$ (top), the gradient norm $\|d^k\|$ (top), and the energy $J(p(w^k))$ (bottom) in computing the positive solution to system (4.2) on Ω_1 (resp. Ω_2); see also Figure 2(a) (resp. Figure 4(a)). Here, k is the iteration number, $w^{k,i}\,(i = 1, 2)$ are the two byproducts as defined in (3.3). The starting point for our iteration is $u_0 = v_0 = (1 - x_1^2)(1 - x_2^2)$ (resp. $u_0 = v_0 = (1.4^2 - x_1^2 - x_2^2)$) with $x = (x_1, x_2) \in \Omega_1$ (resp. $x = (x_1, x_2) \in \Omega_2$). From Figures. 6 and 7, one sees that at the final iteration $\|d^k\|^2 \approx |J(p(w^{k,2})) - J(p(w^{k,1}))|$ for both cases, which agrees with our estimate established in Corollary 2.1.

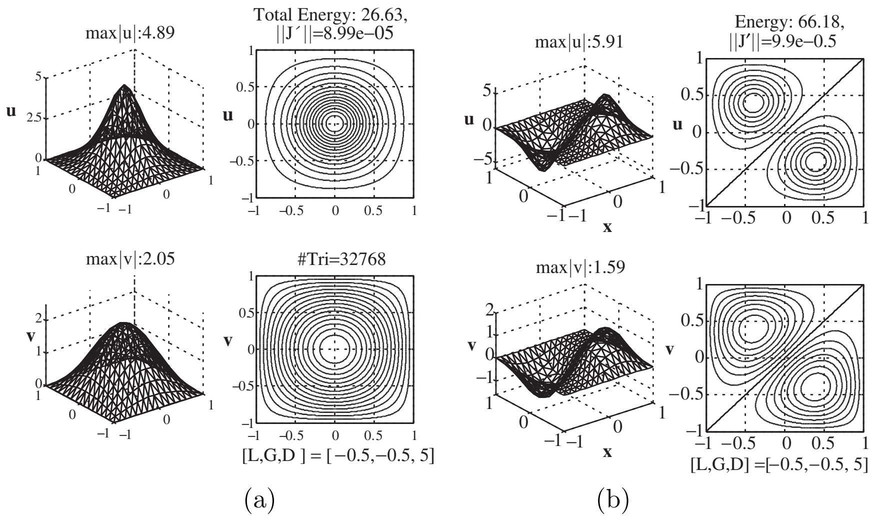

FIGURE 2. Profiles and contours of a positive solution (a) and a 2-peak sign-changing solution (b) to system (4.2) with $p = q = 3, \lambda = \gamma = -0.5, \delta = 5$.

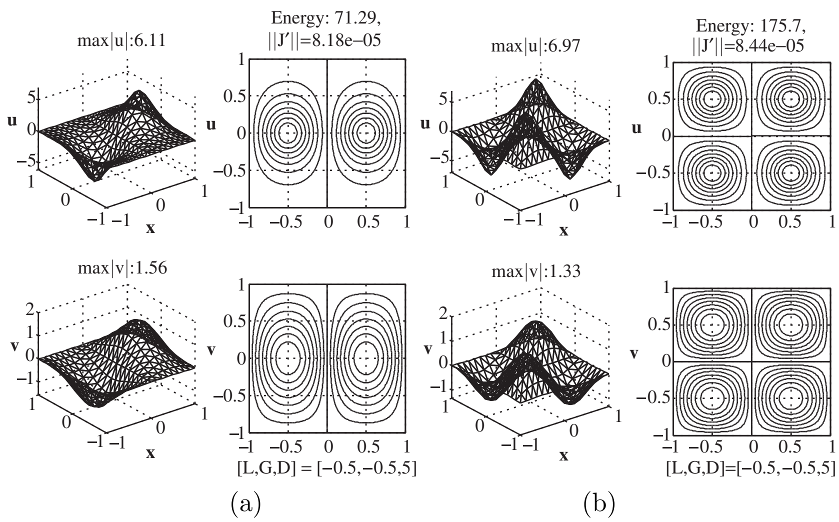

FIGURE 3. Profiles and contours of a 2-peak sign-changing solution (u_2^2, v_2^2) (a) where $L = \text{span}\{u_1\} \times \text{span}\{v_1\}$ and a 4-peak sign-changing solution (u_3, v_3) (b) where $L = \text{span}\{u_1, u_2^2\} \times \text{span}\{v_1, v_2^2\}$ to system (4.2) with $p = q = 3, \lambda = \gamma = -0.5, \delta = 5$.

