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FUNDAMENTALITY OF A CUBIC UNIT u FOR $\mathbb{Z}[u]$

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Abstract. Consider a cubic unit u of positive discriminant. We present a computational proof of the fact that u is a fundamental unit of the order $\mathbb{Z}[u]$ in most cases and determine the exceptions. This extends a similar (but restrictive) result due to E. Thomas.

Introduction

Let $f(X) := X^3 + aX^2 + bX \pm 1 \in \mathbb{Z}[X]$ be irreducible in $\mathbb{Z}[X]$ and with three (distinct) real roots. We think of the order $R := \mathbb{Z}[u]$, obtained by adjoining a root u of $f(X)$, as a subring of the real numbers. It is well known that the group $U^+(R)$ of positive units of R is a free Abelian group of rank 2 and the unit-group of R is $\{-1, 1\} \times U^+(R)$. By a *fundamental unit of* R we mean a unit of R whose absolute value is a member of some free basis of $U^+(R)$. Since u is clearly a unit of R, it is natural to ask when u is a fundamental unit of R. In his investigation [7] of fundamental units of cubic orders using Berwick's algorithm, E. Thomas has defined a useful numerical function of the roots of $f(X)$ which is denoted by $\theta(a, b)$ in the present article. In (3.1) of [7] Thomas proved that if $\theta(a, b) > 2$, then u is a fundamental unit of R; he also indicated the necessity of some such restriction by alluding to the case of $(a, b) = (2n, n^2)$, where $n \geq 3$ is an integer, in which u fails to be a fundamental unit of R and in fact $\theta(2n, n^2) < 2$. This result of Thomas is the cornerstone and the starting point of our investigation. Without any loss, we restrict ourselves to the case where $f(0) = 1$ and $a < b$, throughout the article. Our main theorem is

Theorem. *Let (a, b) be an ordered pair of integers with $a < b$ such that*

$$f_{(a,b)}(X) := X^3 + aX^2 + bX + 1$$

is irreducible in $\mathbb{Z}[X]$ and has three distinct real roots. Let u be a real number with $f_{(a,b)}(u) = 0$ and let $R := \mathbb{Z}[u]$. Then, u is a fundamental unit of R if and only if $(a, b) \neq (2n, n^2)$ for any integer $n \geq 3$ and $(a, b) \neq (5, 6)$.

Our proof is almost entirely computational in nature, involving symbolic as well as (real) numerical computation. In order to estimate the values of $\theta(a, b)$ for the integer pairs (a, b) of interest, we partition their natural domain into 15 parts. Then, Mathematica is harnessed to compute the real extrema of appropriate rational functions of two variables on each of the parts to determine whether $\theta(a, b)$ exceeds

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or does not exceed the magic number 2. Subsequently, the cases for which $\theta(a,b)$ is established to be at most 2 are examined using Sage. Fortunately, most of the exceptions can be treated by applying some special-case results of Thomas from [7] in which he has obtained a complete set of fundamental units of certain cubic orders; only in a small number of sporadic cases do we rely solely on Sage to compute such a complete set of fundamental units for the corresponding R. Our result is analogous to a classical theorem of Nagell (see Satz XXII of [5]) which applies to the case where $f(X) := X^3 + aX^2 + bX \pm 1$ has negative discriminant, i.e., when $f(X)$ has only one real root. Nagell provides an exact determination of all the integer pairs (a,b) for which u is a fundamental unit of R. As in the positive discriminant case, the exceptions consist of a single-parameter family and finitely many other sporadic examples. Nagell's theorem was proved via a new method in 2006 by Louboutin (see Theorem 4 of [2]). Louboutin conjectured in [3] that a similar theorem applied in the case of a totally complex quartic number field. In [6], Park and Lee prove this conjecture. The referee has informed us that "a simpler proof of the results proved in [2] and [6] is provided" in [4].

Of course, in the case of a cubic polynomial with negative discriminant, u is a fundamental unit of R if and only if $|u|$ generates $U^+(R)$. In the positive discriminant case, $\{|u|\}$ has to be extended to obtain a complete set of fundamental units of R. All that can be said in this regard is that an explicit determination of a unit v such that $\{|u|, v\}$ is a basis of $U^+(R)$ appears to be much more difficult (if at all possible) and the present investigation sheds little (if any) light on this matter.

1. Notation and Definitions

- $f_{(a,b)}(X) := X^3 + aX^2 + bX + 1$.
- $S := \{(a,b) \in \mathbb{Z}^2 \mid b > a\}$.
- $T := \{(a,b) \in \mathbb{Z}^2 \mid f_{(a,b)}(X)$ is irreducible in $\mathbb{Z}[X]$ and has three real roots$\}$.
- $\Delta = \Delta(f(X))$ will denote the discriminant of a polynomial $f(X)$.
- $F_>$ will denote the forward image function corresponding to a function F. That is, $F_> : A \mapsto \{F(a) \mid a \in A\}$.
- Let J_-, J, I, I_+ denote the real, open intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, $(1, \infty)$ respectively.

Assuming $(a,b) \in S$, $f_{(a,b)}(X)$ is irreducible if and only if $a + b + 2 \neq 0$ and $f_{(a,b)}(X)$ has three real roots if and only if $\Delta\left(f_{(a,b)}(X)\right)$ is positive.

1.1. Let (a,b) be a member of T and let

$$f_{(a,b)}(X) := (X-u)(X-v)(X-w)$$

where u, v, w are real numbers. The *Thomas number* $\theta(a,b)$ is defined as follows. If each of J_-, J, I, I_+ contains at most one root of $f_{(a,b)}$, then $\theta(a,b) := \infty$. Otherwise, by suitable relabeling, assume that $\{u,v\}$ is contained in (exactly) one of the intervals J_-, J, I, I_+ and define

$$\theta(a,b) := \begin{cases} |u-v|(1-w) & \text{if } \{u,v\} \subset I_+, \\ |u-v|(1-w) & \text{if } \{u,v\} \subset I, \\ -|u-v|(1+w) & \text{if } \{u,v\} \subset J, \\ |u-v|(1+w) & \text{if } \{u,v\} \subset J_-. \end{cases}$$

For the remainder of this subsection, assume $(a,b) \in S \cap T$.

We have $f_{(a,b)}(0) = 1 > 0$ and $f_{(a,b)}(-1) = a - b < 0$. Therefore, there are an odd number of roots in J. Since there cannot be three roots in J, $f_{(a,b)}$ must have precisely one root in J. From now on, call that root w.

If $a > 0$, we can say more. In this case, since $b > a$, $b > 0$ and all of the roots of $f_{(a,b)}$ are negative. Since precisely one is in J, the other two must be in J_- and $\theta(a,b) = |u - v|(1 + w)$ where $\{u, v\} \subset J_-$.

