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VALUES OF SYMMETRIC CUBE L-FUNCTIONS AND FOURIER COEFFICIENTS OF SIEGEL EISENSTEIN SERIES OF DEGREE-3

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ABSTRACT. We obtain formulas for certain weighted sums of values of the symmetric square and triple product L-functions. As a consequence, we get exact values at the right critical point for the symmetric square and symmetric cube L-functions attached to certain cuspforms. We also give applications to Fourier coefficients of modular forms.

1. INTRODUCTION

In [23] Zagier found exact values of the standard degree-2 L-function and the symmetric square L-function attached to the weight 12 and level 1 cuspform $\Delta_{12}(z)$. The values of the symmetric square L-function were given in terms of the square of the Petersson norm of $\Delta_{12}(z)$, powers of π, and explicit rational numbers. From these results and numerical computations, predictions were given in [23] for certain exact values of the symmetric cube and the symmetric fourth power L-functions attached to $\Delta_{12}(z)$.

Using the methods of [23], Dummigan [6] found values of symmetric square L-functions attached to cuspforms of level 1 and weights 12, 16, 18, 20, 22, and 26 at all the critical points in the sense of Deligne [5]. He also studied the rational numbers that occur in these values and related primes that occur in the numerators of the rational parts to Shafarevich-Tate groups. In [15] Katsurada applied the method of pullbacks of Eisenstein series due to Böcherer [1] and Garrett [8, 9, 11] to the study of the values of the symmetric square L-functions. These results illustrate how the rational parts of the values can be naturally expressed in terms of Bernoulli numbers B_n and the generalized class numbers $H(r, n)$ of Cohen [3]. This method was used by Heim [13] to study Ramanujan's τ-function, which gives the Fourier coefficients of $\Delta_{12}(z)$.

In [10] Garrett discovered an integral representation of an L-function attached to three cuspforms by investigating the pullback of a Siegel Eisenstein series of degree-3. Taking the three cuspforms to be identical, special value results were obtained for the symmetric cube L-function. These results for the triple product and the symmetric cube were extended in [12] and [17]. In [20] Mizumoto verified Zagier's predictions for the symmetric cube L-function of $\Delta_{12}(z)$. His method started from Garrett's integral but went via the $2+1$ diagonal and so replaced one

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TABLE 1. Values of symmetric cube L-functions at the right critical point.

κ	$L(2\kappa-2,\mathrm{Sym}^3\Delta_\kappa)L(\kappa-1,\Delta_\kappa)^2/\langle\Delta_\kappa,\Delta_\kappa\rangle^3\pi^{5\kappa-5}$
12	$2^{57}/3^{12}\cdot 5^7\cdot 7^5\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 691^2$
16	$2^{73}\cdot 37/3^{19}\cdot 5^9\cdot 7^7\cdot 11^4\cdot 13^4\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 3617^2$
18	$2^{76}\cdot 59/3^{25}\cdot 5^8\cdot 7^9\cdot 11^5\cdot 13^4\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 43867^2$
20	$2^{81}\cdot 157/3^{28}\cdot 5^{14}\cdot 7^5\cdot 11^4\cdot 13^4\cdot 17^4\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37\cdot 283^2\cdot 617^2$
22	$2^{91}\cdot 38329/3^{34}\cdot 5^{13}\cdot 7^8\cdot 11^6\cdot 13^4\cdot 17^4\cdot 19^4\cdot 23\cdot 29\cdot 31\cdot 37\cdot 41\cdot 131^2\cdot 593^2$
26	$2^{103}\cdot 173\cdot 311/3^{29}\cdot 5^{15}\cdot 7^{10}\cdot 11^9\cdot 13^3\cdot 17^4\cdot 19^4\cdot 23^4\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47\cdot 657931^2$

TABLE 2. Values of symmetric square L-functions at the right critical point.

κ	$L(2\kappa-2,\mathrm{Sym}^2\Delta_\kappa)/\langle\Delta_\kappa,\Delta_\kappa\rangle\pi^{3\kappa-3}$
12	$2^{24}/3^9\cdot 5^4\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 691$
16	$2^{30}/3^{13}\cdot 5^6\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 3617$
18	$2^{32}/3^{17}\cdot 5^5\cdot 7^4\cdot 11^2\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 43867$
20	$2^{32}\cdot 71^2/3^{18}\cdot 5^9\cdot 7^3\cdot 11^2\cdot 13^2\cdot 17^2\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37\cdot 283\cdot 617$
22	$2^{37}\cdot 61^2\cdot 103/3^{21}\cdot 5^8\cdot 7^4\cdot 11^4\cdot 13^2\cdot 17\cdot 19^2\cdot 23\cdot 29\cdot 31\cdot 37\cdot 41\cdot 131\cdot 593$
26	$2^{41}\cdot 163\cdot 187273/3^{26}\cdot 5^{10}\cdot 7^7\cdot 11^4\cdot 13^2\cdot 17^2\cdot 19^2\cdot 23^2\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47\cdot 657931$

of the degree-1 cuspforms with a Klingen Eisenstein series. This avoided the degree-3 Eisenstein series and so avoided certain difficulties that arise by using Garrett's integral directly. Note that Mizumoto's method applies to other critical points as well, and these difficulties are most relevant for obtaining these other values.

In this paper we use the pullback of Siegel Eisenstein series of degree-3 to obtain exact values for certain cuspforms of symmetric cube L-functions at the right critical point. We avoid some difficulties by switching to adelic language in certain places. We first redo the results of [15] for level 1 cuspforms and express the rational parts of the values of certain symmetric square L-functions in terms of the Fourier coefficients of Siegel Eisenstein series of degree-2. This gives an expression for the values in terms of B_n and $H(r,n)$. We use Garrett's integral to express a weighted sum of the values of certain triple L-functions in terms of values of symmetric square L-functions and Fourier coefficients of Eisenstein series of degrees 1 and 3. From the values of the symmetric square L-functions and Katsurada's explicit formula for the Fourier coefficients of Siegel Eisenstein series of degree-3 in [14] we can express the weighted sum in terms of B_n and $H(r,n)$. For certain weights, this gives exact values for the product of a symmetric cube L-function and the square of a standard degree-2 L-function. The method here leads to essentially different computations than those done by Mizumoto for the right critical point, as the latter involve values of twisted degree-1 L-functions.

The values of the symmetric square and the symmetric cube L-functions are given in Tables 1 and 2, respectively. Those of the symmetric square are known and those of the symmetric cube are mostly new. The special value results follow from Theorem 1 and equation (3.9) in Section 3. The precise definitions of the L-functions and of B_n and $H(r,n)$ are given in Section 2.

Theorem 1. *Let $\mathcal{B}_\kappa$ be a basis for the space of cuspforms of weight κ and level 1 consisting of Hecke eigenfunctions and normalized so that their first Fourier coefficients are 1. Let B_n denote the n^{th} Bernoulli number and $H(r,n)$ Cohen's* *generalized class number. Then*

$$\begin{aligned}
&\sum_{f_1,f_2,f_3\in\mathcal{B}_\kappa} \frac{L(2\kappa-2, f_1\otimes f_2\otimes f_3)}{\langle f_1,f_1\rangle\langle f_2,f_2\rangle\langle f_3,f_3\rangle\pi^{5\kappa-5}} \\
&\quad= \frac{2^{8\kappa-9}}{(\kappa-2)!^3(2\kappa-2)!}\Bigg(B_{2\kappa-2}\left(1+\frac{\kappa}{B_\kappa}\right)\left(1+\frac{2\kappa}{B_\kappa}\right) \\
&\qquad - 2(\kappa-1)\left(4+\frac{3\kappa}{B_\kappa}\right)H(\kappa-1,3) - 3(\kappa-1)\left(1+\frac{\kappa}{B_\kappa}\right)H(\kappa-1,4) \\
&\qquad - (-1)^{\kappa/2}(\kappa-1)\frac{B_\kappa B_{2\kappa-2}}{|B_\kappa B_{2\kappa-2}|}(2^{2\kappa-4}+2\cdot 3^{\kappa-1}+2^{\kappa+2}-23)\Bigg).
\end{aligned}$$

For $\kappa\in\{12,16,18,20,22,26\}$ let $\Delta_\kappa(z)$ denote the unique weight κ and level 1 cuspform normalized so that the first Fourier coefficient is 1. We have

$$L(s,\Delta_\kappa\otimes\Delta_\kappa\otimes\Delta_\kappa) = L(s,\mathrm{Sym}^3\Delta_\kappa)L(s-(\kappa-1),\Delta_\kappa)^2,$$

and so Theorem 1 immediately gives the values in Table 1. Let $\Delta_{24,1}(z)$ and $\Delta_{24,2}(z)$ denote the two normalized Hecke eigenfunctions of weight 24. Then Theorem 1 gives a sum of the values of L-functions attached to these cuspforms in (1.1):

$$\begin{aligned}
&\frac{L(46,\mathrm{Sym}^3\Delta_{24,1})L(23,\Delta_{24,1})^2}{\langle\Delta_{24,1},\Delta_{24,1}\rangle^3\pi^{115}} + \frac{3L(46,\mathrm{Sym}^2\Delta_{24,1}\otimes\Delta_{24,2})L(23,\Delta_{24,2})}{\langle\Delta_{24,1},\Delta_{24,1}\rangle^2\langle\Delta_{24,2},\Delta_{24,2}\rangle\pi^{115}} \\
&+\frac{3L(46,\mathrm{Sym}^2\Delta_{24,2}\otimes\Delta_{24,1})L(23,\Delta_{24,1})}{\langle\Delta_{24,2},\Delta_{24,2}\rangle^2\langle\Delta_{24,1},\Delta_{24,1}\rangle\pi^{115}} + \frac{L(46,\mathrm{Sym}^3\Delta_{24,2})L(23,\Delta_{24,2})^2}{\langle\Delta_{24,2},\Delta_{24,2}\rangle^3\pi^{115}} \\
&= \frac{2^{97}\cdot 1194654368793652666374 1}{3^{40}\cdot 5^{19}\cdot 7^{14}\cdot 11^7\cdot 13^6\cdot 17^4\cdot 19^4\cdot 23\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47\cdot 103^2\cdot 2294797^2}.
\end{aligned} \tag{1.1}$$

The values in Table 2 are obtained by equation (3.9) and are all contained in Dummigan's article [6], who used the methods of [23]. As in (1.1) we can also use (3.9) to get the following result for the cuspforms of weight 24:

$$\begin{aligned}
&\frac{L(46,\mathrm{Sym}^2\Delta_{24,1})}{\langle\Delta_{24,1},\Delta_{24,1}\rangle\pi^{69}} + \frac{L(46,\mathrm{Sym}^2\Delta_{24,2})}{\langle\Delta_{24,2},\Delta_{24,2}\rangle\pi^{69}} \\
&= \frac{2^{38}\cdot 59\cdot 691\cdot 2294824233197}{3^{26}\cdot 5^{12}\cdot 7^7\cdot 11^3\cdot 13^4\cdot 17^2\cdot 19^2\cdot 23\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47\cdot 103\cdot 2294797}.
\end{aligned}$$