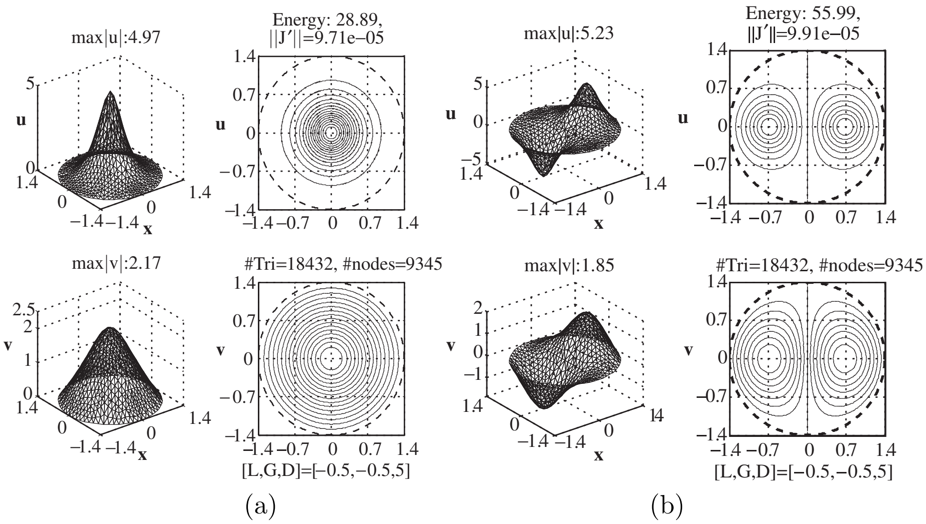

FIGURE 4. Profiles and contours of a positive solution (u_1, v_1) (a) and a 2-peak sign-changing solution (u_2, v_2) (b) where $L = \text{span}\{u_1\} \times \text{span}\{v_1\}$ to system (4.2) with $p = q = 3, \lambda = \gamma = -0.5, \delta = 5$. The dashed circle indicates the boundary of the domain.

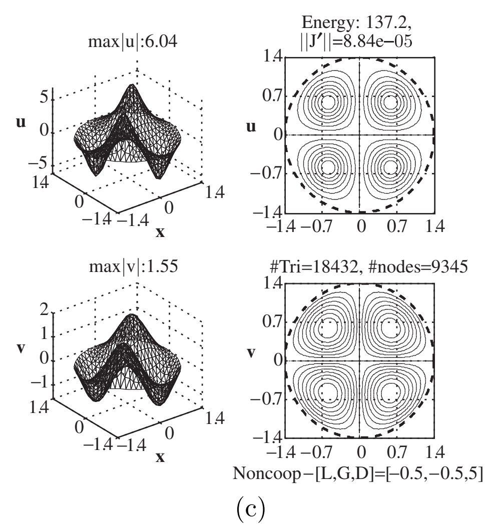

FIGURE 5. Profiles and contours of a 4-peak sign-changing solution (c) to system (4.2) with $p = q = 3, \lambda = \gamma = -0.5, \delta = 5$. The dashed circle indicates the domain boundary.