We can also say more in the cases in which $a < 0$ but $a + b + 2 > 0$. Since $f_{(a,b)}(1) = a + b + 2 > 0$ and $f_{(a,b)}(0) = 1 > 0$, there are an even number of roots in I. Since the product of the roots is -1, there can't be two roots in I and one in J, so there must be no roots in I. Hence, the other two roots are either both in J_- or both in I_+. Since the sum of the roots is equal to $-a > 0$, they are both in I_+, and $\theta(a,b) = |u - v|(1 - w)$ where $\{u, v\} \in I_+$.

2. Partition of S

2.1. **List of parts.**

(1) $A_1 := \big\{\big(2n, n^2\big) \mid n \geq 3\big\}$.
(2) $A_2 := \big\{\big(2n+1, n^2+n\big) \mid n \geq 2\big\}$.
(3) $A_3 := \big\{\big(-2n, n^2-1\big) \mid n \geq 2\big\}$.
(4) $A_4 := \big\{\big(-2n-1, n^2+n\big) \mid n \geq 4\big\}$.
(5) $B_1 := \big\{\big(2n, n^2-m\big) \mid n \geq 3 \text{ and } 1 \leq m \leq n^2-2n-1\big\}$.
(6) $B_2 := \big\{\big(2n+1, n^2+n-m\big) \mid n \geq 3 \text{ and } 1 \leq m \leq n^2-n-2\big\}$.
(7) $B_3 := \big\{\big(-2n, n^2-1-m\big) \mid n \geq 3 \text{ and } 1 \leq m \leq n^2-2n\big\}$.
(8) $B_4 := \big\{\big(-2n-1, n^2+n-m\big) \mid n \geq 2 \text{ and } 1 \leq m \leq n^2-n\big\}$.
(9) $C_1 := \big\{\big(2n, n^2-m\big) \mid n \geq 3 \text{ and } m \leq -1\big\}$.
(10) $C_2 := \big\{\big(2n+1, n^2+n-m\big) \mid n \geq 2 \text{ and } m \leq -1\big\}$.
(11) $C_3 := \big\{\big(-2n, n^2-1-m\big) \mid n \geq 2 \text{ and } m \leq -1\big\}$.
(12) $C_4 := \big\{\big(-2n-1, n^2+n-m\big) \mid n \geq 2 \text{ and } m \leq -1\big\} \cup \{(-5,6), (-7,12)\}$.
(13) $D := \{(-n, n-2-m) \mid n \geq 2 \text{ and } 1 \leq m \leq 2n-3\}$.
(14) $E := \{(-n, n-2) \mid n \geq 2\}$.
(15) $F := \{(n, m) \mid -3 \leq n \leq 4 \text{ and } m > \max(n, -2-n)\}$.

2.2. **They do cover.** Let $(a,b) \in S$ be given. If $a + b + 2 < 0$, then $(a,b) \in D$. (If $a \leq -2$, this comes from the construction of D. Otherwise, $a \geq -1$ and $b + 1 < 0$, which contradicts $b > a$.) If $a + b + 2 = 0$, then $(a,b) \in E$. From now on, assume $a + b + 2 > 0$. If $a > 4$ and a is even, then depending on how b compares to $a^2/4$, (a,b) is in either A_1, B_1, or C_1. If $a > 4$ and a is odd, then depending on how b compares to $(a^2-1)/4$, (a,b) is in either A_2, B_2, or C_2. If $a < -3$ and a is even, then depending on how b compares to $(a^2-4)/4$, (a,b) is in either A_3, B_3, or C_3. If $a < -3$ and a is odd, then depending on how b compares to $(a^2-1)/4$, (a,b) is in either A_4, B_4, or C_4 (being in C_4 unexpectedly if (a,b) is $(-5,6)$ or $(-7,12)$). If $-3 \leq a \leq 4$, then $(a,b) \in F$. Since (a,b) was arbitrary,

$$S \subseteq \bigcup_i A_i \cup \bigcup_i B_i \cup \bigcup_i C_i \cup D \cup E \cup F.$$

The relevant reverse inclusions will be shown throughout the next section.

3. Which sets are in $S \cap T$

3.1. **Sets** A_1, B_1, C_1**.** We claim that $A_1, B_1 \subset S \cap T$ and $C_1 \subset S$, but $C_1 \cap T = \emptyset$.

3.1.1. *Irreducibility and $b > a$.* Assume that $(a,b) = \left(2n, n^2 - m\right) \in A_1 \cup C_1$, with $n \geq 3$ and $m \leq 0$. Then $b - a = (n-3)^2 + 4(n-3) + 3 - m > 0$ and $a + b + 2 = n^2 + 2n + 2 - m > 0$. Now assume that $(a,b) \in B_1$. Then, $b - a \geq n^2 - \left(n^2 - 2n - 1\right) - 2n = 1 > 0$ and $a + b + 2 \geq 2n + n^2 - \left(n^2 - 2n - 1\right) + 2 = 4n + 3 > 0$. Hence, if $(a,b) \in A_1, B_1, C_1$, then $f_{(a,b)}$ is irreducible and $(a,b) \in S$.

3.1.2. *Discriminant for A_1.* For $(a,b) \in A_1$, we have $\Delta\left(f_{(a,b)}(X)\right) = 4k^3 + 36k^2 + 108k + 81 > 0$ where $n = k + 3 \geq 3$. Hence $A_1 \subset T$.

3.1.3. *Discriminant for B_1.* For $(a,b) \in B_1$, $\Delta\left(f_{(a,b)}(X)\right) > 0$ and hence $B_1 \subset T$. We must show that if $n \geq 3$ and $1 \leq m \leq n^2 - 2n - 1$, then

$$\Delta := 4m^3 - 8n^2m^2 + \left(4n^4 - 36n\right)m + 4n^3 - 27$$

is positive.

The real critical points of Δ (as a function of m and n) are at

$$(n,m) = \left(\frac{3}{2}, \quad -\frac{3}{4}\right) \text{ and } (n,m) = (0,0),$$

which are both out of the domain for m and n. Therefore, since Δ is an analytic function of m and n, it suffices to check $\Delta > 0$ at the boundaries: $n = 3$, $n = \infty$, $m = 1$, and $m = n^2 - 2n - 1$.

At $n = 3$,

$$\Delta\mid_{m=1} = 229 > 0 \text{ and } \Delta\mid_{m=2} = 257 > 0.$$

At $n = \infty$, since $m > 0$, we have

$$\lim_{n\to\infty} \Delta > 0.$$

If $n = k + 3 \geq 3$, then at $m = 1$, $\Delta = 4k^4 + 52k^3 + 244k^2 + 456k + 229 > 0$, and at $m = n^2 - 2n - 1$, $\Delta = 16k^4 + 144k^3 + 460k^2 + 612k + 257 > 0$.