Based on numerical computations, Zagier in [23] made several predictions about the exact values of $L(s,\mathrm{Sym}^3\Delta_{12})$ and $L(s,\mathrm{Sym}^4\Delta_{12})$. Among them is

$$\frac{L(22,\mathrm{Sym}^3\Delta_{12})}{C_+^3C_-\pi^{33}} = \frac{2^{16}}{3^8\cdot 5^5\cdot 7^3\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23}$$

where C_+ and C_- are certain periods in the sense of [22] attached to $\Delta_{12}(z)$. With Zagier's normalization we have $C_+C_- = 2^{11}\langle\Delta_{12},\Delta_{12}\rangle$. From [23] we have the value

$$\frac{L(11,\Delta_{12})^2}{C_-^2\pi^{22}} = \frac{2^8}{3^4\cdot 5^2\cdot 7^2\cdot 691^2}.$$

Therefore, the prediction in [23] is equivalent to

$$\frac{L(22,\mathrm{Sym}^3\Delta_{12})L(11,\Delta_{12})^2}{\langle\Delta_{12},\Delta_{12}\rangle^3\pi^{55}} = \frac{2^{57}}{3^{12}\cdot 5^7\cdot 7^5\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 691^2}$$

and this is the first entry of Table 1. Mizumoto in [20] also obtained this value using other methods. Using Maass-Shimura type operators as in [15] we hope to obtain further critical values of these and other symmetric cube L-functions in future work.

For the real part of $s \in \mathbb{C}$ sufficiently large, we can write the respective L-functions of the normalized weight κ and level 1 Hecke eigencuspform $f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}$ in the following ways:

$$L(s,f) = \prod_{p \text{ prime}} (1 - a_f(p)p^{-s} + p^{\kappa-1-2s})^{-1},$$

$$L(s, \mathrm{Sym}^2 f) = \prod_{p \text{ prime}} \left((1 - (a_f(p)^2 - 2p^{\kappa-1})p^{-s} + p^{2\kappa-2-2s})(1 - p^{\kappa-1-s})\right)^{-1},$$

$$L(s, \mathrm{Sym}^3 f) = \prod_{p \text{ prime}} \big((1 - (a_f(p)^3 - 3p^{\kappa-1}a_f(p))p^{-s} + p^{3\kappa-3-2s})(1 - a_f(p)p^{\kappa-1-s} + p^{3\kappa-3-2s})\big)^{-1}.$$

Therefore, for $\Delta_\kappa(z) = \sum_{n=1}^{\infty} \tau_\kappa(n) e^{2\pi i n z}$ we have the ratio

$$\begin{aligned}&\frac{L(2\kappa-2, \mathrm{Sym}^3\Delta_\kappa)L(\kappa-1,\Delta_\kappa)^2\pi^{4\kappa-4}}{L(2\kappa-2,\mathrm{Sym}^2\Delta_\kappa)^3} \\ &= \pi^{4\kappa-4} \prod_{p \text{ prime}} \frac{(1-(\tau_\kappa(p)^2 - 2p^{\kappa-1})p^{-2\kappa+2} + p^{-2\kappa+2})^3(1-p^{-\kappa+1})^3}{(1-(\tau_\kappa(p)^3 - 3p^{\kappa-1}\tau_\kappa(p))p^{-2\kappa+2} + p^{-\kappa+1})(1-\tau_\kappa(p)p^{-\kappa+1} + p^{-\kappa+1})^2}.\end{aligned} \tag{1.2}$$

We calculated (1.2) numerically using 2000 factors in the Euler product for κ as in Tables 1 and 2. We compared these to the values of the ratios obtained using the results in Tables 1 and 2, and found agreement to 47 decimal places.

In Section 2 we define the Eisenstein series and the L-functions that we work with and give the definitions of B_n and $H(r, n)$. In Section 3 we prove an explicit decomposition of the restriction to the diagonal of Siegel Eisenstein series of degree-2 which gives the values in Table 2. We also give certain applications of this formula. We prove a decomposition formula for the restriction to the diagonal of Siegel Eisenstein series of degree-3 in Section 4. This makes precise a general description of the restriction first stated in Section 6 in [10]. We switch to adelic language in certain places of both sections to simplify some computations. In the last section we use Katsurada's formula for the Fourier coefficients of Siegel Eisenstein series of degree-3 in [14] to prove Theorem 1. The numerical computations here were performed using Mathematica 6.0.

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2. Siegel Eisenstein series and L-functions

Let

$$J_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$$

in $n \times n$ blocks. For a commutative ring R let $R^\times$ denote the group of units of R, $M_n(R)$ the set of $n \times n$ matrices with entries in R, and let

$$Sp_n(R) = \{g \in M_{2n}(R) \mid g^T J_n g = J_n\}$$

where g^T is a matrix transpose. Consider the Siegel parabolic subgroup

$$P_{n,0}(R) = \left\{ \begin{pmatrix} A & B \\ 0 & A^{T-1} \end{pmatrix} \in Sp_n(R) \;\middle|\; A \in GL_n(R) \right\},$$

the Klingen parabolic subgroup for $n = 2$,

$$P_{2,1}(R) = \left\{ \begin{pmatrix} a & * & * & * \\ 0 & \alpha & * & \beta \\ 0 & 0 & a^{-1} & 0 \\ 0 & \gamma & * & \delta \end{pmatrix} \in Sp_2(R) \;\middle|\; a \in R^\times, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(R) \right\},$$

and the Borel subgroups for $n = 1$ and 2,

$$B_1(R) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in SL_2(R) \mid a \in R^\times, \quad b \in R \right\},$$

$$B_2(R) = \left\{ \begin{pmatrix} A & B \\ 0 & A^{T-1} \end{pmatrix} \in Sp_2(R) \;\middle|\; A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in GL_2(R) \right\}.$$

Let $n \in \mathbb{Z}_{>0}$ and let $\kappa > n + 2$ be an even integer. The degree-n Siegel upper half-space is

$$\mathfrak{H}_n = \{z \in M_n(\mathbb{C}) \mid z^T = z, \ z = x + iy, \ y > 0\}.$$

Then $Sp_n(\mathbb{Z})$ acts on $\mathfrak{H}_n$ by $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)(z) = (Az + B)(Cz + D)^{-1}$ for $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \in Sp_n(\mathbb{Z})$ and $z \in \mathfrak{H}_n$. Recall that a coprime symmetric pair of matrices $\{C, D\}$ satisfies $CD^T = DC^T$ and the entries are coprime in the sense that if GC and GD are both matrices with integral entries, then so is the matrix G. From Maass [19] this is the same as the ordered pair (C, D) having a completion to an element $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \in Sp_n(\mathbb{Z})$. Note that two coprime symmetric pairs are said to be equivalent if $(C, D) = G(C', D')$ for some $G \in GL_n(\mathbb{Z})$. Then we define the weight κ and level 1 Siegel Eisenstein series of degree-n by

$$E_{n,\kappa}(z) = \sum_{\{C,D\}} |Cz + D|^{-\kappa} \tag{2.1}$$

where $\{C, D\}$ ranges over all representatives of equivalence classes of coprime symmetric pairs of degree-n.

For computational reasons, it will be convenient to temporarily switch to adelic language. Recall that the profinite completion of the integers is $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$ and the ring of integral adeles is $\mathbb{A}_\mathbb{Z} = \mathbb{R} \times \widehat{\mathbb{Z}}$. The ring of rational adeles is $\mathbb{A}_\mathbb{Q} = \mathbb{Q} \otimes \mathbb{A}_\mathbb{Z}$ and $\mathbb{J}_\mathbb{Q} = \mathbb{A}_\mathbb{Q}^\times$ denotes the ideles. Let $\mathbb{A}_0$ denote the finite adeles of $\mathbb{Q}$.

Following [11], we can associate to a holomorphic modular form $f(z)$ on $\mathfrak{H}_n$ a holomorphic modular form on $Sp_n(\mathbb{R})$ and then a holomorphic modular form on $Sp_n(\mathbb{A}_\mathbb{Q})$ in the following way. For $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \in Sp_n(\mathbb{R})$ and $z \in \mathfrak{H}_n$ let $\rho_\kappa(g) = \det(A + iB)^\kappa$ and $\mu(g, z) = \det(Cz + D)^{-1}$. To a modular form $f(z)$ on $\mathfrak{H}_n$ define $\Psi_f(g) = \mu(g, i1_n)^\kappa f(g(i1_n))$. Setting

$$K_\infty = \{g \in Sp_n(\mathbb{R}) \mid g(i1_n) = i1_n\},$$

then $\Psi_f(g)$ is left $Sp_n(\mathbb{Z})$-invariant and right ρ_κ-equivariant with respect to K_∞. We say that $\Psi_f(g)$ is a holomorphic cuspform on $Sp_n(\mathbb{R})$ if $f(z)$ is a holomorphic cuspform on $\mathfrak{H}_n$.

The Strong Approximation Theorem states that $Sp_n(\mathbb{Q})Sp_n(\mathbb{R})$ is dense in $Sp_n(\mathbb{A}_\mathbb{Q})$. Thus the natural injection $Sp_n(\mathbb{Z})\backslash Sp_n(\mathbb{R}) \to Sp_n(\mathbb{Q})\backslash Sp_n(\mathbb{A}_\mathbb{Q})/Sp_n(\mathbb{A}_0)$

is a diffeomorphism. It follows that to a left $Sp_n(\mathbb{Z})$-invariant and right (K_∞, ρ_κ)-equivariant continuous function Ψ_f on $Sp_n(\mathbb{R})$ we can associate a left $Sp_n(\mathbb{Q})$-invariant, right (K_∞, ρ_κ)-equivariant and right $Sp_n(\mathbb{A}_0)$-invariant function on $Sp_n(\mathbb{A}_\mathbb{Q})$. We temporarily label this function $\widetilde{\Psi}_f(g)$ and we say that $\widetilde{\Psi}_f(g)$ is holomorphic of weight κ if $\Psi_f(g)$ (and also f) is holomorphic of weight κ. Similarly, $\widetilde{\Psi}_f(g)$ is a cuspform if $\Psi_f(g)$ (and f) is a cuspform. By abuse of notation, for simplicity we relabel $\widetilde{\Psi}_f(g)$ by $f(g)$ throughout.

