MATHEMATICS OF COMPUTATION Volume 81, Number 278, April 2012, Pages 1149–1161 S 0025-5718(2011)02547-0 Article electronically published on September 30, 2011

EXPLICIT COMPUTATIONS ON THE DESINGULARIZED KUMMER SURFACE

V. G. LOPEZ NEUMANN AND CONSTANTIN MANOIL

ABSTRACT. We find formulas for the birational maps from a Kummer surface \mathcal{K} and its dual \mathcal{K}^* to their common minimal desingularization \mathcal{S} . We show how the nodes of \mathcal{K} and \mathcal{K}^* blow up. Then we give a description of the group of linear automorphisms of \mathcal{S} .

1. INTRODUCTION

In the 19th century a singular surface \mathcal{K} , called the Kummer surface, was attached to a quadratic line complex. A minimal desingularization Σ of \mathcal{K} and a birational map $\kappa_1 : \mathcal{K} \dashrightarrow \Sigma$ were constructed by geometric methods. One may call this the *classical* construction of the Kummer surface, which we recall in Section 4.

Another construction is the following: Let A be an abelian surface, and let σ be the involution of A given by $\sigma(x) = -x$. The quotient $\mathcal{K} = A/\sigma$ has 16 double points, and one defines a K3 surface S to be \mathcal{K} with these 16 nodes blown up ([1], Prop. 8.11). To be consistent with the historical point of view and with our main reference [3], we call S the *desingularized* Kummer surface. If $A = \mathcal{J}(\mathcal{C})$, where $\mathcal{J}(\mathcal{C})$ is the Jacobian of a curve \mathcal{C} of genus 2, then \mathcal{K} is called the Kummer surface belonging to \mathcal{C} . The connection between the two constructions of the Kummer surface is explicitly established in [3], Chapter 17 (see Lemma 4.5).

A desingularization S of \mathcal{K} is constructed explicitly in [3], Chapter 16, by algebraic methods. Denote by \mathcal{K}^* the projective dual of \mathcal{K} . There are birational maps $\kappa : \mathcal{K} \dashrightarrow \mathcal{S}$ and $\kappa^* : \mathcal{K}^* \dashrightarrow \mathcal{S}$ and morphisms extending $\kappa^{-1} : \mathcal{S} \dashrightarrow \mathcal{K}$ and $\kappa^{*-1} : \mathcal{S} \dashrightarrow \mathcal{K}^*$ to all of \mathcal{S} . We denote these extensions also by κ^{-1} and κ^{*-1} . They are minimal desingularizations of \mathcal{K} and \mathcal{K}^* .

Origins. Cassels and Flynn explain that the surface S comes from the behavior of six of the tropes (see Definition 2.3) under the duplication map. The existence of S raises more far-reaching questions. Indeed, if the ground field k is algebraically closed, one has a commutative diagram:



Received by the editor July 3, 2009.

 $\textcircled{\sc constraint}$ ©2011 American Mathematical Society Reverts to public domain 28 years from publication

 $^{2010\} Mathematics\ Subject\ Classification.\ {\it Primary\ 14J28,\ 14M15;\ Secondary\ 14J50.}$

Key words and phrases. Genus 2 curves, Kummer surfaces, line complexes.

where $\mathcal{J}(\mathcal{C})^0$ is the dual of $\mathcal{J}(\mathcal{C})$ as an abelian variety. Here, the maps d_0 and d_0^* depend on the choice of a rational point on \mathcal{C} . Thus the abelian varieties duality matches with the projective one. When k is not algebraically closed, one has to enlarge the ground field to obtain such diagrams, yet \mathcal{S} is a desingularization over k of both \mathcal{K} and \mathcal{K}^* . One may ask if there is a unifying object for $\mathcal{J}(\mathcal{C})$ and $\mathcal{J}(\mathcal{C})^0$, generalizing the abelian varieties duality.

Recent developments. The Jacobian $\mathcal{J}(\mathcal{C})$ can be embedded in \mathbb{P}^{15} and is described by 72 quadratic equations ([4]). More computable objects, \mathcal{S} and its twists, appeared in recent attempts by M. Stoll and N. Bruin, to compute the Mordell-Weil group of $\mathcal{J}(\mathcal{C})$. We give a brief account of it in Section 5 of [8].