3.1.4. *Discriminant for C_1.* For $(a,b) \in C_1$, $\Delta\left(f_{(a,b)}(X)\right) < 0$ and hence $C_1 \cap T = \emptyset$. We must show that if $n \geq 3$ and $m \leq -1$, then Δ, as defined in the previous subsection, is negative. Since the real critical points of Δ are out of the domain and Δ is analytic, it suffices to check $\Delta < 0$ at the boundaries $n = 3$, $n = \infty$, $m = -1$, and $m = -\infty$.

At $n = 3$, if $m = -\ell - 1 \leq -1$, $\Delta = -\left(4\ell^3 + 84\ell^2 + 372\ell + 211\right) < 0$. At $n = \infty$, negativity of m implies

$$\lim_{n\to\infty} \Delta < 0.$$

At $m = -1$, if $n = k + 3 \geq 3$, $\Delta = -\left(4k^4 + 44k^3 + 188k^2 + 336k + 211\right) < 0$. So regardless of what n is,

$$\lim_{m\to-\infty} \Delta < 0.$$

3.2. **Sets A_2, B_2, C_2.** We claim that $A_2, B_2 \subset S \cap T$ and $C_2 \subset S$, but $C_2 \cap T = \emptyset$.

3.2.1. *Irreducibility and $b > a$.* Assume that $(a,b) = \left(2n+1, n^2 + n - m\right) \in A_2 \cup C_2$, with $n \geq 2$ and $m \leq 0$. Then $b - a = (n-2)^2 + 3(n-2) + 1 - m > 0$ and $a + b + 2 = n^2 + 3n + 3 - m > 0$. Now assume that $(a,b) \in B_2$. Then, $b - a \geq n^2 + n - \left(n^2 - n - 2\right) - 2n - 1 = 1 > 0$ and $a + b + 2 \geq 2n + 1 + n^2 + n - \left(n^2 - n - 2\right) + 2 = 4n + 5 > 0$. Hence, if $(a,b) \in A_2, B_2, C_2$, then $f_{(a,b)}$ is irreducible and $(a,b) \in S$.

3.2.2. *Discriminant for A_2.* For $(a,b) \in A_2$, we have $\Delta\left(f_{(a,b)}(X)\right) = k^4 + 14k^3 + 67k^2 + 126k + 49 > 0$ where $n = k + 2 \geq 2$. Hence $A_2 \subset T$.

3.2.3. *Discriminant for B_2.* For $(a,b) \in B_2$, $\Delta\left(f_{(a,b)}(X)\right) > 0$ and hence $B_2 \subset T$. Letting $\alpha(n) := n^4 + 6n^3 + 7n^2 - 6n - 31$, we have

$$\Delta = 4m^3 - \left(8n^2 + 8n - 1\right)m^2 + \left(4n^4 + 8n^3 + 2n^2 - 38n - 18\right)m + \alpha(n).$$

We must show that if $n \geq 3$ and $1 \leq m \leq n^2 - n - 2$, then $\Delta > 0$. The real critical points of Δ (as a function of m and n) are at

$$(n,m) = (1,-1) \text{ and } (n,m) = \left(-\frac{1}{2}, -\frac{1}{4}\right),$$

which are not in the domain of (m,n). Since Δ is an analytic function of m and n, it suffices to check $\Delta > 0$ at the boundaries: $n = 3$, $n = \infty$, $m = 1$, and $m = n^2 - n - 2$.

At $n = 3$, $\Delta > 0$,

$$\Delta\,|_{m=1} = 592,\ \Delta\,|_{m=2} = 761,\ \Delta\,|_{m=3} = 788, \text{ and } \Delta\,|_{m=4} = 697.$$

Since $m > (-1/4)$,

$$\lim_{n\to\infty} \Delta > 0.$$

If $n = k + 3 \geq 3$, then at $m = 1$, $\Delta = 5k^4 + 74k^3 + 397k^2 + 872k + 592 > 0$ and at $m = n^2 - n - 2$, $\Delta = 16k^4 + 176k^3 + 700k^2 + 1188k + 697 > 0$.

3.2.4. *Discriminant for C_2.* For $(a,b) \in C_2$, $\Delta\left(f_{(a,b)}(X)\right) < 0$ and hence $C_2 \cap T = \emptyset$. We must show that if $n \geq 2$ and $m \leq -1$, then Δ, as defined in the previous subsection, is negative. Since the real critical points of Δ are not in the domain and Δ is analytic, it again suffices to check that $\Delta < 0$ at the boundaries $n = 2$, $n = \infty$, $m = -1$, and $m = -\infty$.

At $n = 2$, if $m = -1 - \ell \leq -1$, $\Delta = -\left(4\ell^3 + 59\ell^2 + 148\ell + 44\right) < 0$. Since $m < -1/4$,

$$\lim_{n\to\infty} \Delta < 0.$$

At $m = -1$, if $n = k + 2 \geq 2$, $\Delta = -\left(44 + 108k + 87k^2 + 26k^3 + 3k^4\right) < 0$. Hence

$$\lim_{m\to-\infty} \Delta < 0.$$

3.3. **Sets A_3, B_3, C_3.** We claim that $A_3, B_3 \subset S \cap T$ and $C_3 \subset S$, but $C_3 \cap T = \emptyset$.

3.3.1. *Irreducibility and $b > a$.* Assume that $(a,b) = \left(-2n, n^2 - 1 - m\right) \in A_3 \cup C_3$, with $n \geq 2$ and $m \leq 0$. Then $b - a = (n-2)^2 + 6(n-2) + 7 - m > 0$ and $a + b + 2 = (n-2)^2 + 2(n-2) + 1 - m > 0$. Now assume that $(a,b) \in B_3$. Then, $b - a \geq n^2 - 1 - \left(n^2 - 2n\right) - (-2n) = 4(n-3) + 11 > 0$ and $a + b + 2 \geq 2n + n^2 - \left(n^2 - 2n - 1\right) + 2 = 1 > 0$. Hence, if $(a,b) \in A_3, B_3, C_3$, then $f_{(a,b)}$ is irreducible and $(a,b) \in S$.

3.3.2. *Discriminant for A_3.* For $(a,b) \in A_3$, we have $\Delta\left(f_{(a,b)}(X)\right) = 4k^4 + 28k^3 + 64k^2 + 84k + 49 > 0$ where $n = k + 2 \geq 2$. Hence $A_3 \subset T$.