For $p = \left(\begin{smallmatrix} A & B \\ 0 & A^{T-1} \end{smallmatrix}\right) \in P_{n,0}(\mathbb{A}_\mathbb{Q})$ we define $\epsilon_{n,\kappa}(p) = |\det A|_\mathbb{J}^\kappa$. For $g \in Sp_n(\mathbb{A}_\mathbb{Q})$ let $\epsilon_{n,\kappa}$ be a smooth function so that $\epsilon_{n,\kappa}(pg) = \epsilon_{n,\kappa}(p)\epsilon_{n,\kappa}(g)$. The adelic version of the Eisenstein series (2.1) is

$$E_{n,\kappa}(g) = \sum_{\gamma \in P_{n,0}(\mathbb{Q})\backslash Sp_n(\mathbb{Q})} \epsilon_{n,\kappa}(\gamma g)$$

for $g \in Sp_n(\mathbb{A}_\mathbb{Q})$.

Consider the diagonal embedding $\iota : \mathfrak{H}_1 \times \cdots \times \mathfrak{H}_1 \hookrightarrow \mathfrak{H}_n$ where

$$\iota(z_1, \ldots, z_n) = \begin{pmatrix} z_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z_n \end{pmatrix}.$$

The respective embedding of groups is $\iota : SL_2(R) \times \cdots \times SL_2(R) \hookrightarrow Sp_n(R)$ where

$$\iota(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}) = \begin{pmatrix} a_1 & & & b_1 & & \\ & \ddots & & & \ddots & \\ & & a_n & & & b_n \\ c_1 & & & d_1 & & \\ & \ddots & & & \ddots & \\ & & c_n & & & d_n \end{pmatrix}.$$

These embeddings are compatible in the sense that for $g \in SL_2(\mathbb{Z}) \times \cdots \times SL_2(\mathbb{Z})$ and $z \in \mathfrak{H}_1 \times \cdots \times \mathfrak{H}_1$ we have $\iota(g(z)) = \iota(g)(\iota(z))$. Note that for $g = (g_1, \ldots, g_n) \in B_1(\mathbb{A}_\mathbb{Q}) \times \cdots \times B_1(\mathbb{A}_\mathbb{Q})$ then $\epsilon_{n,\kappa}(\iota(g)) = \epsilon_{1,\kappa}(g_1) \cdots \epsilon_{1,\kappa}(g_n)$, and note that this holds also for $g \in SL_2(\mathbb{A}_\mathbb{Q}) \times \cdots \times SL_2(\mathbb{A}_\mathbb{Q})$. Similarly, for $\iota : Sp_2(\mathbb{A}_\mathbb{Q}) \times SL_2(\mathbb{A}_\mathbb{Q}) \hookrightarrow Sp_3(\mathbb{A}_\mathbb{Q})$ and $(g_1, g_2) \in Sp_2(\mathbb{A}_\mathbb{Q}) \times SL_2(\mathbb{A}_\mathbb{Q})$ we have $\epsilon_{3,\kappa}(\iota(g_1, g_2)) = \epsilon_{2,\kappa}(g_1)\epsilon_{1,\kappa}(g_2)$.

Set $q = e^{2\pi i z}$ and let $f(z) = \sum_{n=1}^\infty a_f(n) q^n$ be a normalized cuspform of weight κ and level 1 that is an eigenfunction of the Hecke operators at all primes. We define the standard degree-2 L-function attached to $f(z)$ by

$$L(s, f) = \prod_{p \text{ prime}} \left((1 - \alpha_p p^{-s})(1 - \alpha'_p p^{-s})\right)^{-1},$$

which converges for $\mathrm{Re}(s) > (\kappa + 1)/2$. For any prime p, $\{\alpha_p, \alpha'_p\}$ are the Satake parameters of $f(z)$ where $\alpha_p \alpha'_p = p^{\kappa-1}$ and $\alpha_p + \alpha'_p = a_f(p)$. Via the Mellin transform, this L-function has a holomorphic continuation to $\mathbb{C}$ with a functional equation and critical set $\{1, \ldots, \kappa - 1\}$ in the sense of [5].

For the real part of s sufficiently large, the symmetric power L-functions of $f(z)$ are defined by the Euler products

$$L(s, \mathrm{Sym}^n f) = \prod_{p \text{ prime}} \prod_{j=0}^{n} (1 - \alpha_p^{n-j} \alpha_p'^j p^{-s})^{-1}$$

and, in particular,

$$L(s, \mathrm{Sym}^2 f) = \prod_{p \text{ prime}} \left((1-\alpha_p^2 p^{-s})(1-p^{\kappa-1-s})(1-\alpha_p'^2 p^{-s})\right)^{-1}$$

and

$$L(s, \mathrm{Sym}^3 f) = \prod_{p \text{ prime}} \left((1-\alpha_p^3 p^{-s})(1-\alpha_p^2\alpha_p' p^{-s})(1-\alpha_p\alpha_p'^2 p^{-s})(1-\alpha_p'^3 p^{-s})\right)^{-1}.$$

These last two are known to have analytic continuations to $\mathbb{C}$ from [1] and [16, 17], for example. The critical set of both L-functions is $\{1, \ldots, 2\kappa-2\}$.

For 3 cuspforms f_1, f_2, f_3 of weight κ and respective Satake parameters $\{\alpha_{p,j}, \alpha'_{p,j}\}$ for $j = 1, 2, 3$ we define the triple product L-function by

$$\begin{aligned} &L(s, f_1 \otimes f_2 \otimes f_3) \\ &\quad = \prod_{p \text{ prime}} \big((1-\alpha_{p,1}\alpha_{p,2}\alpha_{p,3}p^{-s})(1-\alpha_{p,1}\alpha_{p,2}\alpha'_{p,3}p^{-s})(1-\alpha_{p,1}\alpha'_{p,2}\alpha_{p,3}p^{-s}) \\ &\qquad\qquad \times (1-\alpha'_{p,1}\alpha_{p,2}\alpha_{p,3}p^{-s})(1-\alpha_{p,1}\alpha'_{p,2}\alpha'_{p,3}p^{-s})(1-\alpha'_{p,1}\alpha_{p,2}\alpha'_{p,3}p^{-s}) \\ &\qquad\qquad \times (1-\alpha'_{p,1}\alpha'_{p,2}\alpha_{p,3}p^{-s})(1-\alpha'_{p,1}\alpha'_{p,2}\alpha'_{p,3}p^{-s})\big)^{-1}. \end{aligned}$$

From [10] this has a meromorphic continuation, a functional equation, and critical set $\{\kappa, \ldots, 2\kappa-2\}$. Note that since $\alpha_{p,j}\alpha'_{p,j} = p^{\kappa-1}$ we have the decompositions

$$L(s, f \otimes f \otimes f) = L(s, \mathrm{Sym}^3 f) L(s-(\kappa-1), f)^2$$

and

$$L(s, f_1 \otimes f_1 \otimes f_2) = L(s, \mathrm{Sym}^2 f_1 \otimes f_2) L(s-(\kappa-1), f_2)$$

where

$$\begin{aligned} L(s, \mathrm{Sym}^2 f_1 \otimes f_2) = \prod_{p \text{ prime}} &\big((1-\alpha_{p,1}^2\alpha_{p,2}p^{-s})(1-\alpha_{p,1}^2\alpha'_{p,2}p^{-s})(1-\alpha'^2_{p,1}\alpha_{p,2}p^{-s}) \\ &\times (1-\alpha'^2_{p,1}\alpha'_{p,2}p^{-s})(1-\alpha_{p,2}p^{\kappa-1-s})(1-\alpha'_{p,2}p^{\kappa-1-s})\big)^{-1}. \end{aligned}$$

The Bernoulli numbers B_n are rational numbers defined by the generating function

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

From [3] the generalized class numbers $H(r, n)$ are rational numbers and $H(r, 0) = \zeta(1-2r) = -B_{2r}/2r$. For $n > 0$ and $D = (-1)^r n$ let $\chi_D(r) = \left(\frac{D}{r}\right)$ and $L(s, \chi_D) = \sum_{k=1}^{\infty} \chi_D(k) k^{-s}$. Define

$$h(r, n) = (-1)^{[r/2]}(r-1)! n^{r-1/2} 2^{1-r} \pi^{-r} L(r, \chi_D)$$

for $D \equiv 0, 1 \pmod 4$, and 0 otherwise. Then the generalized class numbers are defined by

$$H(r, n) = \sum_{d^2 | n} h\left(r, \frac{n}{d^2}\right).$$

If D is a discriminant of a quadratic field extension, then $H(r, n) = L(1-r, \chi_D) = -\frac{B_{r,\chi_D}}{r}$ where

$$B_{r,\chi_D} = |D|^{r-1} \sum_{j=0}^{|D|} \chi_D(j) B_r\left(\frac{j}{|D|}\right)$$

is a generalized Bernoulli number and

$$B_r(x) = \sum_{k=0}^{r} \binom{r}{k} B_k x^{r-k}$$

is a Bernoulli polynomial.

3. Restriction of Siegel Eisenstein series of degree-2 and applications

In this section we give a decomposition formula for the restriction to the diagonal of Siegel Eisenstein series of degree-2. In essence this result is already known; see [1] and [9] for example. Let $w_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and for $g \in GL_2(R)$ let $g^\natural = w_2 g w_2$ and let $f^\natural(g) = \overline{f(g^\natural)}$. It is formal that $f^\natural$ is an eigenfunction of the Hecke operators if f is. Throughout, for a group G we write G^2 for $G \times G$, G^3 for $G \times G \times G$, G^Δ for the diagonal $\{(g,g) \mid g \in G\} \subset G^2$, and $G^{\Delta'}$ for $\{(g, g^\natural) \mid g \in G\} \subset G^2$. Recall that by a cuspform on $SL_2(\mathbb{A}_\mathbb{Q})$ we mean in the sense of Section 2. That is, $f(g)$ is the cuspform $\widetilde{\Psi}_f(g)$ associated to a cuspform $f(z)$ on $\mathfrak{H}_1$.

Lemma 1 ([2, 9])**.** *The double coset space $P_{2,0}(\mathbb{Q})\backslash Sp_2(\mathbb{Q})/\iota(SL_2(\mathbb{Q})^2)$ has representatives 1_4 and $\xi = \left(\begin{smallmatrix} 1_2 & 0_2 \\ w_2 & 1_2 \end{smallmatrix}\right)$. The respective isotropy groups in $\iota(SL_2(\mathbb{Q})^2)$ are $\iota(B_1(\mathbb{Q})^2)$ and $\iota(SL_2(\mathbb{Q})^{\Delta'})$.*

The following integral representation has a generalization to $n \in \mathbb{Z}_{>0}$, but the $n = 1$ case is the one relevent for our purposes.