Cassels and Flynn already suggested that the 2-Selmer group could be investigated by using twists of S. In 2007 A. Logan and R. van Luijk ([7]) and P. Corn ([2]) made use of twists of S to find specific curves with nontrivial 2-torsion elements in the Tate-Shafarevich groups of their Jacobians.

Our results and structure of this paper. In Section 2 we give a background. The rare relevant facts, not included in this paper, are contained in [3] and [8].

This paper is structured along two computational ideas. First, to profit from the algebraic construction of S in [3] in order to describe its linear automorphism. Second, to link S and Σ and thus bring line complexes into the picture. Our results achieve part of the program suggested in [3], at the end of Chapter 16.

In Section 3 we complete the construction in [3], enlarge the set of points where $\kappa : \mathcal{K} \dashrightarrow \mathcal{S}$ is explicitly defined and find formulas for κ . These formulas allow one to describe how each singularity of \mathcal{K} and \mathcal{K}^* blows up (see Lemma 3.6).

In Section 4 we define an explicit isomorphism Θ between Σ and S. We observe that well-known projective isomorphisms $W_i : \mathcal{K} \longrightarrow \mathcal{K}^*$ lift to Σ to correspondences given by line complexes of degree 1. Using Θ we show that the W_i lift to S to commuting involutions $\varepsilon^{(i)}$ (Corollary 4.12). With the formulas for κ at hand, one can now find formulas for κ^* .

In Section 5 we give a description of the group of linear automorphisms of \mathcal{S} .

2. Preliminaries

We work over a field k with $char(k) \neq 2$ and with more than 5 elements. By a curve, we mean a smooth projective irreducible variety of dimension 1. *Throughout this paper*, C will be a curve of genus 2. Such a curve admits an affine model:

(2.1)
$$C': Y^2 = F(X) = f_0 + f_1 X + \dots + f_6 X^6 \in k[X], \quad f_6 \neq 0$$

and F has distinct roots $\theta_1, \ldots, \theta_6$. The points $\mathfrak{a}_i = (\theta_i, 0)$ are the Weierstrass points. We denote by ∞^{\pm} the points at infinity on the completion \mathcal{C} of \mathcal{C}' . For a given point $\mathfrak{r} = (x, y)$ on \mathcal{C} , the *conjugate of* \mathfrak{r} *under the* $\pm Y$ *involution* is the point $\overline{\mathfrak{r}} = (x, -y)$. For a divisor $\mathfrak{X} = \sum n_i \mathfrak{r}_i$ we denote by $\overline{\mathfrak{X}} = \sum n_i \overline{\mathfrak{r}}_i$. The class of a divisor \mathfrak{X} is denoted by $[\mathfrak{X}]$ and a divisor in the canonical class by $K_{\mathcal{C}}$.

One can regard a point of the Jacobian $J(\mathcal{C})$ as the class of a divisor $\mathfrak{X} = \mathfrak{r} + \mathfrak{u}$, where $\mathfrak{r} = (x, y)$, $\mathfrak{u} = (u, v)$ is a pair of points on \mathcal{C} . Starting from this, Flynn constructs a projective embedding of the Jacobian as follows ([4]). For a point $\mathfrak{X} = {\mathfrak{r}, \mathfrak{u}}$ on $\mathcal{C}^{(2)}$ (symmetric product) with $\mathfrak{r} = (x, y)$, $\mathfrak{u} = (u, v)$, define:

$$\sigma_0 = 1, \quad \sigma_1 = x + u, \quad \sigma_2 = xu, \quad \beta_0 = \frac{F_0(x, u) - 2yv}{(x - u)^2},$$

DUAL KUMMER SURFACE

where

$$F_0(x,u) = 2f_0 + f_1(x+u) + 2f_2xu + f_3xu(x+u) + 2f_4(xu)^2 + f_5(xu)^2(x+u) + 2f_6(xu)^3.$$

The Jacobian is then the projective locus of $\mathbf{z} = (z_0 : \ldots : z_{15})$ in \mathbb{P}^{15} , where $z_0 = \delta$; $z_1 = \gamma_1$; $z_2 = \gamma_0$; $z_i = \beta_{5-i}$, i = 3, 4, 5; $z_i = \alpha_{9-i}$, $i = 6, \ldots, 9$; $z_i = \sigma_{14-i}$, $i = 10, \ldots, 14$; and $z_{15} = \rho$. For the definition of the functions α , β , etc. and details, see [3], Chapter 2.