3.3.3. *Discriminant for B_3.* For $(a,b) \in B_3$, $\Delta\left(f_{(a,b)}(X)\right) > 0$ and hence $B_3 \subset T$. Letting $\alpha(n) := 4n^4 - 4n^3 - 8n^2 + 36n - 23$, we have

$$\Delta = 4m^3 + \left(12 - 8n^2\right)m^2 + \left(4n^4 - 16n^2 + 36n + 12\right)m + \alpha(n).$$

We must show that if $n \geq 3$ and $1 \leq m \leq n^2 - 2n$, then $\Delta > 0$.

The real critical points of Δ (as a function of m and n) are at

$$(n,m) = \left(-\frac{3}{2}, -\frac{7}{4}\right) \text{ and } (n,m) = (0,-1).$$

These are not in the domain for m and n. Since Δ is an analytic function of m and n, it suffices to check that $\Delta > 0$ at the boundaries: $n = 3$, $n = \infty$, $m = 1$, and $m = n^2 - 2n$.

At $n = 3$,

$$\Delta\mid_{m=1} = 473 > 0,\ \Delta\mid_{m=2} = 621 > 0, \text{ and } \Delta\mid_{m=3} = 697 > 0.$$

Since $m > -1$,

$$\lim_{n\to\infty} \Delta > 0.$$

If $n = k + 3 \geq 3$, then at $m = 1$, $\Delta = 8k^4 + 92k^3 + 364k^2 + 636k + 473 > 0$ and at $m = n^2 - 2n$, $\Delta = 16k^4 + 176k^3 + 700k^2 + 1188k + 697 > 0$.

3.3.4. *Discriminant for C_3.* For $(a,b) \in C_3$, $\Delta\left(f_{(a,b)}(X)\right) < 0$ and hence $C_3 \cap T = \emptyset$. We must show that if $n \geq 2$ and $m \leq -1$, then Δ, as defined in the previous subsection, is negative. At $m = -1$, $\Delta = -\left(4n^3 + 27\right)$, which is negative for $n \geq 2$. As before, assuming $m \leq -2$, it suffices to check that $\Delta < 0$ at the boundaries $n = 2$, $n = \infty$, $m = -2$, and $m = -\infty$.

At $n = 2$, if $m = -\ell - 2 \leq -2$, $\Delta = -\left(4\ell^3 + 44\ell^2 + 212\ell + 231\right) < 0$. Since $m < -1$,

$$\lim_{n\to\infty} \Delta < 0.$$

At $m = -2$, $\Delta = -\left(4n^4 + 4n^3 + 8n^2 + 36n + 31\right)$, which is negative for $n \geq 2$. Hence

$$\lim_{m\to-\infty} \Delta < 0.$$

3.4. A_4, B_4, C_4**.** We claim that $A_4, B_4 \subset S \cap T$ and $C_4 \subset S$, but $C_4 \cap T = \emptyset$.

3.4.1. *Irreducibility and $b > a$.* Assume that $(a,b) = \left(-2n-1, n^2+n-m\right) \in A_4 \cup C_4$, with $n \geq 2$ and $m \leq 0$. Then $b - a = (n-2)^2 + 7(n-2) + 11 - m > 0$ and $a + b + 2 = (n-2)^2 + 3(n-2) + 3 - m > 0$. Now assume that $(a,b) \in B_4$. Then, $b - a \geq n^2 + n - \left(n^2 - n\right) - (-2n-1) = 4n + 1 > 0$ and $a + b + 2 \geq -2n - 1 + n^2 + n - \left(n^2 - n\right) + 2 = 1 > 0$. Hence, if $(a,b) \in A_4, B_4, C_4$, then $f_{(a,b)}$ is irreducible and $(a,b) \in S$.

3.4.2. *Discriminant for A_4.* For $(a,b) \in A_4$, we have $\Delta\left(f_{(a,b)}(X)\right) = k^4 + 14k^3 + 67k^2 + 126k + 49 > 0$ where $n = k + 4 \geq 4$. Hence $A_4 \subset T$.

3.4.3. *Discriminant for B_4.* For $(a,b) \in B_4$, $\Delta\left(f_{(a,b)}(X)\right) > 0$ and hence $B_4 \subset T$. Letting $\alpha(n) := n^4 - 2n^3 - 5n^2 + 6n - 23$, we have

$$\Delta = 4m^3 - \left(8n^2 + 8n - 1\right) m^2 + \left(4n^4 + 8n^3 + 2n^2 + 34n + 18\right) m + \alpha(n).$$

We must show that if $n \geq 2$ and $1 \leq m \leq n^2 - n$, $\Delta > 0$.

The real critical points of Δ, as a function of m and n, are at

$$(n,m) = (-2,-1) \text{ and } (n,m) = \left(-\frac{1}{2}, -\frac{1}{4}\right),$$

which are outside the domain of (n,m). Since Δ is an analytic function of m and n, it suffices to check that $\Delta > 0$ at the boundaries: $n = 2$, $n = \infty$, $m = 1$, and $m = n^2 - n$.

At $n = 2$,

$$\Delta \mid_{m=1} = 148 > 0 \text{ and } \Delta \mid_{m=2} = 257 > 0.$$

Since $m > (-1/4)$,

$$\lim_{n \to \infty} \Delta > 0.$$

If $n = k + 2 \geq 2$, then at $m = 1$, $\Delta = 5k^4 + 46k^3 + 145k^2 + 220k + 148 > 0$ and at $m = n^2 - n$, $\Delta = 16k^4 + 144k^3 + 460k^2 + 612k + 257 > 0$.

3.4.4. *Discriminant for C_4.* For $(a,b) \in C_4$, $\Delta\left(f_{(a,b)}(X)\right) < 0$ and hence $C_4 \cap T = \emptyset$. We must show that if $n \geq 2$ and $m \leq -1$, then Δ, as defined in the previous subsection, is negative. In addition, we must note that

$$(-5)(6)(18 + (-5)(6)) - 4\left((-5)^3 + 6^3\right) - 27 = -31 < 0$$

as well as

$$(-7)(12)(18 + (-7)(12)) - 4\left((-7)^3 + 12^3\right) - 27 = -23 < 0.$$

As before, since the real critical points of Δ are not in the domain, it suffices to check $\Delta < 0$ at the boundaries $n = 2$, $n = \infty$, $m = -1$, and $m = -\infty$ (aside from the two sporadic points).

At $n = 2$, $\Delta = -\left(4\ell^3 + 59\ell^2 + 328\ell + 304\right) < 0$, where $m = -\ell - 1 \leq -1$. Since $m < (-1/4)$,

$$\lim_{n \to \infty} \Delta < 0.$$

At $m = -1$, $\Delta = -\left(3n^4 + 10n^3 + 15n^2 + 36n + 44\right)$, which is negative for $n \geq 2$.

3.5. **Sets** D, E, F**.** We claim that $D \subset S \cap T$ but $E \cap T, F \cap T = \emptyset$.