Theorem 2 ([2, 8])**.** *For $f \in \mathcal{B}_\kappa$ we have*

$$\langle E_{2,\kappa}(z,*), f\rangle = \frac{(-1)^{\kappa/2}\pi 2^{3-\kappa}}{(\kappa-1)\zeta(\kappa)\zeta(2\kappa-2)} L(2\kappa-2, \mathrm{Sym}^2 f) f^\natural(z).$$

From Lemma 1 we have $P_{2,0}(\mathbb{Q})\backslash Sp_2(\mathbb{Q}) = \iota(SL_2(\mathbb{Q})^2) \sqcup \xi\iota(SL_2(\mathbb{Q})^2)$ and therefore we get the decomposition of the adelic Eisenstein series

$$\begin{aligned} E_{2,\kappa}(\iota(g_1,g_2)) = & \sum_{\gamma \in B_1(\mathbb{Q})^2\backslash SL_2(\mathbb{Q})^2} \epsilon_{2,\kappa}(\gamma\iota(g_1,g_2)) \\ & + \sum_{\gamma \in SL_2(\mathbb{Q})^{\Delta'}\backslash SL_2(\mathbb{Q})^2} \epsilon_{2,\kappa}(\xi\gamma\iota(g_1,g_2)). \end{aligned} \tag{3.1}$$

As

$$B_1(\mathbb{Q})^2\backslash SL_2(\mathbb{Q})^2 \cong (B_1(\mathbb{Q})\backslash SL_2(\mathbb{Q})) \times (B_1(\mathbb{Q})\backslash SL_2(\mathbb{Q}))$$

and setting $\gamma = (\gamma_1, \gamma_2)$ for $\gamma_j \in B_1(\mathbb{Q})\backslash SL_2(\mathbb{Q})$ we have $\gamma\iota(g_1,g_2) = (\gamma_1,\gamma_2)\iota(g_1,g_2) = \iota(\gamma_1 g_1, \gamma_2 g_2)$. Furthermore, we have $\epsilon_{2,\kappa}(\iota(\gamma_1 g_1, \gamma_2 g_2)) = \epsilon_{1,\kappa}(\gamma_1 g_1)\epsilon_{1,\kappa}(\gamma_2 g_2)$. Therefore,

$$\begin{aligned} \sum_{\gamma \in B_1(\mathbb{Q})^2\backslash SL_2(\mathbb{Q})^2} \epsilon_{2,\kappa}(\gamma\iota(g_1,g_2)) &= \sum_{(\gamma_1,\gamma_2) \in B_1(\mathbb{Q})^2\backslash SL_2(\mathbb{Q})^2} \epsilon_{2,\kappa}(\iota(\gamma_1 g_1, \gamma_2 g_2)) \\ &= \sum_{\gamma_1,\gamma_2 \in B_1(\mathbb{Q})\backslash SL_2(\mathbb{Q})} \epsilon_{1,\kappa}(\gamma_1 g_1)\epsilon_{1,\kappa}(\gamma_2 g_2) \\ &= \left(\sum_{\gamma_1 \in B_1(\mathbb{Q})\backslash SL_2(\mathbb{Q})} \epsilon_{1,\kappa}(\gamma_1 g_1)\right)\left(\sum_{\gamma_2 \in B_1(\mathbb{Q})\backslash SL_2(\mathbb{Q})} \epsilon_{1,\kappa}(\gamma_2 g_2)\right) \\ &= E_{1,\kappa}(g_1)E_{1,\kappa}(g_2). \end{aligned} \tag{3.2}$$

From [9] we have the following.

Lemma 2. *$\sum_{\gamma \in SL_2(\mathbb{Q})^{\Delta'} \backslash SL_2(\mathbb{Q})^2} \epsilon_{2,\kappa}(\xi\gamma\iota(g_1,g_2))$ is a cuspform on $SL_2(\mathbb{A}_\mathbb{Q})$ of weight κ in each of the variables g_1 and g_2.*

Thus we can write

$$\sum_{\gamma \in SL_2(\mathbb{Q})^{\Delta'} \backslash SL_2(\mathbb{Q})^2} \epsilon_{2,\kappa}(\xi\gamma\iota(g_1,g_2)) = \sum_{f_1 \in \mathcal{B}_\kappa} c(f_1)(g_2) f_1(g_1)$$

where $c(f_1)(g_2)$ is a cuspform of weight κ in $g_2 \in SL_2(\mathbb{A}_\mathbb{Q})$. Thus $c(f_1)(g_2) = \sum_{f_2 \in \mathcal{B}_\kappa} c(f_1,f_2) f_2(g_2)$. This gives

$$\sum_{\gamma \in SL_2(\mathbb{Q})^{\Delta'} \backslash SL_2(\mathbb{Q})^2} \epsilon_{2,\kappa}(\xi\gamma\iota(g_1,g_2)) = \sum_{f_1,f_2 \in \mathcal{B}_\kappa} c(f_1,f_2) f_1(g_1) f_2(g_2)$$

for constants $c(f_1,f_2)$. By Theorem 2 and Lemma 2 we have for any $f_1 \in \mathcal{B}_\kappa$,

$$\begin{aligned}
&\frac{(-1)^{\kappa/2}\pi 2^{3-\kappa}}{(\kappa-1)\zeta(\kappa)\zeta(2\kappa-2)} L(2\kappa-2, \mathrm{Sym}^2 f_1) f_1^\natural(g_1) = \langle E_{2,\kappa}(g_1,\cdot), f_1\rangle \\
&= \left\langle \sum_{\gamma \in SL_2(\mathbb{Q})^{\Delta'} \backslash SL_2(\mathbb{Q})^2} \epsilon_{2,\kappa}(\xi\gamma\iota(g_1,\cdot)), f_1 \right\rangle = \sum_{f_2 \in \mathcal{B}_\kappa} c(f_1,f_2)\langle f_1,f_1\rangle f_2(g_2).
\end{aligned}$$

It follows that

$$c(f_1,f_2) = \frac{(-1)^{\kappa/2}\pi 2^{3-\kappa}}{(\kappa-1)\zeta(\kappa)\zeta(2\kappa-2)} \frac{L(2\kappa-2,\mathrm{Sym}^2 f_1)}{\langle f_1,f_1\rangle}$$

if $f_2 = f_1^\natural$ and 0 otherwise. Thus

$$\begin{aligned}
&\sum_{\gamma \in SL_2(\mathbb{Q})^{\Delta'} \backslash SL_2(\mathbb{Q})^2} \epsilon_{2,\kappa}(\xi\gamma\iota(g_1,g_2)) \\
&\quad = \frac{(-1)^{\kappa/2}\pi 2^{3-\kappa}}{(\kappa-1)\zeta(\kappa)\zeta(2\kappa-2)} \sum_{f \in \mathcal{B}_\kappa} \frac{L(2\kappa-2,\mathrm{Sym}^2 f)}{\langle f,f\rangle} f(g_1) f^\natural(g_2).
\end{aligned} \tag{3.3}$$

Applying (3.2) and (3.3), the decomposition in (3.1) can be precisely written

$$\begin{aligned}
&E_{2,\kappa}(\iota(g_1,g_2)) = E_{1,\kappa}(g_1)E_{1,\kappa}(g_2) \\
&\quad + \frac{(-1)^{\kappa/2}\pi 2^{3-\kappa}}{(\kappa-1)\zeta(\kappa)\zeta(2\kappa-2)} \sum_{f \in \mathcal{B}_1} \frac{L(2\kappa-2,\mathrm{Sym}^2 f)}{\langle f,f\rangle} f(g_1) f^\natural(g_2).
\end{aligned} \tag{3.4}$$

The Fourier expansion of a (classical) Siegel Eisenstein series of degree-2 is

$$E_{2,\kappa}(z) = \sum_{\substack{T \in \Lambda \\ T \geq 0}} A_{2,\kappa}(T) e^{2\pi i \mathrm{Tr}(Tz)}$$

where Tr is the matrix trace and $\Lambda = \{\left(\begin{smallmatrix} a & b/2 \\ b/2 & c\end{smallmatrix}\right) \mid a,b,c \in \mathbb{Z}\}$. Let $T = \left(\begin{smallmatrix} a & b/2 \\ b/2 & c\end{smallmatrix}\right)$. From [7] we have

$$A_{2,\kappa}(T) = \frac{2}{\zeta(3-2\kappa)\zeta(1-\kappa)} \sum_{d|(a,b,c)} d^{\kappa-1} H\left(\kappa-1, \frac{4ac-b^2}{d^2}\right) \tag{3.5}$$

where $H(r,n)$ is the generalized class number defined in Section 2. By applying Siegel's Φ-operator (see the proof of Lemma 6 in Section 5 for a discussion of this operator) we have

$$A_{2,\kappa}\begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix} = A_{1,\kappa}(n) = \frac{2\sigma_{\kappa-1}(n)}{\zeta(1-\kappa)}.$$

Note that formula (3.5) reduces to this for $T = \left(\begin{smallmatrix} n & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

Set $q_j = e^{2\pi i z_j}$. Restricting the Fourier expansion above we get

$$\begin{aligned} E_{2,\kappa}(\iota(z_1,z_2)) &= \sum_{\substack{T\in\Lambda \\ T\geq 0}} A_{2,\kappa}(T) e^{2\pi i \operatorname{Tr}(T\iota(z_1,z_2))} = \sum_{\substack{T\in\Lambda \\ T\geq 0}} A_{2,\kappa}(T) q_1^{n_1} q_2^{n_2} \\ &= \sum_{n_1,n_2\geq 0} \left(\sum_{T\in\Lambda(n_1,n_2)} A_{2,\kappa}(T) \right) q_1^{n_1} q_2^{n_2} \end{aligned} \tag{3.6}$$

where $\Lambda(n_1,n_2) = \left\{ \left(\begin{smallmatrix} n_1 & b/2 \\ b/2 & n_2 \end{smallmatrix}\right) \in \Lambda \;\middle|\; 4n_1n_2 - b^2 \geq 0 \right\}$. Taking the Fourier expansions of the terms in the right-hand side of (3.4) (in classical language)

$$\begin{aligned} E_{2,\kappa}(\iota(z_1,z_2)) &= \left(1 + \frac{2}{\zeta(1-\kappa)} \sum_{n_1=0}^{\infty} \sigma_{\kappa-1}(n_1) q_1^{n_1}\right) \left(1 + \frac{2}{\zeta(1-\kappa)} \sum_{n_2=0}^{\infty} \sigma_{\kappa-1}(n_2) q_2^{n_2}\right) \\ &+ \frac{\pi 2^{\kappa-3}}{(\kappa-1)} \frac{(-1)^{\kappa/2}}{\zeta(\kappa)\zeta(2\kappa-2)} \sum_{f\in\mathcal{B}_\kappa} \frac{L(2\kappa-2,\operatorname{Sym}^2 f)}{\langle f,f\rangle} \left(\sum_{n_1=1}^{\infty} a_f(n_1) q_1^{n_1}\right) \left(\sum_{n_2=1}^{\infty} a_f(n_2) q_2^{n_2}\right). \end{aligned} \tag{3.7}$$

Setting $z_2 = 0$ in (3.7) and applying (3.6) we get

$$\sum_{n_1=0}^{\infty} \left(\sum_{T\in\Lambda(n_1,0)} A_{2,\kappa}(T) \right) q_1^{n_1} = E_{1,\kappa}(z_1).$$