Definition 2.1. The Kummer surface \mathcal{K} belonging to a curve of genus 2, is the projective locus in \mathbb{P}^3 of the elements $\xi = (\xi_1 : \xi_2 : \xi_3 : \xi_4) = (\sigma_0 : \sigma_1 : \sigma_2 : \beta_0)$.

The equation of the Kummer surface is

(2.2)
$$\mathcal{K}: \quad K = K_2 \xi_4^2 + K_1 \xi_4 + K_0 = 0,$$

where the K_i are forms of degree 4-i in ξ_1, ξ_2, ξ_3 ([3], formula (3.1.9)). The natural map from $J(\mathcal{C})$ to \mathcal{K} given by

$$(z_0:\ldots:z_{15}) \mapsto (z_{14}:z_{13}:z_{12}:z_5) = (\xi_1:\ldots:\xi_4)$$

is 2 to 1; the ramification points correspond to divisor classes $[\mathfrak{X}]$ with $[\mathfrak{X}] = [\overline{\mathfrak{X}}]$. The images of these classes are the 16 nodes on \mathcal{K} : $N_0 = (0:0:0:1)$ corresponds to $[K_c]$ and 15 nodes N_{ij} correspond to classes $[\mathfrak{X}_{ij}] = [\mathfrak{a}_i + \mathfrak{a}_j]$ with $i \neq j$.

Definition 2.2. The dual Kummer surface $\mathcal{K}^* \subset (\mathbb{P}^3)^{\checkmark} = \mathbb{P}^3$ is the projective dual of \mathcal{K} , i.e., to the point $\xi \in \mathcal{K}$ corresponds the point $\eta \in \mathcal{K}^*$ such that $\eta = (\eta_1 : \eta_2 : \eta_3 : \eta_4) \in (\mathbb{P}^3)^{\checkmark}$ gives the tangent plane to \mathcal{K} at ξ .

Definition 2.3. There are 6 planes T_i containing the 6 nodes N_0 and N_{ij} , $j \neq i$ and 10 planes T_{ijk} containing the 6 nodes N_{mn} for $\{m, n\} \subset \{i, j, k\}$ or $\{m, n\} \cap$ $\{i, j, k\} = \emptyset$. These are the *tropes*; they cut conics on \mathcal{K} . They correspond to the 16 singular points of \mathcal{K}^* . The equations of the T_i are:

(2.3)
$$T_i: \quad \theta_i^2 \xi_1 - \theta_i \xi_2 + \xi_3 = 0,$$

Lemma 2.4. For each Weierstrass point $\mathfrak{a}_i = (\theta_i, 0)$ there is a projective map $W_i : \mathcal{K} \longrightarrow \mathcal{K}^*$ induced by the addition of \mathfrak{a}_i to a divisor of degree 2. One has $W_i(N_0) = T_i$ (singular point of \mathcal{K}^*). Furthermore, $W_i^{-1} \circ W_j(N_0) = N_{ij}$.

Proof. See [3], Lemma 4.5.1.

3. The desingularized Kummer surface

We recall the facts from [3], Chapter 16 we need, keeping the notation there. Let $\mathbf{p} = (p_0 : \ldots : p_5)$, where the p_j are indeterminates, and put $P(X) = \sum_{0}^{5} p_j X^j$. Let S be the projective locus of the \mathbf{p} for which $P(X)^2$ is congruent to a quadratic in X modulo F(X). Put

(3.1)
$$P_j(X) = \prod_{i \neq j} (X - \theta_i) = \sum_{k=0}^5 h_{jk} X^k$$

and $\omega_j = P_j(\theta_j) \neq 0$. Since $\theta_i \neq \theta_j$ for $i \neq j$, we have $\omega_j \neq 0$ and the P_j span the vector space of polynomials of degree at most 5. We have

(3.2)
$$P(X) = \sum_{j} \pi_{j} P_{j}(X), \quad \text{where} \quad \pi_{j} = \frac{P(\theta_{j})}{\omega_{j}}.$$

The K3 surface S is the complete intersection in \mathbb{P}^5 of the quadrics S_0 , S_1 , S_2 where

(3.3)
$$S_i: S_i = 0$$
 and $S_i = \sum_j \theta_j^i \omega_j \pi_j^2$ for $i = 0, 1, 2$.