3.5.1. *Irreducibility and $b > a$.* Assume that $(a,b) = (-n, n-2-m) \in D$, with $n \geq 2$ and $-1 \leq m \leq 2n - 3$. Then, $b - a \geq n - 2 - (2n-3) - (-n) = 1 > 0$ and $a + b + 2 \leq -n + n - 2 - (1) + 2 = -1 < 0$. Hence, if $(a,b) \in D$, then $f_{(a,b)}$ is irreducible and $(a,b) \in S$. Since $(-n) + (n-2) + 2 = 0$, $E \cap T = \emptyset$.

3.5.2. *Discriminant for D.* For $(a,b) \in D$, $\Delta\left(f_{(a,b)}(X)\right) > 0$ and hence $D \subset T$. Letting $\alpha(n) := n^4 - 4n^3 + 10n^2 - 12n + 5$, we have

$$\Delta = 4m^3 + \left(n^2 - 12n + 24\right)m^2 - \left(2n^3 - 16n^2 + 30n - 48\right)m + \alpha(n).$$

We must show that if $n \geq 2$ and $1 \leq m \leq 2n-3$, then $\Delta > 0$.

The real critical points of Δ, as a function of m and n, are at

$$(n,m) = (-3,-8) \text{ and } (n,m) = (0,-2),$$

which are not in the domain of m and n. Again, since Δ is an analytic function of m and n, it suffices to check that $\Delta > 0$ at the boundaries: $n = 2$, $n = \infty$, $m = 1$, and $m = 2n-3$.

At $n = 2$, $\Delta = 4m^3 + 4m^2 + 36m + 5$, which is positive for $m \geq 1$. Also, we clearly have

$$\lim_{n\to\infty} \Delta > 0.$$

If $n = k+2 \geq 2$, then at $m = 1$, $\Delta = k^4 + 2k^3 + 15k^2 + 14k + 49 > 0$ and at $m = 2n-3$, $\Delta = k^4 + 14k^3 + 67k^2 + 126k + 49 > 0$.

3.5.3. *Discriminant for F.* In the eight cases where $-3 \leq n \leq 4$, $\Delta\left(f_{n,m}(X)\right)$ is a cubic polynomial in the variable m, which is negative for $m > \max(n, -2-n)$. Thus, $F \cap T = \emptyset$.

4. Sets with $\theta(a,b) > 2$

4.1. $\theta_>(B_1) \subseteq (2,\infty]$**.** By 1.1, since $2n > 0$, $f_{(a,b)}$ has two roots in J_-. Since the leading coefficient of $f_{(a,b)}$ is positive, $f_{(a,b)}$ assumes only positive values between u and v. To get bounds on $|u-v|$, we find places where $f_{(a,b)}$ is positive.

We handle $m = 1$ separately. First note that $\theta > 2$ in the following three cases:

n	m	$\theta \approx$
3	1	2.039
4	1	2.101
5	1	2.107

Next, we show that $\theta\left(2n, n^2-1\right) > 2$ for $n = k+6 \geq 6$. Note that since $-n+2+\left(2/n^2\right) = -k-4+\left(2/(k+6)^2\right) < 0$,

$$-n-1-\frac{2}{n^2} < -n+1+\frac{2}{n^2} < -1.$$

We have

$$f_{(2n,n^2-1)}\left(-n+1+\frac{2}{n^2}\right) = \frac{n^6-4n^5+4n^4-4n^3+12n^2+8}{n^6}$$

and

$$f_{(2n,n^2-1)}\left(-n-1-\frac{2}{n^2}\right) = \frac{n^6-4n^5-4n^4-4n^3-12n^2-8}{n^6}$$
$$= \frac{k^6+32k^5+416k^4+2780k^3+9852k^2+16704k+9064}{(k+6)^6} > 0.$$

Now

$$\frac{n^6-4n^5+4n^4-4n^3+12n^2+8}{n^6} > \frac{n^6-4n^5-4n^4-4n^3-12n^2-8}{n^6} > 0.$$

Therefore,

$$|u-v| > 2 + \frac{4}{n^2}.$$

To prove that $\theta(a,b) > 2$, note that

$$f_{(2n,n^2-1)}\left(\frac{2}{2+4/n^2}-1\right) = \frac{-n^6+8n^3+12n^2+16n+8}{\left(n^2+2\right)^3}$$
$$= \frac{-\left(k^6+36k^5+540k^4+4312k^3+19284k^2+45632k+44392\right)}{\left(k^2+12k+38\right)^3} < 0.$$

For the remainder of this subsection, assume that $m \geq 2$ and $n = k+3 \geq 3$. To get a general bound on $|u-v|$, we must first prove that

$$\alpha(m,n) < \beta(m,n) < -1$$

where

$$\alpha(m,n) := \frac{\left(-2n^2+5\right)m - 5n^5 + 5n^4 + 20n^3 + 7n^2 + 10n + 5}{5n^2\left(n^2-2n-2\right)}$$

and

$$\beta(m,n) := \frac{\left(n^3-3n^2-1\right)m - n^5 + 3n^4 - n^3 + 2n^2 - 2n - 1}{n^2\left(n^2-2n-2\right)}.$$

Since $n > 1+\sqrt{3}$, the denominators in the above two expressions are positive. The first of these inequalities follows from

$$\left(-n^3+\frac{13}{5}n^2+2\right)m - 2n^4 + n^3 + \frac{17}{5}n^2 + 4n + 2$$
$$= -\left(\left(\frac{5k^3+32k^2+57k+8}{5}\right)m + \frac{10k^4+115k^3+478k^2+823k+452}{5}\right) < 0.$$

Next, we prove the second inequality:

$$\left(n^3-3n^2-1\right)m - n^5 + 3n^4 - n^3 + 2n^2 - 2n - 1 < -n^2\left(n^2-2n-2\right).$$

This is equivalent to

$$\left(n^3-3n^2-1\right)m - n^5 + 4n^4 - 3n^3 - 2n - 1 < 0.$$

At $n=3$, this reduces to $-(m+7) < 0$. For $n = k+4 \geq 4$ and $m = n^2-2n-1-\ell \leq n^2-2n-1$, it reduces to

$$-\left(\left(15+24k+9k^2+k^3\right)\ell + k^4 + 14k^3 + 70k^2 + 144k + 96\right) < 0.$$

Therefore, the above chain of inequalities holds.

Define

$$h := f_{(2n,n^2-m)}\left(\alpha\left(m,n\right)\right).$$

The real critical points of h, as a rational function of m and n, all have $n < 3$. Therefore, it suffices to check that $h > 0$ on the boundaries $n=3$, $n=\infty$, $m=2$, and $m = n^2-2n-1$. At $n=3$, $m = 2 = 3^2 - 2*3 - 1$ and $h = \frac{147}{125} > 0$. Since $m > 1$,

$$\lim_{n\to\infty} h > 0.$$

At $m=2$, the largest real zero of h is $n \approx 2.938$ and we already know that $h > 0$ at $n=3$ and $m=2$. Finally, at $m = n^2-2n-1$, the largest real zero of h is $n \approx 2.931$, and at $n=3$ and $m = 3^2-2*3-1 = 2$, we already know that $h > 0$.