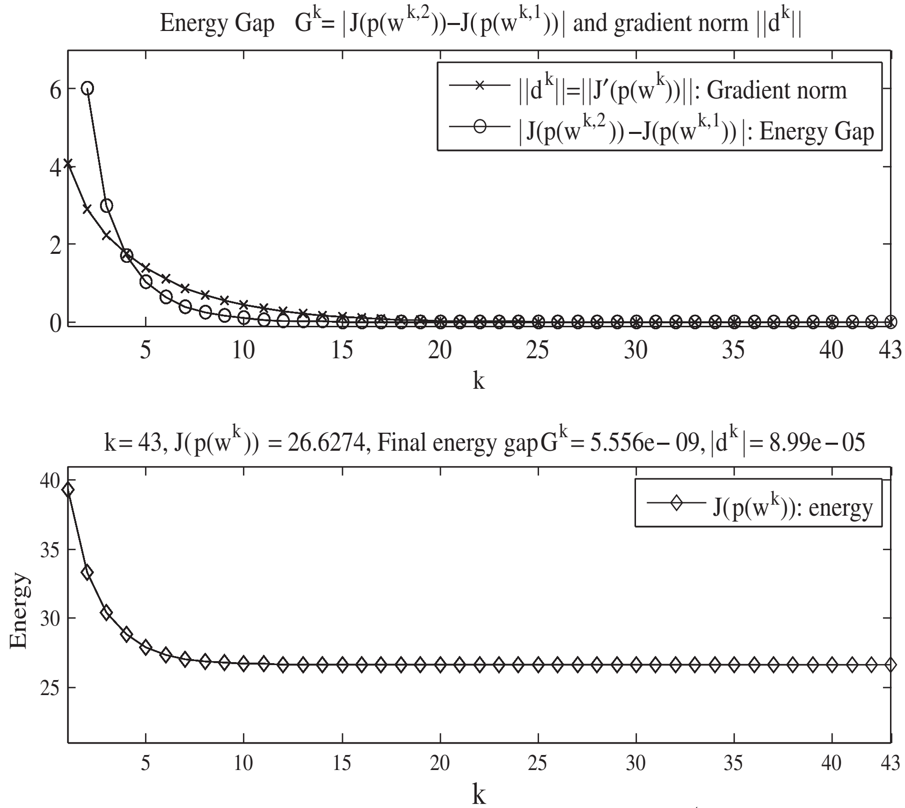

FIGURE 6. Convergence test on the positive solution (see also Figure 2(a)) to system (4.2) with $p = q = 3, \lambda = \gamma = -0.5, \delta = 5, \Omega = (-1,1)^2$: convergence of the energy gap $|J(p(w^{k,2})) - J(p(w^{k,1}))|$ and the gradient norm $\|d^k\|$ (top), convergence of the energy $J(p(w^k))$ (bottom). The x-axis (k) represents the iteration number.

4.2. **Noncooperative systems of indefinite type.** In this section we consider a noncooperative system of the form (4.1) where the nonlinear term $G(x;u,v)$ is indefinite (i.e., sign-changing). Due to this indefinite nature, none of the existence results in [3, 7, 8, 9, 15] is applicable. However, we have numerically found several solutions to such systems and discovered some interesting phenomena.

Example 4.2. Choose $N = 2$ (or $\Omega \subset \mathbb{R}^2$) and $G(x;u,v) \equiv G(u,v) = \frac{1}{p+1}|u|^{p+1} - \frac{1}{q+1}|v|^{q+1}$ with $p, q > 1$. System (4.1) becomes

$$(4.25)\qquad \begin{cases} -\Delta u = \lambda u - \delta v + |u|^{p-1}u, & x \in \Omega, \\ -\Delta v = \delta u + \gamma v + |v|^{q-1}v, & x \in \Omega, \end{cases} \qquad u = v = 0,\ x \in \partial\Omega,$$

to which the associated energy functional

$$(4.26)\quad J(u,v) = \frac{1}{2}\int_\Omega \Big[(|\nabla u|^2 - |\nabla v|^2) - (\lambda u^2 - 2\delta uv - \gamma v^2)\Big]dx - \int_\Omega G(u,v)dx$$

is well defined in $H = H_0^1(\Omega) \times H_0^1(\Omega)$ and of class $C^2(H,\mathbb{R})$.