Equating (3.6) and (3.7) and setting $n_2 = 1$ gives us

$$\begin{aligned} &\sum_{n_1=0}^{\infty} \left(\sum_{T\in\Lambda(n_1,1)} A_{2,\kappa}(T) \right) q_1^{n_1} \\ &= E_{1,\kappa}(z_1) \frac{2}{\zeta(1-\kappa)} + \frac{\pi 2^{3-\kappa}}{(\kappa-1)} \frac{(-1)^{\kappa/2}}{\zeta(\kappa)\zeta(2\kappa-2)} \sum_{f\in\mathcal{B}_\kappa} \frac{L(2\kappa-2,\operatorname{Sym}^2 f)}{\langle f,f\rangle} f(z_1). \end{aligned}$$

From (3.5) we can write

$$\sum_{T\in\Lambda(n_1,1)} A_{2,\kappa}(T) = \frac{2}{\zeta(3-2\kappa)\zeta(1-\kappa)} \sum_{|b|=0}^{[\sqrt{4n_1}]} H(\kappa-1, 4n_1-b^2).$$

For $n_1 \geq 1$ this gives

$$\begin{aligned} \frac{2}{\zeta(3-2\kappa)\zeta(1-\kappa)} \sum_{|b|=0}^{[\sqrt{4n_1}]} H(\kappa-1,4n_1-b^2) &= \left(\frac{2}{\zeta(1-\kappa)}\right)^2 \sigma_{\kappa-1}(n_1) \\ &+ \frac{(-1)^{\kappa/2}\pi 2^{3-\kappa}}{(\kappa-1)\zeta(\kappa)\zeta(2\kappa-2)} \sum_{f\in\mathcal{B}_\kappa} \frac{L(2\kappa-2,\operatorname{Sym}^2 f)}{\langle f,f\rangle} a_f(n_1). \end{aligned} \tag{3.8}$$

Now take $n_1 = 1$ and apply the values $\zeta(n) = (-1)^{n/2+1} 2^{n-1} B_n \pi^n / n!$ and $\zeta(1-n) = -B_n/n$ for n even to (3.8). For $\mathcal{B}_\kappa$ and B_n as in Theorem 1 we then have

$$\sum_{f \in \mathcal{B}_\kappa} \frac{L(2\kappa-2, \mathrm{Sym}^2 f)}{\langle f, f\rangle \pi^{3\kappa-3}} = \frac{2^{4\kappa-5}}{(\kappa-2)!(2\kappa-2)!} \times \left(\frac{\kappa B_{2\kappa-2}}{B_\kappa} + B_{2\kappa-2} - 2(\kappa-1)H(\kappa-1,3) - (\kappa-1)H(\kappa-1,4) \right). \tag{3.9}$$

This is essentially in [15] but is used in the proof of Theorem 1, so we include it here for completeness. Taking $\kappa \in \{12, 16, 18, 20, 22, 26\}$ gives Table 2.

For $\kappa = 4, 6, 8, 10, 14$, (3.8) gives

$$\sum_{|b|=0}^{[\sqrt{4n}]} H(\kappa-1, 4n-b^2) = \frac{2\zeta(3-2\kappa)}{\zeta(1-\kappa)} \sigma_{\kappa-1}(n) = \frac{\kappa B_{2\kappa-2}}{(\kappa-1)B_\kappa} \sigma_{\kappa-1}(n).$$

Note that the $\kappa = 4, 6$ cases of this are on p. 277 of [3] and the $\kappa = 8, 10$ cases are in [7].

Recall from Section 1 that $\Delta_{24,j}(z) = \sum_{n=1}^{\infty} \tau_{24,j}(n) q^n$ for $j = 1, 2$ are the two normalized Hecke eigencuspforms of weight 24 where $\tau_{24,j}(1) = 1$ and $\tau_{24,1}(2) = 540 + 12\sqrt{144169}$ and $\tau_{24,2}(2) = 540 - 12\sqrt{144169}$. From (3.8) we get the values

$$\frac{L(46, \mathrm{Sym}^2 \Delta_{24,1})}{\langle \Delta_{24,1}, \Delta_{24,1}\rangle \pi^{69}} = \frac{2^{27}\big(-2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11^3 \cdot 17 \cdot 19 \cdot 59 \cdot 691 \cdot 144169 \cdot 2294824233197 + 47 \cdot 19661 \cdot 294062653 \cdot 432927907 \cdot 5332396711\sqrt{144169}\big)}{3^{28} \cdot 5^{14} \cdot 7^9 \cdot 11^6 \cdot 13^4 \cdot 17^3 \cdot 19^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 103 \cdot 144169 \cdot 2294797},$$

where replacing $\Delta_{24,2}(z)$ for $\Delta_{24,1}(z)$ on the left-hand side gives the (Galois) conjugate of the right-hand side.

As a further application of (3.8) we obtain the formula

$$23 \cdot 691 \sum_{|b|=0}^{[\sqrt{4n}]} H(11, 4n-b^2) = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 131 \cdot 593 \sigma_{11}(n) + 2^8 \cdot 3^4 \cdot 5^2 \cdot 7^2 \tau_{12}(n).$$

This implies the well-known congruence $\tau_{12}(n) \equiv \sigma_{11}(n) \pmod{691}$ due to Ramanujan. See [13] for details where this was obtained independently. Note that in a similar way we can also obtain further congruences for higher weights as in [4].

4. Restriction of Siegel Eisenstein series of degree-3

In this section we prove a precise decomposition formula for Siegel Eisenstein series of degree-3 restricted to the diagonal. Such a decomposition was first described in Section 6 in [10], but there only the last term below was given explicitly.

Proposition 1. *Let $E_{3,\kappa}(z)$ be the Siegel Eisenstein series of degree-3 and let $\mathcal{B}_\kappa$ be as in Theorem 1. Then*

$$\begin{aligned}E_{3,\kappa}(\iota(z_1,z_2,z_3)) &= E_{1,\kappa}(z_1)E_{1,\kappa}(z_2)E_{1,\kappa}(z_3)\\ &+ \frac{(-1)^{\kappa/2}\pi 2^{3-\kappa}}{(\kappa-1)\zeta(\kappa)\zeta(2\kappa-2)}\Big(E_{1,\kappa}(z_1)\sum_{f\in\mathcal{B}_\kappa}\frac{L(2\kappa-2,\mathrm{Sym}^2 f)}{\langle f,f\rangle}f(z_2)f^\natural(z_3)\\ &+ E_{1,\kappa}(z_2)\sum_{f\in\mathcal{B}_\kappa}\frac{L(2\kappa-2,\mathrm{Sym}^2 f)}{\langle f,f\rangle}f(z_3)f^\natural(z_1) + E_{1,\kappa}(z_3)\sum_{f\in\mathcal{B}_\kappa}\frac{L(2\kappa-2,\mathrm{Sym}^2 f)}{\langle f,f\rangle}f(z_1)f^\natural(z_2)\Big)\\ &+ \frac{(-1)^{\kappa/2}2^{8-5\kappa}\pi^{3-2\kappa}(\kappa-2)!^3}{(\kappa-1)!\zeta(\kappa)\zeta(2\kappa-2)}\sum_{f_1,f_2,f_3\in\mathcal{B}_\kappa}\frac{L(2\kappa-2,f_1\otimes f_2\otimes f_3)}{\langle f_1,f_1\rangle\langle f_2,f_2\rangle\langle f_3,f_3\rangle}f_1(z_1)f_2(z_2)f_3(z_3).\end{aligned}$$

Let ι denote the diagonal embedding (either of degree 2 or 3 depending on the context). For $g_1, g_2 \in SL_2(\mathbb{Q})$ we define the following embeddings of $SL_2(\mathbb{Q})^2$ into $SL_2(\mathbb{Q})^3$, $\iota_{12}(g_1,g_2) = \iota(g_1, g_1^\natural, g_2)$, $\iota_{13}(g_1,g_2) = \iota(g_1,g_2,g_1^\natural)$, and $\iota_{23}(g_1,g_2) = \iota(g_2,g_1,g_1^\natural)$. Let $w_2' = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$.

Lemma 3. *The double coset space*

$$P_{3,0}(\mathbb{Q})\backslash Sp_3(\mathbb{Q})/\iota(SL_2(\mathbb{Q})^3) \tag{4.1}$$

has 5 orbits, with representatives $1_6, \iota(1_2,\xi), \widetilde{\xi}, \widetilde{\xi}\iota(1_2,\gamma_1), \widetilde{\xi}\iota(w_2',\gamma_2)$ *where*

$$\gamma_1 = \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\ \gamma_2 = \begin{pmatrix}1&0&0&0\\1&1&0&0\\0&0&1&-1\\0&0&0&1\end{pmatrix},\ \textit{and}\ \widetilde{\xi} = \begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&1&1&0&0\\0&0&0&0&1&0\\1&0&0&0&0&1\end{pmatrix}.$$

The respective isotropy groups inside $\iota(SL_2(\mathbb{Q})^3)$ are $\iota(B_1(\mathbb{Q})^3)$, $\iota_{12}(SL_2(\mathbb{Q}), B_1(\mathbb{Q}))$, $\iota_{13}(SL_2(\mathbb{Q}), B_1(\mathbb{Q}))$, $\iota_{23}(SL_2(\mathbb{Q}), B_1(\mathbb{Q}))$, and

$$\Theta(\mathbb{Q}) = \left\{\begin{pmatrix}a & b_1\\ 0 & a^{-1}\end{pmatrix}\times\begin{pmatrix}a & b_2\\ 0 & a^{-1}\end{pmatrix}\times\begin{pmatrix}a & b_3\\ 0 & a^{-1}\end{pmatrix}\ \Big|\ a\in\mathbb{Q}^\times,\ b_1,b_2,b_3\in\mathbb{Q},\quad b_1+b_2+b_3=0\right\}.$$

Proof. From Proposition 3.1 of [9] we have that

$$P_{3,0}(\mathbb{Q})\backslash Sp_3(\mathbb{Q})/\iota(SL_2(\mathbb{Q})\times Sp_2(\mathbb{Q}))$$

has 2 orbits, with representatives 1_6 and $\widetilde{\xi}$ and respective isotropy groups $\iota(B_1(\mathbb{Q})\times P_{2,0}(\mathbb{Q}))$ and

$$H_1(\mathbb{Q}) = \left\{\iota\left(\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}^\natural, \begin{pmatrix}a&x&y&y'\\0&\alpha&y'&\beta\\0&0&a^{-1}&0\\0&\gamma&x'&\delta\end{pmatrix}\right)\ \Bigg|\ \begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\in SL_2(\mathbb{Q}),\ a\in\mathbb{Q}^\times,\ x,x',y,y'\in\mathbb{Q}\right\}.$$