Next, define

$$i := f_{(2n,n^2-m)}\left(\beta\left(m,n\right)\right).$$

There is only one real critical point of i with $n \geq 3$, namely $(n,m) \approx (3.726, 8.49)$ where $m > n^2 - 2n - 1$. Therefore, it suffices to check that $i > 0$ on the boundaries $n = 3$, $n = \infty$, $m = 2$, and $m = n^2 - 2n - 1$. At $n = 3$, $m = 2 = 3^2 - 2 * 3 - 1$ and $i = 3 > 0$. Again, since $m > 1$,

$$\lim_{n\to\infty} i > 0.$$

At $m = 2$, the largest real zero of i is $n \approx 2.771$ and we already know that $i > 0$ at $m = 2$ and $n = 3$. At $m = n^2 - 2n - 1$, i has one real root, i.e., $(9/4)$, and at $n = 3$ and $m = 3^2 - 2 * 3 - 1 = 2$, we already know that $i > 0$.

Finally, for $n \geq 3$, define

$$j := \beta(m,n) - \alpha(m,n).$$

We aim to show that $|u - v|\,(1 + w) > 2$. Since $|u - v| > j > 0$, it suffices to show $1 + w > (2/j)$, or, equivalently, $w > (2/j) - 1$. Since w is the largest root of the cubic $f := f_{(2n,n^2-m)}$ and the other roots are less than -1, it suffices to show that $f((2/j) - 1) < 0$ and $(2/j) \geq 0$. The second of these inequalities holds since $j > 0$. For the first inequality, note that the real critical points of $f\left((2/j) - 1\right)$ (as a function of m and n) all have $n < 3$ or $m < 0$. It suffices to check that $f\left((2/j) - 1\right) < 0$ at the boundary values $m = 2$, $m = n^2 - 2n - 1$, $n = 3$, and $n = \infty$. Since $m > 1$,

$$\lim_{n\to\infty} f\left(\frac{2}{j} - 1\right) < 0.$$

When $n = 3$, $m = 2$ and $f\left((2/j) - 1\right) = -\frac{1}{216} < 0$. When $m = 2$, $f\left((2/j) - 1\right)$ reduces to a function of n whose largest real zero is $n \approx 2.997$. Since, as already seen, $f\left((2/j) - 1\right)$ is negative at $n = 3, m = 2$, it is negative for all $n \geq 3$ assuming $m = 2$. When $m = n^2 - 2n - 1$ and $n = k + 3 \geq 3$,

$$f\left(\frac{2}{j} - 1\right) = \frac{-\left(625k^3 + 2800k^2 + 3540k + 8\right)}{\left(5k + 12\right)^3} < 0.$$

4.2. $\theta_{>}\left(B_2\right) \subseteq (2, \infty]$**.** By Subsection 1.1, since $2n + 1 > 0$, $f_{(a,b)}$ has two roots in J_{-}. Since the leading coefficient of $f_{(a,b)}$ is positive, $f_{(a,b)}$ assumes only positive values between u and v. To get bounds on $|u - v|$, we find places where $f_{(a,b)}$ is positive. To begin with, using $n \geq 3$ and $1 \leq m \leq n^2 - n - 2$, it is easy to see that we have

$$\alpha(m,n) < \beta(m,n) < -1,$$

where

$$\alpha(m,n) := \frac{\left(-2n^3 + 5n^2 + 2\right)m - 2n^5 - n^4 + 11n^3 + 2n^2 + 2n + 4}{2n^2\left(n^2 - n - 3\right)}$$

and

$$\beta(m,n) := \frac{\left(2n^3 - 5n^2 - 2\right)m - 2n^5 + 3n^4 + 3n^3 + 4n^2 - 2n - 4}{2n^2\left(n^2 - n - 3\right)}.$$

Now we proceed to get bounds on $|u - v|$.

Define

$$h := f_{(2n+1,n^2+n-m)}\left(\alpha\left(m,n\right)\right).$$

Then h has eight real critical points, but none of them are in the relevant domain. Therefore, it suffices to check that $h > 0$ at $n = 3$, $n = \infty$, $m = 1$, and $m =$

n^2-n-2. At $n=3$ and $m=1,2,3,4=3^2-3-2$, $h=\frac{6247}{5832},\frac{85987}{19683},\frac{1209727}{157464},11>0$, respectively. At $n=\infty$, the limit value is positive since $m-(3/4)$ is positive. At $m=1$, the only real zero of h is about 0.749, and we know it is positive at $n=3$. At $m=n^2-n-2$, the real zeros of h are all less than 3, and h is positive at $n=3$.

Define

$$i:=f_{(2n+1,n^2+n-m)}\left(\beta\left(m,n\right)\right).$$

None of the three real critical points of i are in the domain for m and n. Therefore, we only need to check at $n=3$, $n=\infty$, $m=1$, and $m=n^2-n-2$. At $n=3$ and $m=1,2,3,4=3^2-3-2$, $i=\frac{6047}{5832},\frac{51281}{19683},\frac{617579}{157464},5>0$, respectively. At $n=\infty$, the limit value is positive since $m-(3/4)$ is positive. At $m=1$, the largest real zero of i is 2, and since we know that i is positive at $n=3$, it is positive for all $n\geq 3$. At $m=n^2-n-2$, we have $i=4n-7>0$. For $n\geq 3$, define

$$j:=\beta(m,n)-\alpha(m,n)$$

so that

$$|u-v|>j>0.$$

We aim to show that $|u-v|\,(1+w)>2$. It suffices to show that $w>(2/j)-1$. Since w is the largest root of the cubic $f:=f_{(2n+1,n^2+n-m)}$ and the other roots are less than -1, it suffices to show that $(2/j)>0$ and $f\left((2/j)-1\right)<0$. The first inequality follows from the fact that $j>0$. Now, if $m=1$ and $n=k+3\geq 3$,

$$f\left(\frac{2}{j}-1\right)=\frac{-\left(k^5+13k^4+66k^3+157k^2+158k+31\right)}{\left(k^2+6k+10\right)^3}<0.$$

From now on, assume $m\geq 2$. The eight real critical points of $f\left((2/j)-1\right)$ are all outside the domain for m and n. Therefore, it suffices to check the boundaries: $n=\infty$, $n=3$, $m=2$, $m=n^2-n-2$. Again, since $m>1$,

$$\lim_{n\to\infty}f\left(\frac{2}{j}-1\right)<0.$$

At $n=3$ and $m=2,3,4$, $f\left((2/j)-1\right)=-\frac{205743}{300763},-\frac{47657}{50653},-\frac{25}{27}<0$, respectively. At $m=2$, $f\left((2/j)-1\right)$ reduces to a function which is negative when $n=3$ and whose seven real roots are all less than 3. At $m=n^2-n-2$, if $n=k+3\geq 3$,

$$f\left(\frac{2}{j}-1\right)=\frac{-\left(24k^3+92k^2+106k+25\right)}{\left(2k+3\right)^3}<0.$$

4.3. $\theta_>\left(B_3\right)\subseteq(2,\infty]$**.** By Subsection 1.1, since $-2n<0$, $f_{(a,b)}$ has two roots in I_+. Since the leading coefficient of $f_{(a,b)}$ is positive, $f_{(a,b)}$ assumes only negative values between u and v. To get bounds on $|u-v|$, we find places where $f_{(a,b)}$ is negative.