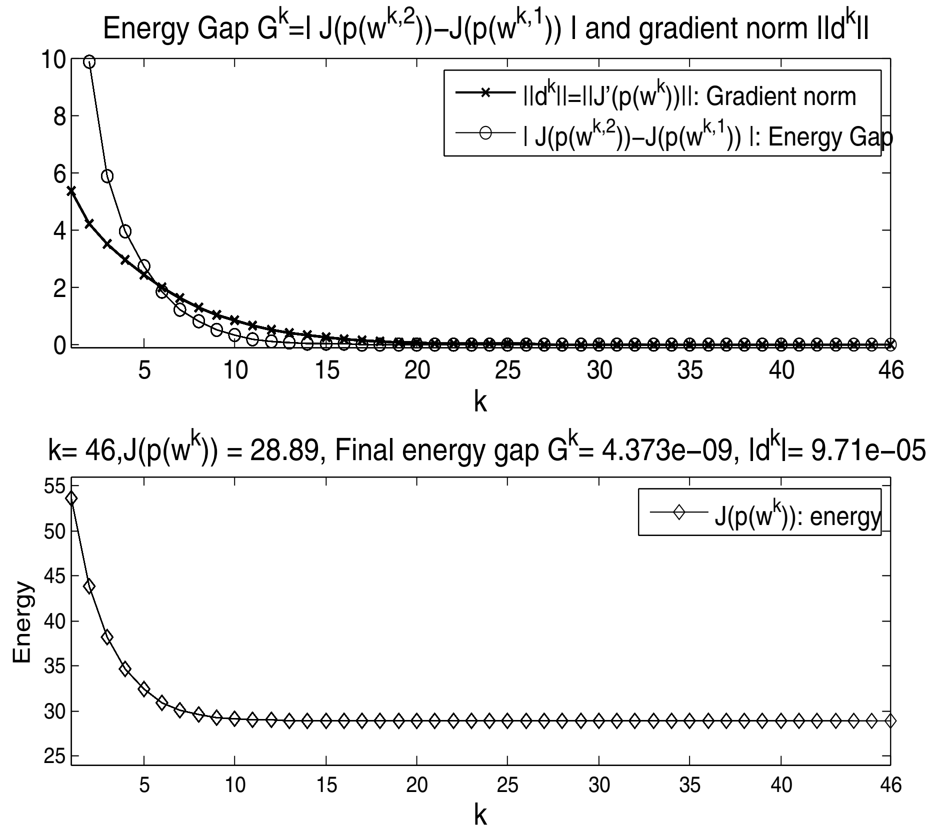

FIGURE 7. Convergence test on the positive solution (see also Figure 4(a)) to system (4.2) with $p = q = 3, \lambda = \gamma = -0.5, \delta = 5, \Omega = \{x \in \mathbb{R}^2 : |x| < 1.4\}$: convergence of the energy gap $|J(p(w^{k,2}))-J(p(w^{k,1}))|$ and the gradient norm $\|d^k\|$ (top), convergence of the energy $J(p(w^k))$ (bottom). The x-axis (k) represents the iteration number.

One sees that for this particular example, both the *asymptotic noncrossing* conditions $(F_4^{\pm})$ and *crossing* conditions (F_5)–(F_6) in Section 4.1 fail due to

$$\liminf_{|(u,v)|\to\infty} \frac{2G(x;u,v)}{|(u,v)|^2} = -\infty, \ \limsup_{|(u,v)|\to\infty} \frac{2G(x;u,v)}{|(u,v)|^2} = \infty \text{ and } \limsup_{|(u,v)|\to 0} \frac{2G(x;u,v)}{|(u,v)|^2} = 0.$$

Proposition 4.4. *Any critical point of J in* (4.26) *has an infinite Morse index.* ▮

Proposition 4.5. *For every critical point (u,v) of J in* (4.26)*, it follows that*

$$J(u,v) = \int_\Omega \Big[(\frac{1}{2} - \frac{1}{p+1})|u|^{p+1} + (\frac{1}{q+1} - \frac{1}{2})|v|^{q+1} \Big] dx.$$

Proof. Refer to the proof of Proposition 4.2. □

For this type of noncooperative system, so far we cannot give a general result on the existence of a local L-$\perp$ selection (which eventually boils down to the existence of nontrivial solutions to a system of nonlinear algebraic equations and hence is very difficult to solve). Instead, similar to Theorem 4.1, we establish a separation result for the case $L = \{0\} \times \{0\}$. As before, let σ_1 be the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$ and $\mathcal{M}_0$ be the solution set

$$\mathcal{M}_0 = \Big\{\tilde{p}(u,v) \neq (0,0) : \|(u,v)\| = 1\Big\}.$$

Theorem 4.3. *Assume $\delta > 0, \lambda, \gamma < \sigma_1$. Then there exists some $\alpha > 0$ such that*

$$\textit{dist}(\mathcal{M}_0, (0,0)) \geq \alpha > 0. \tag{4.27}$$

Consequently $(0,0) \notin \overline{\mathcal{M}}_0$ and solutions to system (4.25) found by the LMMOM are nontrivial.