Consider

$$\begin{aligned}
&\iota(B_1(\mathbb{Q}) \times P_{2,0}(\mathbb{Q}))\backslash\iota(SL_2(\mathbb{Q}) \times Sp_2(\mathbb{Q}))/\iota(SL_2(\mathbb{Q})^3) \\
&\qquad\qquad \cong B_1(\mathbb{Q}) \times P_{2,0}(\mathbb{Q})\backslash SL_2(\mathbb{Q}) \times Sp_2(\mathbb{Q})/SL_2(\mathbb{Q}) \times \iota(SL_2(\mathbb{Q})^2) \\
&\qquad\qquad \cong P_{2,0}(\mathbb{Q})\backslash Sp_2(\mathbb{Q})/\iota(SL_2(\mathbb{Q})^2). \qquad (4.2)
\end{aligned}$$

From Lemma 1, the last double coset of (4.2) has 2 orbits, with representatives 1_4 and $\xi = \left(\begin{smallmatrix} 1_2 & 0_2 \\ w_2 & 1_2 \end{smallmatrix}\right)$ and respective isotropy groups $\iota(B_1(\mathbb{Q})^2)$ and $\iota(SL_2(\mathbb{Q})^{\Delta'})$. Thus the double coset on the left-hand side of (4.2) has 2 orbits with representatives $\iota(1_2, 1_4) = 1_6$ and $\iota(1_2, \xi)$ and respective isotropy groups $\iota(B_1(\mathbb{Q})^3)$ and $\iota_{23}(SL_2(\mathbb{Q}) \times B_1(\mathbb{Q}))$. It follows that the double coset (4.1) has these representatives (so far) with the respective isotropy groups $\iota(B_1(\mathbb{Q})^3)$ and $\iota_{23}(B_1(\mathbb{Q}) \times SL_2(\mathbb{Q}))$.

Consider

$$H_1(\mathbb{Q})\backslash\iota(SL_2(\mathbb{Q}) \times Sp_2(\mathbb{Q}))/\iota(SL_2(\mathbb{Q})^3) \cong P_{2,1}(\mathbb{Q})\backslash Sp_2(\mathbb{Q})/\iota(SL_2(\mathbb{Q})^2). \qquad (4.3)$$

Note that

$$P_{2,1}(\mathbb{Q})\backslash Sp_2(\mathbb{Q}) \cong \left\{x' \mid x' \in \mathbb{Q}\right\}\backslash\left\{(x, y, z, w) \in \mathbb{Q}^4 \mid x, y, z, w \ \text{ not all } \ 0\right\}.$$

By the right action of $\iota(SL_2(\mathbb{Q})^2)$ on this space, we can assume that $(z, w) = (0, 0)$. Thus this action transforms the above space into

$$\{x \mid x \in \mathbb{Q}\}\backslash \left\{(ax, a'y, bx, b'y) \mid (x, y), (a, b), (a', b') \neq (0, 0)\right\}.$$

If $x \neq 0$ and $y = 0$, then the right action gives the representative $(1, 0, 0, 0)$. If $x = 0$ and $y \neq 0$, we get $(0, 1, 0, 0)$, and if $x, y \neq 0$, then we get $(1, 1, 0, 0)$. These correspond to the following representatives of (4.3),

$$1_4, \ \gamma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ \gamma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

respectively. See also Proposition 2.4 of [10]. These representatives have the respective isotropy groups $\iota(B_1(\mathbb{Q}), SL_2(\mathbb{Q}))$, $\iota(SL_2(\mathbb{Q}), B_1(\mathbb{Q}))$, and

$$H_2(\mathbb{Q}) = \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b' \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & a^{-1} \end{pmatrix} \;\middle|\; a \in \mathbb{Q}^\times, \quad b, b' \in \mathbb{Q} \right\}.$$

Thus the isotropy groups of (4.3) are written as $\iota_{13}(SL_2(\mathbb{Q}), B_1)$, $\iota_{12}(SL_2(\mathbb{Q}), B_1)$, and $\Theta(\mathbb{Q})$ with the representatives as in the statement of the theorem. $\square$

Lemma 3 implies the decomposition

$$\begin{aligned}
P_{3,0}(\mathbb{Q})\backslash Sp_3(\mathbb{Q}) =&\iota(SL_2(\mathbb{Q})^3) \sqcup \iota(1_2, \xi)\iota(SL_2(\mathbb{Q})^3) \sqcup \widetilde{\xi}\iota(SL_2(\mathbb{Q})^3) \\
&\sqcup \widetilde{\xi}\iota(1_2, \gamma_1)\iota(SL_2(\mathbb{Q})^3) \sqcup \widetilde{\xi}\iota(w_2', \gamma_2)\iota(SL_2(\mathbb{Q})^3).
\end{aligned}$$

As in Section 3 this implies the following decomposition of the pullback of $E_{3,\kappa}(g)$,

$$
\begin{aligned}
E_{3,\kappa}(\iota(g_1,g_2,g_3)) = & \sum_{\gamma\in B_1(\mathbb{Q})^3\backslash SL_2(\mathbb{Q})^3} \epsilon_{3,\kappa}(\gamma\iota(g_1,g_2,g_3)) \\
& + \sum_{\gamma\in\iota_{23}(SL_2(\mathbb{Q}),B_1(\mathbb{Q}))\backslash SL_2(\mathbb{Q})^3} \epsilon_{3,\kappa}(\iota(1_2,\xi)\gamma\iota(g_1,g_2,g_3)) \\
& + \sum_{\gamma\in\iota_{13}(SL_2(\mathbb{Q}),B_1(\mathbb{Q}))\backslash SL_2(\mathbb{Q})^3} \epsilon_{3,\kappa}(\widetilde{\xi}\gamma\iota(g_1,g_2,g_3)) \\
& + \sum_{\gamma\in\iota_{12}(SL_2(\mathbb{Q}),B_1(\mathbb{Q}))\backslash SL_2(\mathbb{Q})^3} \epsilon_{3,\kappa}(\widetilde{\xi}\iota(1_2,\gamma_1)\gamma\iota(g_1,g_2,g_3)) \\
& + \sum_{\gamma\in\Theta(\mathbb{Q})\backslash SL_2(\mathbb{Q})^3} \epsilon_{3,\kappa}(\widetilde{\xi}\iota(w_2',\gamma_2)\gamma\iota(z_1,z_2,z_3)).
\end{aligned} \tag{4.4}
$$

We consider each term in (4.4). Following Section 2 we have that

$$
\sum_{\gamma\in B_1(\mathbb{Q})^3\backslash SL_2(\mathbb{Q})^3} \epsilon_{3,\kappa}(\gamma\iota(g_1,g_2,g_3)) = E_{1,\kappa}(g_1)E_{1,\kappa}(g_2)E_{1,\kappa}(g_3). \tag{4.5}
$$

Note that

$$
\iota_{23}(SL_2(\mathbb{Q}),B_1(\mathbb{Q}))\backslash\iota(SL_2(\mathbb{Q})^3) \cong (B_1(\mathbb{Q})\backslash SL_2(\mathbb{Q})) \times \left(SL_2(\mathbb{Q})^{\Delta'}\backslash SL_2(\mathbb{Q})^2\right).
$$

Thus for the second term in (4.4) we have

$$
\begin{aligned}
& \sum_{\gamma\in\iota_{23}(SL_2(\mathbb{Q}),B_1(\mathbb{Q}))\backslash SL_2(\mathbb{Q})^3} \epsilon_{3,\kappa}(\iota(1_2,\xi)\gamma\iota(g_1,g_2,g_3)) \\
& \quad = \sum_{(\gamma_1,\gamma_2)\in(B_1(\mathbb{Q})\backslash SL_2(\mathbb{Q}))\times\left(SL_2(\mathbb{Q})^{\Delta'}\backslash SL_2(\mathbb{Q})^2\right)} \epsilon_{3,\kappa}(\iota(\gamma_1 g_1,\xi\gamma_2\iota(g_2,g_3)) \\
& \quad = \sum_{(\gamma_1,\gamma_2)\in(B_1(\mathbb{Q})\backslash SL_2(\mathbb{Q}))\times\left(SL_2(\mathbb{Q})^{\Delta'}\backslash SL_2(\mathbb{Q})^2\right)} \epsilon_{1,\kappa}(\gamma_1 g_1)\epsilon_{2,\kappa}(\xi\gamma_2\iota(g_2,g_3)) \\
& \quad = \sum_{\gamma\in B_1(\mathbb{Q})\backslash SL_2(\mathbb{Q})} \epsilon_{1,\kappa}(\gamma g_1) \times \sum_{\gamma\in SL_2(\mathbb{Q})^{\Delta'}\backslash SL_2(\mathbb{Q})^2} \epsilon_{2,\kappa}(\xi\gamma\iota(g_2,g_3)).
\end{aligned}
$$

From (3.3) we have

$$
\begin{aligned}
& \sum_{\gamma\in SL_2(\mathbb{Q})^{\Delta'}\backslash SL_2(\mathbb{Q})^2} \epsilon_{2,\kappa}(\xi\gamma\iota(g_2,g_3)) \\
& \qquad = \frac{\pi 2^{\kappa-3}}{(\kappa-1)\zeta(\kappa)\zeta(2\kappa-2)} \sum_{f\in\mathcal{B}_\kappa} \frac{L(2\kappa-2,\mathrm{Sym}^2 f)}{\langle f,f\rangle} f(g_2)f^\natural(g_3)
\end{aligned}
$$

and $\sum_{\gamma\in B_1(\mathbb{Q})\backslash SL_2(\mathbb{Q})} \epsilon_{1,\kappa}(\gamma g_1) = E_{1,\kappa}(g_1)$. Thus this term is

$$
E_{1,\kappa}(g_1)\frac{\pi 2^{\kappa-3}}{(\kappa-1)\zeta(\kappa)\zeta(2\kappa-2)} \sum_{f\in\mathcal{B}_\kappa} \frac{L(2\kappa-2,\mathrm{Sym}^2 f)}{\langle f,f\rangle} f(g_2)f^\natural(g_3). \tag{4.6}
$$

The representative $\widetilde{\xi}\iota(1_2,\gamma_1)$ can be given as $\iota(\xi,1_2)$ and for $g_1,g_2,g_3\in SL_2(\mathbb{A}_\mathbb{Q})$ we can write $\widetilde{\xi}\iota(g_1,g_2,g_3)$ as $\iota(\xi(g_1,g_3),g_2)$. Thus we can apply the same analysis as above to the third and the fourth terms in (4.4). This gives

$$E_{1,\kappa}(g_2)\frac{\pi 2^{\kappa-3}}{(\kappa-1)\zeta(\kappa)\zeta(2\kappa-2)}\sum_{f\in\mathcal{B}_\kappa}\frac{L(2\kappa-2,\mathrm{Sym}^2 f)}{\langle f,f\rangle}f(g_3)f^\natural(g_1),$$

$$E_{1,\kappa}(g_3)\frac{\pi 2^{\kappa-3}}{(\kappa-1)\zeta(\kappa)\zeta(2\kappa-2)}\sum_{f\in\mathcal{B}_\kappa}\frac{L(2\kappa-2,\mathrm{Sym}^2 f)}{\langle f,f\rangle}f(g_1)f^\natural(g_2), \tag{4.7}$$

respectively.