First, using $n\geq 3$ and $1\leq m\leq n^2-2n$, it is easy to see that we have

$$1<\alpha\left(m,n\right)<\beta\left(m,n\right),$$

where

$$\alpha\left(m,n\right):=\frac{-3m+2n^4-6n^3+n^2+4n+4}{2n\left(n^2-2n-1\right)}$$

and

$$\beta\left(m,n\right):=\frac{3m+2n^4-2n^3-5n^2-4n-4}{2n\left(n^2-2n-1\right)}.$$

Define

$$h := f_{(-2n,n^2-1-m)}\left(\alpha\left(m,n\right)\right).$$

All six real critical points of h have $n < 3$, so it suffices to check that $h < 0$ at the boundary values $n = 3$, $n = \infty$, $m = 1$, and $n^2 - 2n$. At $n = 3$ and $m = 1, 2, 3$, $h = -\frac{37}{216}, -\frac{989}{1728}, -\frac{17}{27}$, respectively. Since m is positive,

$$\lim_{n\to\infty} h < 0.$$

At $m = 1$ and $n = k + 3 \geq 3$,

$$h = \frac{-\left(8k^4 + 72k^3 + 218k^2 + 234k + 37\right)}{8\left(k+3\right)^3} < 0.$$

At $m = n^2 - 2n$ and $n = k + 3 \geq 3$,

$$h = \frac{-\left(k^6 + 15k^5 + 90k^4 + 269k^3 + 396k^2 + 228k + 17\right)}{\left(k+3\right)^3} < 0.$$

Define

$$i := f_{(-2n,n^2-1-m)}\left(\beta\left(m,n\right)\right).$$

The six real critical points of i all have $n < 3$, so it suffices to check the (same as in the above paragraph) four boundary cases. At $n = 3$ and $m = 1, 2, 3$, $i = -\frac{359}{216}, -\frac{5851}{1728}, -\frac{127}{27}$, respectively. Since m is positive

$$\lim_{n\to\infty} i < 0.$$

If $n = k + 3 \geq 3$, then at $m = 1$,

$$i = \frac{-\left(8k^4 + 88k^3 + 354k^2 + 606k + 359\right)}{8\left(k+3\right)^3} < 0,$$

and at $m = n^2 - 2n$,

$$i = \frac{-\left(k^6 + 17k^5 + 120k^4 + 441k^3 + 856k^2 + 750k + 127\right)}{\left(k+3\right)^3} < 0.$$

Let

$$j := \beta\left(m,n\right) - \alpha\left(m,n\right)$$

so that

$$|u - v| > j.$$

Since $w \in J$, we have $-w \in I$ and $1 - w > 1$. Therefore, if we could show that $j \geq 2$, we would be done. Now $j - 2$ has no real critical points, so it suffices to check $j \geq 2$ at the boundary: $n = 3$, $n = \infty$, $m = 1$, and $m = n^2 - 2n$. At $n = 3$, $j - 2 = (6m - 2)/12$, which is obviously positive since $m \geq 1$. Also,

$$\lim_{n\to\infty} j = 2.$$

At $m = 1$, $j - 2 = 1/n > 0$. At $m = n^2 - 2n$, $j - 2 = 4/n > 0$.

4.4. $\theta_{>}(B_4) \subseteq (2,\infty]$**.** By Subsection 1.1, since $-2n-1<0$, $f_{(a,b)}$ has two roots in I_+. Since the leading coefficient of $f_{(a,b)}$ is positive, $f_{(a,b)}$ assumes only negative values between u and v. To get bounds on $|u-v|$, we find places where $f_{(a,b)}$ is negative. We postpone the special cases of $2 \le n \le 4$ to be handled separately. From now on, assume $n \ge 5$ unless otherwise specified.

First, using $n \ge 5$ and $1 \le m \le n^2-n$, it is easy to see that we have

$$1 < \alpha(m,n) < \beta(m,n),$$

where

$$\alpha(m,n) := \frac{(-2n+5)m + 2n^3 - 3n^2 + n - 4}{2(n^2-n-1)}$$

and

$$\beta(m,n) := \frac{(2n-5)m + 2n^3 + n^2 - 7n + 2}{2(n^2-n-1)}.$$

Define

$$h := f_{(-2n-1,n^2+n-m)}(\alpha(m,n)).$$

Neither of the two real critical points of h have $n>4$, so it suffices to check that $h<0$ at the boundary values $n=5$, $n=\infty$, $m=1$, and $m=n^2-n$. At $n=5$, the three real zeros of h are outside the domain of m, and $h=-(1/8)<0$ when $m=1$. Since $m>(3/4)$,

$$\lim_{n\to\infty} h < 0.$$

If $n=k+5\ge 5$, then at $m=1$, $h=-(2k+1)/8<0$ and at $m=n^2-n$, $h=-(4k+15)<0$.

Next, define

$$i := f_{(-2n-1,n^2+n-m)}(\beta(m,n)).$$

None of the six real critical points of i have $n>4$, so it suffices to check that $i<0$ at the boundary values $n=5$, $n=\infty$, $m=1$, and $m=n^2-n$. At $n=5$, the three real zeros of i are outside the domain of m, and $i=-(5/8)<0$ when $m=1$. Since $m>(3/4)$,

$$\lim_{n\to\infty} i < 0.$$

If $n=k+5\ge 5$, then at $m=1$, $i=-(2k+5)/8<0$ and at $m=n^2-n$, $i=-\left(4k^2+18k+19\right)<0$.

We know that

$$|u-v| > \beta(m,n) - \alpha(m,n).$$

If $n=k+5\ge 5$ and $m=\ell+1\ge 1$, the difference on the right reduces to

$$2 + \frac{(2k+5)\ell}{k^2+9k+19} \ge 2.$$

Now since $w \in J$, we have $1-w>1$ and it follows that $\theta(a,b)>2$.