Proof. For convenience, we borrow some notation used in equations (4.8)–(4.9), namely,

$$\begin{aligned} a_0 &= \delta \int \bar{u}\bar{v}dx,\ a_1 = \int [|\nabla \bar{u}|^2 - \lambda \bar{u}^2]dx,\ a_2 = \int |\bar{u}|^{p+1}dx, \\ b_1 &= \int [|\nabla \bar{v}|^2 - \gamma \bar{v}^2]dx,\ b_2 = \int |\bar{v}|^{q+1}dx \end{aligned}$$

for every unit vector $(\bar{u},\bar{v}) \in H$. By definition, if $\tilde{p}$ is an L-$\perp$ selection of J in (4.26) with respect to $L = \{0\} \times \{0\}$, then $\tilde{p}(\bar{u},\bar{v}) = (t\bar{u}, s\bar{v})$ with $(0,0) \neq (t,s) \in \mathbb{R}^2$ satisfies the system

$$\frac{\partial J}{\partial t}(t\bar{u}, s\bar{v}) = ta_1 + sa_0 - |t|^{p-1}ta_2 = 0, \tag{4.28}$$

$$\frac{\partial J}{\partial s}(t\bar{u}, s\bar{v}) = ta_0 - sb_1 + |s|^{q-1}sb_2 = 0. \tag{4.29}$$

Thus it suffices to prove that $\exists\, \alpha > 0$ s.t. $\|\tilde{p}(\bar{u},\bar{v})\| = \|(t\bar{u}, s\bar{v})\| \geq \alpha$, for any $\tilde{p}(\bar{u},\bar{v}) \in \mathcal{M}_0$.

We have two cases: (i) $a_0 = 0$ and (ii) $a_0 \neq 0$.

Case (i): One can see that equations (4.28) and (4.29) are actually decoupled. The fact $\|(\bar{u},\bar{v})\| = 1$ implies that at least one component of the vector $(\bar{u},\bar{v})$ must be nonzero. Then, using the Poincaré and Sobolev inequalities and following the lines in the proof of Lemma 4.1, one can easily complete the proof for this case.

Case (ii): Clearly, $\bar{u} \neq 0, \bar{v} \neq 0$. Then, $a_2, b_2 > 0$. By the Poincaré inequality, $a_1, b_1 > 0$. Multiplying (4.28) by t and (4.29) by s, and then subtracting one from another yields

$$t^2a_1 - |t|^{p+1}a_2 + s^2b_1 - |s|^{q+1}b_2 = 0 \quad \text{or} \quad t^2a_1 + s^2b_1 = |t|^{p+1}a_2 + |s|^{q+1}b_2. \tag{4.30}$$

With $\|\bar{u}\|_2^2 = \int |\nabla \bar{u}|^2dx$ and $\|\bar{v}\|_2^2 = \int |\nabla \bar{v}|^2dx$, applying the Poincaré and Sobolev inequalities gives

$$c_p|t|^{p+1}\|\bar{u}\|_2^{p+1} \geq |t|^{p+1}\int |\bar{u}|^{p+1}dx = |t|^{p+1}a_2, \tag{4.31}$$

$$c_q|s|^{q+1}\|\bar{v}\|_2^{q+1} \geq |s|^{q+1}\int |\bar{v}|^{q+1}dx = |s|^{q+1}b_2 \tag{4.32}$$

and

$$t^2 a_1 = t^2 \int [|\nabla \bar{u}|^2 - \lambda \bar{u}^2] dx \geq t^2 (1 - \frac{\lambda}{\sigma_1}) \|\bar{u}\|_2^2, \tag{4.33}$$

$$s^2 b_1 = s^2 \int [|\nabla \bar{v}|^2 - \gamma \bar{v}^2] dx \geq s^2 (1 - \frac{\gamma}{\sigma_1}) \|\bar{v}\|_2^2, \tag{4.34}$$

for some constants (independent of $\bar{u}$ and $\bar{v}$) $c_p, c_q > 0$, which, together with (4.30), lead to

$$\begin{aligned} c_p |t|^{p+1} \|\bar{u}\|_2^{p+1} + c_q |s|^{q+1} \|\bar{v}\|_2^{q+1} &\geq |t|^{p+1} a_2 + |s|^{q+1} b_2 \\ \geq \quad t^2 (1 - \frac{\lambda}{\sigma_1}) \|\bar{u}\|_2^2 + s^2 (1 - \frac{\gamma}{\sigma_1}) \|\bar{v}\|_2^2 &\geq \min \left\{ 1 - \frac{\lambda}{\sigma_1}, 1 - \frac{\gamma}{\sigma_1} \right\} \|(t\bar{u}, s\bar{v})\|^2. \end{aligned} \tag{4.35}$$