We now consider the last term. Note that Mizumoto [20] states that the identity in Theorem 1.3 in [10] should be multiplied by 4. This is due to a non-archimedean computation in Section 3 of [10]. In particular, in the notation of [10], for $\tilde{\gamma}\in P_{2,1}(\mathbb{Z})\backslash Sp_2(\mathbb{Z})$ we can write $\tilde{\gamma}=\xi\iota_{1,1}(\gamma,A_{\epsilon,\nu}\gamma')$ for $\gamma,\gamma'\in P_{1,0}(\mathbb{Z})\backslash Sp_1(\mathbb{Z})$ and $A_{\epsilon,\nu}=\left(\begin{smallmatrix}\nu/\epsilon & 0\\ 0 & \epsilon/\nu\end{smallmatrix}\right)$ for $\epsilon,\nu\in\mathbb{Z}$ (and not just in $\mathbb{Z}_{>0}$). Thus the expression (3.6.2) in [10] should read

$$\begin{aligned}&\zeta(2s+2k)^{-1}D_\psi^{(2)}(s+2k-1)\\ &\qquad\iint_{(\mathfrak{H}_1/SL_2(\mathbb{Z}))^2}\sum_{\substack{\epsilon,\nu\in\mathbb{Z}-\{0\}\\ \epsilon,\nu\text{ relatively prime}}}\ \sum_{\gamma,\gamma'\in P_{1,0}(\mathbb{Z})\backslash Sp_1(\mathbb{Z})}\mu(\gamma,z_1)^{-2\kappa}\\ &\qquad\mu(\gamma',z_2)^{-2\kappa}\varphi_s^*(\epsilon^2\gamma z_1+\nu^2\gamma' z_2)\overline{f(z_1)\varphi(z_2)}(y_1y_2)^{2\kappa-2}\,dx_1\,dx_2\,dy_1\,dy_2\end{aligned}$$

where f,φ are elliptic cuspforms of weight 2κ and $D_\psi^{(2)}$ is a certain Dirichlet series attached to an elliptic cuspform ψ of weight 2κ. Here we consider cuspforms of weight κ, so this implies the following integral representation.

Theorem 3 ([10]). *Let $f_1,f_2,f_3\in\mathcal{B}_\kappa$, then*

$$\begin{aligned}&\big\langle\big\langle\langle E_{3,\kappa}(\iota(\cdot,\cdot,\cdot),f_1\rangle,f_2\big\rangle,f_3\big\rangle\\ &\qquad=\frac{(-1)^{\kappa/2}2^{8-4s-10\kappa}\pi^{3-s-2\kappa}\Gamma(s+\kappa-1)^3\Gamma(s+2\kappa-2)}{\Gamma(2s+2\kappa-2)\Gamma(s+\kappa)}\\ &\qquad\quad\times\frac{L(s+2\kappa-2,f_1\otimes f_2\otimes f_3)}{\zeta(2s+\kappa)\zeta(4s+2\kappa-2)}.\end{aligned}$$

Furthermore, as a consequence of this and [10] we have the result.

Lemma 4. $\sum_{\gamma\in\Theta(\mathbb{Q})\backslash SL_2(\mathbb{Q})^3}\epsilon_{3,\kappa}(\widetilde{\xi}\iota(w_2',\gamma_2)\gamma\iota(g_1,g_2,g_3))$ *is a cuspform on $SL_2(\mathbb{A}_\mathbb{Q})$ in each of the variables g_1, g_2 and g_3.*

Following Section 3, from the integral representation of Theorem 3 and Lemma 4 we have that the last term in (4.4) is

$$\begin{aligned}\sum_{\gamma\in\Theta(\mathbb{Q})\backslash SL_2(\mathbb{Q})^3}\epsilon_{3,\kappa}(\widetilde{\xi}\iota(w_2',\gamma_2)\gamma\iota(g_1,g_2,g_3))&=\frac{(-1)^{\kappa/2}2^{8-5\kappa}\pi^{3-2\kappa}(\kappa-2)!^3}{(\kappa-1)!\zeta(\kappa)\zeta(2\kappa-2)}\\ \times\sum_{f_1,f_2,f_3\in\mathcal{B}_\kappa}&\frac{L(2\kappa-2,f_1\otimes f_2\otimes f_3)}{\langle f_1,f_1\rangle\langle f_2,f_2\rangle\langle f_3,f_3\rangle}f_1(g_1)f_2(g_2)f_3(g_3).\end{aligned}\tag{4.8}$$

Substituting (4.5), (4.6), (4.7), and (4.8) into (4.4) and switching to classical language gives Proposition 1.

5. Proof of Theorem 1

We consider the Fourier expansion of both sides of Proposition 1 in the variables $q_j = e^{2\pi i n_j z}$ for $j = 1, 2, 3$. Equating the first Fourier coefficients and determining their exact values will give Theorem 1. For simplicity we denote the matrix $\begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix}$ by (a, b, c).

Lemma 5. *For positive definite $T \in M_3\left(\frac{1}{2}\mathbb{Z}\right)$ let $A_{3,\kappa}(T)$ be the T^{th} Fourier coefficent of the Siegel Eisenstein series $E_{3,\kappa}$. Then*

$$\begin{aligned} A_{3,\kappa}(0,0,0) &= \frac{(-1)^{\kappa/2} 2^3 \kappa(\kappa-1)}{|B_\kappa B_{2\kappa-2}|}(2^{2\kappa-4}-1), \\ A_{3,\kappa}(0,0,1/2) &= \frac{(-1)^{\kappa/2} 2^3 \kappa(\kappa-1)}{|B_\kappa B_{2\kappa-2}|}(3^{\kappa-2}-1), \\ A_{3,\kappa}(0,1/2,1/2) &= \frac{(-1)^{\kappa/2} 2^3 \kappa(\kappa-1)}{|B_\kappa B_{2\kappa-2}|}(2^{\kappa-2}-1). \end{aligned}$$

Proof. These formulas follow from Katsurada's explicit formula for the Fourier coefficients of a Siegel Eisenstein series of degree-3 from [14]. More precisely, for T a positive definite half-integral matrix over $\mathbb{Z}$ of degree-3, Theorem 1.1 from [14] gives the formula

$$A_{3,\kappa}(T) = \frac{(-1)^{\kappa/2} 2^{5-\kappa} \kappa(\kappa-1)(\det(2T))^{\kappa-2}}{|B_\kappa B_{2\kappa-2}|} \prod_{p \mid 4\det(T)} F_{p,\kappa}(T)$$

where B_i is the i^{th} Bernoulli number and $F_{p,\kappa}(T)$ is a polynomial in p defined on p. 203 of [14]. It is straightforward that for $T = (0,0,0)$ we have $p = 2$ and $F_{2,\kappa}(0,0,0) = 2^{2\kappa-4} - 1$. In a similar way we have $F_{3,\kappa}(0,0,1/2) = 3^{\kappa-2} - 1$ and $F_{2,\kappa}(0,1/2,1/2) = 2^{\kappa-1} - 1$. Also, $F_{p,\kappa}(T) = 1$ for all other primes and T's as above, and this gives the result. Also see the tables from [21]. $\square$

The Fourier coefficents of the right-hand side of Proposition 1 at $n_1 = n_2 = n_3 = 1$ are readily computed from the special value results and (3.9) in Section 3. We use Lemma 5 to compute this Fourier coefficient on the left-hand side of Proposition 1.

Lemma 6. *The Fourier coefficient of the $q_1 q_2 q_3$-term of $E_{3,\kappa}(\iota(z_1, z_2, z_3))$ is*

$$\begin{aligned} &-\frac{2^3\kappa}{B_\kappa} + \frac{2^3\kappa(\kappa-1)}{B_\kappa B_{2\kappa-2}}\left(2^3 H(\kappa-1,3) + 3H(\kappa-1,4)\right) \\ &\qquad + \frac{(-1)^{\kappa/2} 2^3 \kappa(\kappa-1)}{|B_\kappa B_{2\kappa-2}|}\left(2^{2\kappa-4} + 2\cdot 3^{\kappa-1} + 2^{\kappa+2} - 23\right). \end{aligned}$$

Proof. We have

$$\begin{aligned} E_{3,\kappa}(\iota(z_1,z_2,z_3)) &= \sum_{\substack{T\in\Lambda \\ T\geq 0}} A_{3,\kappa}(T) e^{2\pi i \mathrm{Tr}(T\iota(z_1,z_2,z_3))} \\ &= \sum_{n_1,n_2,n_3\geq 0} \left(\sum_{T\in\Lambda(n_1,n_2,n_3)} A_{3,\kappa}(T) \right) q_1^{n_1} q_2^{n_2} q_3^{n_3} \end{aligned}$$

where

$$\Lambda(n_1,n_2,n_3)=\left\{T=\begin{pmatrix}n_1 & a & b\\ a & n_2 & c\\ b & c & n_3\end{pmatrix}\ \middle|\ T\geq 0,\ 2a,2b,2c,n_1,n_2,n_3\in\mathbb{Z}\right\}.$$

Thus the $q_1q_2q_3$-Fourier coefficient of $E_{3,\kappa}(\iota(z_1,z_2,z_3))$ is $\sum_{T\in\Lambda(1,1,1)}A_{3,\kappa}(T)$ where

$$\Lambda(1,1,1)=\left\{T=\begin{pmatrix}1 & a & b\\ a & 1 & c\\ b & c & 1\end{pmatrix}\ \middle|\ T\geq 0,\ 2a,2b,2c\in\mathbb{Z}\right\}. \tag{5.1}$$

Let $\vec{x}=(x,y,z)\in\mathbb{R}^3$, so for $T\in\Lambda(1,1,1)$ we have

$$\vec{x}T\vec{x}^t=x^2+y^2+z^2+2axy+2bxz+2cyz\geq 0.$$

Taking $(x,y,z)=(-1,1,0)$ gives $1\geq a$ and taking $(x,y,z)=(1,1,0)$ gives $a\geq -1$. By symmetry we have $1\geq a,b,c\geq -1$ in (5.1). We distinguish 3 cases.

(i) At least one of a,b,c is ± 1.

If $a=1$, then $\vec{x}T\vec{x}^t=(x+y)^2+z^2+2z(bx+cy)\geq 0$. Taking $x=-y=1$ we get $z^2+2z(b-c)\geq 0$, so for $z>0$ this implies that $z\geq 2(b-c)$. This holds for all real $z>0$ and so we must have $b=c$. If $z<0$, then $z\leq 2(b-c)$ and we get $b=c$. Similarly, if $b=1$, then $a=c$ and if $c=1$, then $a=b$.