Finally, it only remains to check the values of $\theta\left(-2n-1, n^2+n-m\right)$ when $2 \le n \le 4$ and $1 \le m \le n^2-n$. The following four 3-column tables exhibit our computation of these values:

n	m	$\theta \approx$
2	1	2.097
2	2	3.147

n	m	$\theta \approx$
3	1	2.109
3	2	3.027
3	3	3.744
3	4	4.366
3	5	4.934
3	6	5.473

n	m	$\theta \approx$
4	1	2.120
4	2	2.985
4	3	3.656
4	4	4.229
4	5	4.739
4	6	5.207

n	m	$\theta \approx$
4	7	5.645
4	8	6.060
4	9	6.450
4	10	6.849
4	11	7.232
4	12	7.614

Evidently, $\theta\left(-2n-1, n^2+n-m\right) > 2$ in each of these special cases.

4.5. $\theta_>(D) \subseteq (2,\infty]$. In this case $a+b+2<0$ by Subsection 3.5.1. Therefore, $f_{(a,b)}(1)<0$ and $f_{(a,b)}(0)>0$ and there is exactly one root of $f_{(a,b)}$ in I (there cannot be three). Since $f_{(a,b)}$ has exactly one root in two different intervals (see Subsection 1.1), $\theta(a,b)=\infty$.

5. Proof of the main theorem

Proof. If $(a,b)=(2n,n^2)$ for some integer $n \geq 3$, then since (a,b) is in A_1, the requirement that (a,b) is in $S \cap T$ has already been verified. Observe that since

$$u = -(u+n)^{-2}$$

in this case, u is clearly not a fundamental unit of R. Likewise, if $(a,b)=(5,6)$, then

$$u = -(u^2+3u+1)^2$$

and hence it is not a fundamental unit of R.

Consider the family A_2 of pairs $(a,b)=(2n+1,n^2+n)$ where n is an integer at least 2. We have $(a,b)=(5,6)$ when $n=2$. Assuming $n \geq 3$, we can apply (3.9) of [6] to the minimal polynomial of $u+n+1$ and conclude that $\{u+n+1, u+n\}$ is a complete set of fundamental units of R. From

$$u \;=\; -(u+n+1)^{-1}(u+n)^{-1}$$

it follows that u is a fundamental unit of R.

Next we deal with the family A_3 of pairs $(a,b)=(-2n,n^2-1)$ where $n \geq 2$ is an integer. When $n=2$, the set $\{u-2, u^2-2u\}$ is a complete set of fundamental units of R according to Sage. Since

$$u = (u^2-2u)(u-2)^{-1},$$

clearly u is a fundamental unit of R. Assuming $n \geq 3$, we can apply (3.9) of [6] to the minimal polynomial of $-(u-n-1)$ and conclude that $\{u-n-1, u-n+1\}$ is a complete set of fundamental units of R. Observe that

$$u = -(u-n-1)^{-1}(u-n+1)^{-1}$$

and hence u is a fundamental unit of R.

Now, consider the family A_4. Here $(a,b)=(-2n-1,n^2+n)$ for integers $n \geq 4$. When $n=4$, we have $(a,b)=(-9,20)$ and $\{u^2-5u+2, u^2-4u-1\}$ is a complete fundamental set of units of the corresponding order R according to Sage. It is straightforward to verify that

$$u = (u^2-5u+2)^{-4}(u^2-4u-1)^{-1}$$

and hence u is indeed a fundamental unit of R. Assuming $n \geq 5$, we can apply (3.6) of [6] to the minimal polynomial of $-(u-n-1)^{-1}$ and conclude that $\{-u(u-n), 1-u(u-n)\}$ is a complete fundamental set of units of R. It is straightforward to verify that

$$u \;=\; -(u(u-n))^2(1-u(u-n))^{-1}$$

and hence u is a fundamental unit of R.

As shown in the previous section, $\theta(a,b) > 2$ for those integer pairs (a,b) in $S \cap T$ which are not contained in any of the parts A_1, A_2, A_3, A_4. In fact, $\theta_>(A_i) \subseteq (-\infty, 2)$ for $1 \leq i \leq 4$. Given $\theta(a,b) > 2$, Theorem (3.1) of [6] shows that u is a fundamental unit of the corresponding cubic order R. Thus our assertion is established. □

6. A note on calculations

Mathematica was used for most of the calculations in this paper. Apart from the standard Mathematica functions `Factor`, `Simplify`, `NSolve`, `Limit`, etc., the following code was used to approximate $\theta(a,b)$ in some (finitely many) cases:

```
case[a_,b_]:=
 Module[{roots=x/.NSolve[x^3+a*x^2+b*x+1==0,x]},
  u=roots[[1]];v=roots[[2]];w=roots[[3]];
  If[((a-b)*(a+b+2)!=0&&4*(a^3+b^3)+27-a*b*(18+a*b)<0),
   If[w>1&&1>v>0&&u<0,{0},
    If[(w>1&&v>1)||(1>w>0&&1>v>0),{1,u,v,w},
     If[(0>w>-1&&0>v>-1),{2,u,v,w},{3,u,v,w}]]],{-1}]];
theta[a_,b_]:=
 Module[{c=case[a, b]},
  Switch[c[[1]],-1,"Undefined",0,Infinity,
  1,Abs[c[[3]]-c[[4]]]*(1-c[[2]]),
  2,Abs[c[[3]]-c[[4]]]*(-1-c[[2]]),
  3,Abs[c[[2]]-c[[3]]]*(1+c[[4]])]]
```

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References

[1] H. Cohen, *A Course in Computational Algebraic Number Theory,* Graduate Texts in Mathematics 138, Springer-Verlag (2000). MR1228206 (94i:11105)

[2] S. Louboutin, *The class-number one problem for some real cubic number fields with negative discriminants,* J. Number Theory 121 (2006), 30–39. MR2268753 (2007k:11189)

[3] S. Louboutin, *The fundamental unit of some quadratic, cubic or quartic orders,* J. Ramanujan Math. Soc. 23, No.2 (2008), 191-210. MR2432797 (2009h:11175)

[4] S. Louboutin, *On some cubic or quartic algebraic units,* J. Number Theory, to appear.

[5] T. Nagell, *Zur Theorie der kubishen Irrationalitaten,* Acta Math. 55 (1930), 33–65. MR1555314

[6] S.-M. Park and G.-N. Lee, *The class number one problem for some totally complex quartic number fields,* J. Number Theory 129 (2009), 1138–1349. MR2521477

[7] E. Thomas, *Fundamental units for orders in certain cubic number fields*, J. Reine Angew. Math. 310 (1979), 33–55. MR546663 (81b:12009)

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