With the fact $\|t\bar{u}\|_2^{p+1} \leq \|(t\bar{u}, s\bar{v})\|^{q+1}$ and $\|s\bar{v}\|_2^{q+1} \leq \|(t\bar{u}, s\bar{v})\|^{q+1}$, we obtain

$$\begin{aligned} (c_p + c_q)(\|(t\bar{u}, s\bar{v})\|^{p+1} + \|(t\bar{u}, s\bar{v})\|^{q+1}) &\geq c_p \|t\bar{u}\|_2^{p+1} + c_q \|s\bar{v}\|_2^{q+1} \\ &\geq \min \left\{ 1 - \frac{\lambda}{\sigma_1}, 1 - \frac{\gamma}{\sigma_1} \right\} \|(t\bar{u}, s\bar{v})\|^2 \end{aligned} \tag{4.36}$$

or

$$(\|(t\bar{u}, s\bar{v})\|^{p-1} + \|(t\bar{u}, s\bar{v})\|^{q-1}) \geq \frac{1}{c_p + c_q} \min \left\{ 1 - \frac{\lambda}{\sigma_1}, 1 - \frac{\gamma}{\sigma_1} \right\}.$$

Since $p, q > 1$ and $c_p, c_q, \sigma_1, \gamma, \lambda$ are independent of $\bar{u}, \bar{v}$, we conclude that $\exists \alpha > 0$ s.t.

$$\|\tilde{p}(\bar{u}, \bar{v})\| = \|(t\bar{u}, s\bar{v})\| \geq \alpha, \forall \tilde{p}(\bar{u}, \bar{v}) \in \mathcal{M}_0. \tag{4.37}$$ □

Similarly, the gradient $d \equiv (d_1, d_2) = \nabla J(u, v)$ of J in (4.26) is solved from the following two linear elliptic equations

$$\begin{cases} \Delta d_1 = \Delta u + \lambda u - \delta v + |u|^{p-1} u & x \in \Omega, \\ \Delta d_2 = -(\Delta v + \delta u + \gamma v + |v|^{q-1} v) & x \in \Omega, \end{cases} \quad d_1 = d_2 = 0, \ x \in \partial\Omega. \tag{4.38}$$

At the end of this section, we present the first few numerical solutions to system (4.25) with $\lambda = -0.5, \gamma = -1, \delta = 0.5$ on the domains: $\Omega_1 = (-2, 2)^2, \Omega_2 = (-3, 3)^2$ and $\Omega_3 =$ a disk of radius 3. For all cases, $L = \{0\} \times \{0\}$; meanwhile, symmetries have been used particularly to find the sign-changing solutions shown in Figures 9–11 as well as the nonradial positive solution shown in Figure 10(b) in order to make our method more efficient [14].

On Ω_1, we found a unique positive solution which is symmetric w.r.t. both x- and y-axes and two sign-changing solutions. Their profiles are similar to the solutions of system (4.25) on Ω_2 as shown in Figures 9(a), 9(b) and 10(a), respectively, and therefore are omitted here.

On Ω_2, we surprisingly found two asymmetric positive solutions to system (4.25) as shown in Figure 8(a)-(b) with relatively smaller energy than that of the symmetric positive one shown in Figure 9(a). Due to the symmetry of the problem, any asymmetric solution becomes a new solution after a rotation by $\frac{\pi}{2}, \pi, \frac{3}{2}\pi$. Since there is no explicit appearance of the space variable x in system (4.25), such asymmetric positive solutions do not exist for its analogous single equation problem due to the well-known Gidas-Ni-Nirenberg theorem. To further confirm this new phenomenon and eliminate a possible corner effect of the domain Ω_2, we repeated our experiments on Ω_3. Besides the radial positive solution and the sign-changing solution as shown in Figure 11, we found an asymmetric positive solution as depicted in Figure 10(b). Since such an asymmetric solution is always a solution after a rotation by any angle, we actually obtained a one-parameter family of degenerate asymmetric positive solutions.