If $a=-1$, then $x=y=1$ implies $b=-c$ and similarly for $b=-1$ or $c=-1$. Thus the possible choices for T in this case are $(1,a,a)$, $(a,1,a)$, $(a,a,1)$, $(-1,a,-a)$, $(a,-1,-a)$, $(a,-a,-1)$ where $a\in\{0,\pm 1/2,\pm 1\}$. Note that all of these matrices are degenerate. Thus the possible matrices in $\Lambda(1,1,1)$ satisfying condition (i) are

$$\begin{aligned}&(1,0,0),(1,1,1),(1,1/2,1/2),(1,-1,-1),(1,-1/2,-1/2),(0,1,0),(1/2,1,1/2),\\&(-1/2,1,-1/2),(-1,1,-1),(0,0,1),(1/2,1/2,1),(-1/2,-1/2,1),(-1,-1,1),\\&(-1,0,0),(-1,1/2,-1/2),(-1,-1/2,1/2),(0,-1,0),(-1/2,-1,1/2),\\&(1/2,-1,-1/2),(0,0,-1),(1/2,-1/2,-1),(-1/2,1/2,-1).\end{aligned}$$

Denote the the 2×2 matrix $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ by T_1' and $\left(\begin{smallmatrix}1&c\\c&1\end{smallmatrix}\right)$ by T_c' for $c\in\{0,\pm 1/2\}$. For $T_j'=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ let $T_j=\left(\begin{smallmatrix}a&b&0\\c&d&0\\0&0&0\end{smallmatrix}\right)$. It is elementary that the matrices above are respectively equivalent to

$$\begin{aligned}&T_0,T_1,T_{1/2},T_1,T_{-1/2},T_0,T_1,T_{1/2},T_{-1/2},T_0,T_1,T_{1/2},T_{-1/2},T_0,T_{1/2},T_{-1/2},\\&T_0,T_{-1/2},T_{1/2},T_0,T_{1/2},T_{-1/2}.\end{aligned} \tag{5.2}$$

The Fourier coefficients $A_{3,\kappa}(T_j)$ for all of these T_j's can be determined by Siegel's Φ-operator. Recalling the definition from [18] , let $z=\left(\begin{smallmatrix}z'&0\\0&it\end{smallmatrix}\right)\in\mathfrak{H}_n$ where $z'\in\mathfrak{H}_{n-1}$ and $t\in\mathbb{R}$. Then Siegel's Φ-operator on a modular form $f(z)$ of degree-n and weight κ is

$$(\Phi f)(z')=\lim_{t\to\infty}f\begin{pmatrix}z'&0\\0&it\end{pmatrix}.$$

The operator maps modular forms of degree-n to modular forms of degree-$(n-1)$ and is surjective for $\kappa>2n$. For a modular form of degree-n with Fourier expansion $f(z)=\sum_{T\geq 0}A_{n,\kappa}(T)e^{2\pi i\mathrm{Tr}(Tz)}$ we have

$$(\Phi f)(z')=\sum_{T'\geq 0}A_{n,\kappa}\begin{pmatrix}T'&0\\0&0\end{pmatrix}e^{2\pi i\mathrm{Tr}(T'z')}.$$

From Theorem 2 on p. 72 of [18] we have that Φ takes Siegel Eisenstein series of degree-n to Siegel Eisenstein series of degree-$(n-1)$. It follows that we have $A_{3,\kappa}(T_j) = A_{2,\kappa}(T'_j)$.

Letting $\Lambda_{(i)} = \{T \in \Lambda(1,1,1) \mid T \text{ satisfies } (i)\}$, then from (5.2) this gives

$$\begin{aligned}\sum_{T\in\Lambda_{(i)}} A_{3,\kappa}(T) &= 6A_{3,\kappa}(T_0) + 4A_{3,\kappa}(T_1) + 12A_{3,\kappa}(T_{1/2}) \\ &= 6A_{2,\kappa}(T'_0) + 4A_{2,\kappa}(T'_1) + 12A_{2,\kappa}(T'_{1/2}).\end{aligned} \tag{5.3}$$

(ii) None of a, b, c is ± 1 and at least one is 0.

This condition gives the possible matrices

$$(0,0,0), (0,0,\pm 1/2), (0,\pm 1/2, 0), (\pm 1/2, 0, 0), (0, \pm 1/2, \pm 1/2),$$
$$(\pm 1/2, 0, \pm 1/2), (\pm 1/2, \pm 1/2, 0).$$

Note that if the triple (a', b', c') is a permutation of the triple (a, b, c), then the respective matrices are equivalent. Also note that $(0, 0, -1/2)$ is equivalent to $(0, 0, 1/2)$ and $(0, \pm 1/2, \pm 1/2)$ is equivalent to $(0, 1/2, 1/2)$. It follows that for $\Lambda_{(ii)} = \{T \in \Lambda(1,1,1) \mid T \text{ satisfies } (ii)\}$ we have

$$\sum_{T\in\Lambda_{(ii)}} A_{3,\kappa}(T) = A_{3,\kappa}(0,0,0) + 6A_{3,\kappa}(0,0,1/2) + 12A_{3,\kappa}(0,1/2,1/2). \tag{5.4}$$

(iii) None of a, b, c is 0 or ± 1.

Let $\mathrm{sgn}(x)$ denote the sign of x. It is easy to see that for $T = (a, b, c)$ satisfying (iii) that $\mathrm{sgn}(a)\mathrm{sgn}(b)\mathrm{sgn}(c) = -1$ if and only if $\det(T) = 0$. In these cases we have the possible T's,

$$(1/2, 1/2, -1/2), (1/2, -1/2, 1/2), (-1/2, 1/2, 1/2), (-1/2, -1/2, -1/2).$$

These are all equivalent to $\widetilde{T}_{1/2}$. In the cases where $\mathrm{sgn}(a)\mathrm{sgn}(b)\mathrm{sgn}(c) = 1$ we have

$$(1/2, 1/2, 1/2), (1/2, -1/2, -1/2), (-1/2, 1/2, -1/2), (-1/2, -1/2, 1/2),$$

and these are all equivalent to $(0, 1/2, 1/2)$. Setting

$$\Lambda_{(iii)} = \{T \in \Lambda(1,1,1) \mid T \text{ satisfies } (iii)\}$$

we get

$$\begin{aligned}\sum_{T\in\Lambda_{(iii)}} A_{3,\kappa}(T) &= 4A_{3,\kappa}(T_{1/2}) + 4A_{3,\kappa}(0,1/2,1/2) \\ &= 4A_{2,\kappa}(T'_{1/2}) + 4A_{3,\kappa}(0,1/2,1/2).\end{aligned} \tag{5.5}$$

As $\Lambda(1,1,1) = \Lambda_{(i)} \sqcup \Lambda_{(ii)} \sqcup \Lambda_{(iii)}$, then (5.3), (5.4), and (5.5) give

$$\begin{aligned}\sum_{T\in\Lambda(1,1,1)} A_{3,\kappa}(T) =& 6A_{2,\kappa}(T'_0) + 4A_{2,\kappa}(T'_1) + 16A_{2,\kappa}(T'_{1/2}) \\ &+ A_{3,\kappa}(0,0,0) + 6A_{3,\kappa}(0,0,1/2) + 16A_{3,\kappa}(0,1/2,1/2).\end{aligned} \tag{5.6}$$

TABLE 3. The Fourier coeffiicent of the term $n_1 = n_2 = n_3 = 1$ of the restriction of the weight κ Siegel Eisenstein series of degree-3, $E_{3,\kappa}(\iota(z_1, z_2, z_3))$.

κ	$q_1q_2q_3$ Fourier coefficient of $E_{3,\kappa}(\iota(z_1, z_2, z_2))$
4	13824000
6	-128024064
8	110592000
10	-18399744
12	54446805504000/53678953
14	-13824
16	1819899560558592000/6232699579062017
18	-44969531539968/6651496075469717
20	9322484674060800000/26926798278409918871
22	-3309606048905127411236352/1180857452724874941654444953
24	13608303486944159980965888000/61295489051232438959582100912 37
26	-19008718781109314728173603 84/1302853928405875096112334202 0031

We have from (3.5) that

$$\begin{aligned} A_{2,\kappa}(T'_1) &= \frac{2}{\zeta(1-\kappa)}, \\ A_{2,\kappa}(T'_0) &= \frac{2}{\zeta(3-2\kappa)\zeta(1-\kappa)} H(\kappa-1,4), \\ A_{2,\kappa}(T'_{1/2}) &= \frac{2}{\zeta(3-2\kappa)\zeta(1-\kappa)} H(\kappa-1,3). \end{aligned}$$

Applying this and Lemma 6 to (5.6) gives the result. $\square$

We give some of the values of this coefficient of the pullback of $E_{3,\kappa}(z)$ in Table 3. Note that for $\kappa \in \{4, 6, 8, 10, 14\}$ the values in Table 3 obtained fom Lemma 5 are equal to $(-2\kappa/B_\kappa)^3$.

We take the Fourier expansion of both sides of Proposition 1 and equate the $q_1q_2q_3$-Fourier coefficients. From Lemma 6 this gives us

$$\begin{aligned} -\frac{2^3\kappa}{B_\kappa} &+ \frac{2^3\kappa(\kappa-1)}{B_\kappa B_{2\kappa-2}} \left(2^3 H(\kappa-1,3) + 3H(\kappa-1,4)\right) \\ &+ \frac{(-1)^{\kappa/2} 2^3 \kappa(\kappa-1)}{|B_\kappa B_{2\kappa-2}|} (2^{2\kappa-4} + 2\cdot 3^{\kappa-1} + 2^{\kappa+2} - 23) \\ &= \left(-\frac{2\kappa}{\zeta(1-\kappa)}\right)^3 - \frac{(-1)^{\kappa/2}\pi 2^{3-\kappa}\cdot 3}{(\kappa-1)\zeta(\kappa)\zeta(2\kappa-2)} \frac{2\kappa}{\zeta(1-\kappa)} \sum_{f\in\mathcal{B}_\kappa} \frac{L(2\kappa-2, \mathrm{Sym}^2 f)}{\langle f, f\rangle} \\ &+ \frac{(-1)^{\kappa/2} 2^{8-5\kappa}\pi^{3-2\kappa}(\kappa-1)!^3}{(\kappa-1)!\zeta(\kappa)\zeta(2\kappa-2)} \sum_{f_1,f_2,f_3\in\mathcal{B}_\kappa} \frac{L(2\kappa-2, f_1\otimes f_2\otimes f_3)}{\langle f_1, f_1\rangle\langle f_2, f_2\rangle\langle f_3, f_3\rangle}. \end{aligned}$$

We then apply the relevant critical values of $\zeta(s)$ and the result from (3.9). Solving for the weighted sum of the triple product L-functions proves Theorem 1.

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