It is a minimal desingularization of \mathcal{K} and of \mathcal{K}^* . Here the S_i are quadratic forms in \mathbf{p} with coefficients in $\mathbb{Z}[f_1, \ldots, f_6]$.

The following theorems hold ([3], Theorems 16.5.1 and 16.5.3):

Theorem 3.1. There is a birational map $\kappa : \mathcal{K} \to \mathcal{S}$ defined for general $\xi \in \mathcal{K}$ as follows: Let $\mathfrak{X} = \{(x, y), (u, v)\}$ correspond to ξ . Put G(X) = (X - x)(X - u) and let M(X) be the cubic determined by the property that Y - M(X) vanishes twice on \mathfrak{X} . Let $P(X) = \sum_{0}^{5} p_{j} X^{j}$ be determined by $GP \equiv M \mod F$. Then $\kappa(\xi)$ is the point with projective coordinates $(p_{0} : \ldots : p_{5})$.

Let $\kappa^* : \mathcal{K}^* \dashrightarrow \mathcal{S}$ be the birational map defined in [3], Theorem 16.5.2.

Theorem 3.2. Let $\xi \in \mathcal{K}$ and $\eta \in \mathcal{K}^*$ be dual, that is, η gives the tangent to \mathcal{K} at ξ . Then $\kappa(\xi) = \kappa^*(\eta)$.

Our first result is the following.

Lemma 3.3. The map $\kappa : \mathcal{K} \dashrightarrow \mathcal{S}$ from Theorem 3.1 is given by the formulas listed below.

Proof. The problem is to make effective the method given in [3], Chapter 16. For completeness and due to typing errors there, we recall it in [8], in the Appendix. As presented in [3], the method works for a general element $[\mathfrak{X}] = [(x, y) + (u, v)]$, where $yv \neq 0$ and $x \neq u$.

First, we put $y^2 = F(x)$, $v^2 = F(u)$, $yv = [F_0(x, u) - \beta_0(x - u)^2]/2$ in the coefficients of P(X), then as the resulting coefficients are symmetric functions of x and u, we express them in terms of $\xi_2 = x + u$ and $\xi_3 = xu$. Finally, we homogenize the formulas with respect to $\xi_1 = 1$, ξ_2 , ξ_3 , ξ_4 . One first obtains

$$\kappa(\xi) = \left(\tilde{p}_0(\xi) : \ldots : \tilde{p}_5(\xi)\right),\,$$

where

(3.4)
$$\tilde{p}_j(\xi) = \alpha_j K_2 + \beta_j (K_1 \xi_4 + K_0), \text{ for } 0 \le j \le 5.$$

Here α_j and β_j are homogeneous forms in ξ of degree 4 and 2, respectively, and the K_j are those in (2.2). Taking $p_j(\xi) = (\tilde{p}_j(\xi) - \beta_j K)/K_2 = \alpha_j - \beta_j \xi_4^2$, we obtain formulas of degree 4 for κ , which will be defined also for $K_2 = 0$, extending κ to images of divisor classes $[\mathfrak{X}] = [(x, y) + (u, v)]$ with x = u and $y = v \neq 0$. However, the formulas do not work for points with F'(x) = 0 and for the image of $[2\infty^+]$. We will treat the case y = 0 or v = 0 in connection with nodes and tropes.

Polynomial definition of κ .

$$p_{0} = -f_{3}f_{6}\xi_{1}\xi_{3}^{3} + 1/2f_{5}^{2}\xi_{2}\xi_{3}^{3} - 2\xi_{3}^{3}f_{4}f_{6}\xi_{2} + 2\xi_{3}^{2}\xi_{1}^{2}f_{1}f_{6} - \xi_{3}^{2}\xi_{1}^{2}f_{5}f_{2}$$

$$-2\xi_{3}^{2}\xi_{1}\xi_{2}f_{6}f_{2} - 1/2\xi_{3}^{2}\xi_{1}f_{5}\xi_{4} - 1/2\xi_{3}^{2}\xi_{1}\xi_{2}f_{5}f_{3} - \xi_{3}^{2}\xi_{2}^{2}f_{6}f_{3} - 2\xi_{3}^{2}\xi_{2}f_{6}\xi_{4}$$