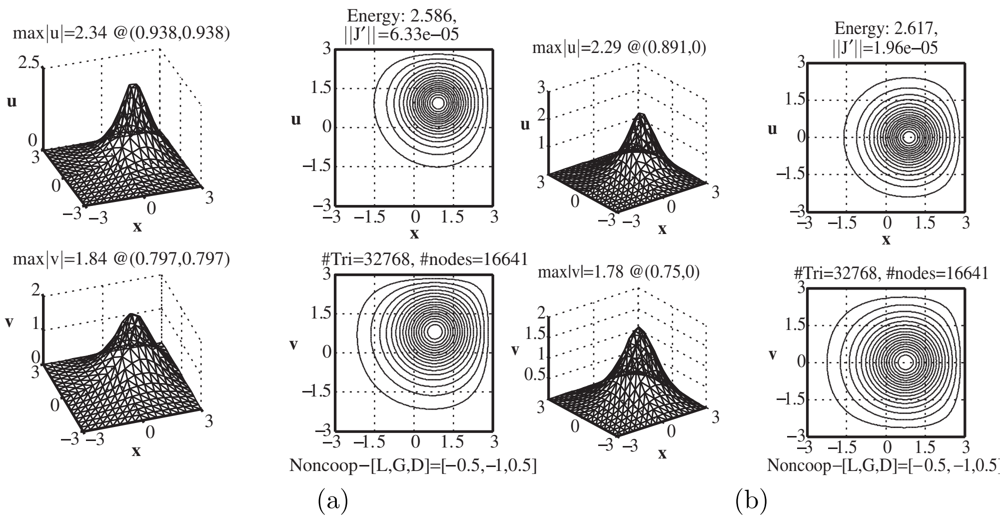

FIGURE 8. Profiles and contours of two asymmetric positive solutions (a) and (b) to system (4.25) with $\lambda = -0.5, \gamma = -1, \delta = 0.5, \Omega = (-3, 3) \times (-3, 3)$.

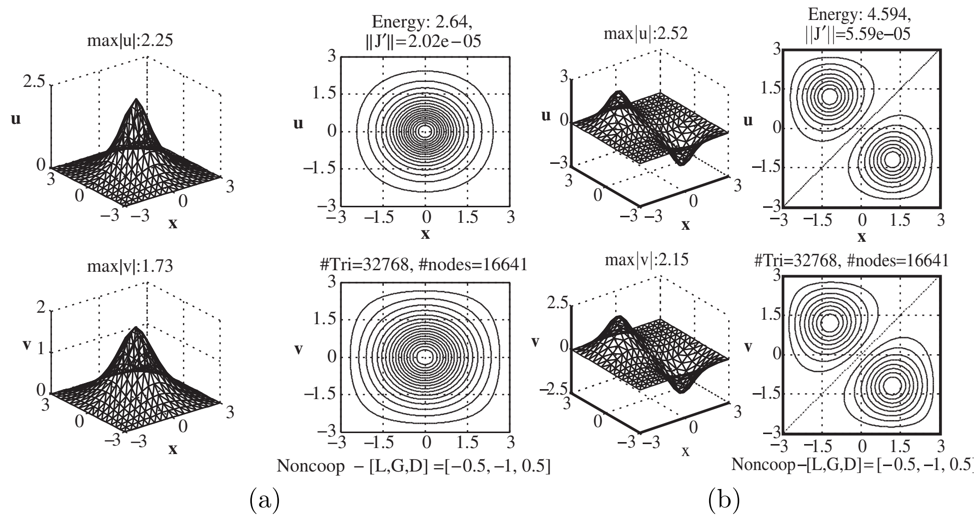

FIGURE 9. Profiles and contours of a symmetric positive solution (a) and a sign-changing solution (b) to system (4.25) with $\lambda = -0.5, \gamma = -1, \delta = 0.5, \Omega = (-3, 3) \times (-3, 3)$.

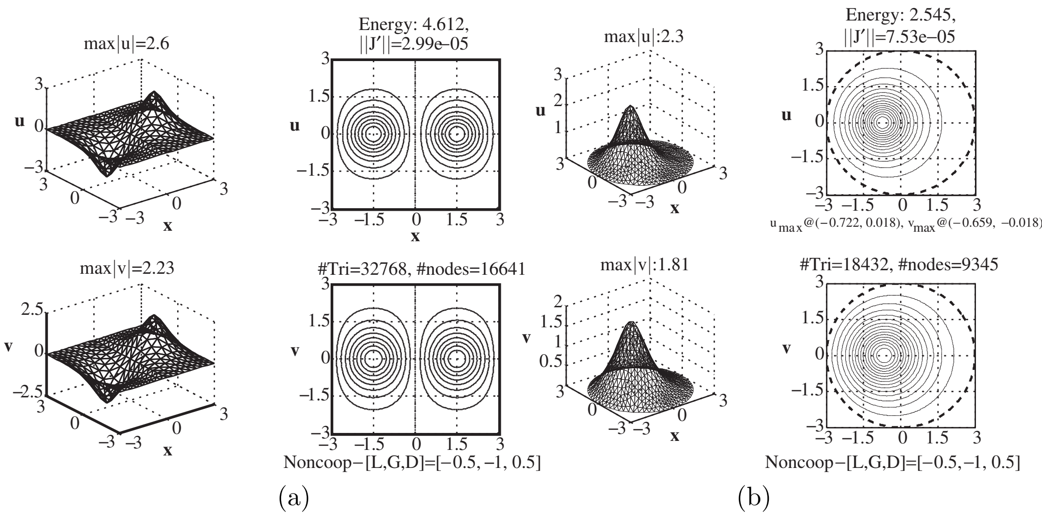

FIGURE 10. Profiles and contours of a second sign-changing solution (a) to system (4.25) with $\lambda = -0.5, \gamma = -1, \delta = 0.5, \Omega = (-3,3)\times(-3,3)$ and an asymmetric positive solution (b) to system (4.25) with $\lambda = -0.5, \gamma = -1, \delta = 0.5, \Omega =$ a disk of radius 3.

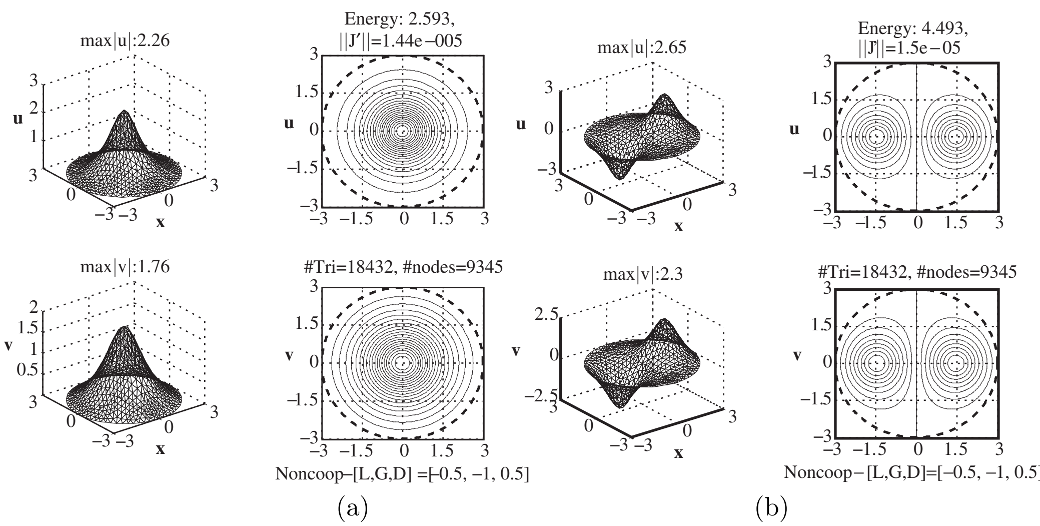

FIGURE 11. Profiles and contours of a radial positive solution (a) and a sign-changing solution (b) to system (4.25) with $\lambda = -0.5, \gamma = -1, \delta = 0.5, \Omega =$ a disk of radius 3. The dashed circle indicates the domain boundary.

In conclusion, we have developed a mathematically justified numerical method to solve noncooperative systems for their multiple solutions in an order and carried out numerical experiments in solving systems (4.2) and (4.25) on both square and radial domains. In particular, asymmetric positive solutions to system (4.25) are numerically found, possibly as a result of a bifurcation from the symmetric positive ones w.r.t. the domains. Hopefully, this new numerical finding will promote some theoretical verification on such phenomenon. In a subsequent paper, we will continue to study this new method including its convergence analysis.

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Institute for Mathematics & Its Application, University of Minnesota, Minneapolis, Minnesota 55455

E-mail address: `xchen@ima.umn.edu`

Department of Mathematics, Texas A&M University, College Station, Texas 77843

E-mail address: `jzhou@math.tamu.edu`