$$-1/2\xi_{3}f_{3}\xi_{4}\xi_{1}^{2} - 3/2\xi_{3}\xi_{1}^{2}\xi_{2}f_{5}f_{1} - \xi_{3}\xi_{2}f_{4}\xi_{4}\xi_{1} - 3\xi_{3}\xi_{1}\xi_{2}^{2}f_{6}f_{1}$$

$$-1/2\xi_{3}\xi_{2}^{2}f_{5}\xi_{4} + f_{1}f_{2}\xi_{1}^{4} + \xi_{1}^{3}\xi_{2}f_{1}f_{3} + 3/2\xi_{1}^{3}f_{1}\xi_{4} + \xi_{2}^{2}f_{4}f_{1}\xi_{1}^{2} + \xi_{2}^{3}f_{5}f_{1}\xi_{1}$$

$$+\xi_{2}^{4}f_{1}f_{6} - 1/2\xi_{1}\xi_{2}\xi_{4}^{2}$$

$$p_{1} = 2\xi_{1}^{4}f_{2}^{2} - 2\xi_{3}\xi_{1}\xi_{2}^{2}f_{6}f_{2} + 1/2\xi_{1}^{2}\xi_{2}^{2}f_{5}f_{1} - 1/2\xi_{1}^{4}f_{3}f_{1} + 2\xi_{2}^{4}f_{2}f_{6} + 3\xi_{1}^{3}\xi_{4}f_{2}$$

$$+ 1/2\xi_{3}f_{3}^{2}\xi_{1}^{3} + 1/2\xi_{2}^{3}f_{5}\xi_{4} + \xi_{3}\xi_{1}^{2}\xi_{2}f_{4}f_{3} - 1/2\xi_{3}^{2}\xi_{2}^{2}f_{5}^{2} + 3/2\xi_{3}\xi_{1}\xi_{2}^{2}f_{5}f_{3}$$

$$+ 2\xi_{3}^{2}f_{4}f_{6}\xi_{2}^{2} - \xi_{3}\xi_{1}^{2}\xi_{2}f_{5}f_{2} + \xi_{3}\xi_{2}^{2}f_{6}\xi_{4} + 2\xi_{3}\xi_{2}^{3}f_{6}f_{3} + \xi_{1}^{2}\xi_{4}^{2} + 2\xi_{1}^{3}\xi_{2}f_{2}f_{3}$$

$$- \xi_{3}\xi_{1}^{2}\xi_{2}f_{6}f_{1} + 2\xi_{1}^{2}f_{2}f_{4}\xi_{2}^{2} + 3/2\xi_{1}^{2}\xi_{2}f_{3}\xi_{4} + \xi_{1}f_{4}\xi_{4}\xi_{2}^{2} + \xi_{1}\xi_{2}^{3}f_{6}f_{1} + 2\xi_{1}\xi_{2}^{3}f_{5}f_{2}$$

$$+ 2\xi_{3}^{2}\xi_{1}^{2}f_{6}f_{2} - 1/2\xi_{3}^{2}\xi_{1}^{2}f_{5}f_{3} + \xi_{3}^{2}\xi_{4}f_{6}\xi_{1}$$

$$p_{2} = 2\xi_{1}^{2}\xi_{2}^{2}f_{4}f_{3} - f_{6}f_{5}\xi_{1}\xi_{3}^{3} + \xi_{3}^{2}\xi_{1}^{2}f_{3}f_{6} - \xi_{3}^{2}\xi_{1}^{2}f_{4}f_{5} + \xi_{3}^{2}\xi_{1}f_{5}^{2}\xi_{2}$$

$$+ \xi_{3}f_{1}f_{6}\xi_{1}^{3} + \xi_{3}\xi_{1}^{3}f_{3}f_{4} - 2\xi_{3}f_{5}f_{2}\xi_{1}^{3} + 2\xi_{3}\xi_{1}^{2}\xi_{2}f_{4}^{2} - 2\xi_{3}f_{5}\xi_{4}\xi_{1}^{2}$$

$$- \xi_{1}^{4}f_{1}f_{4} + 2\xi_{1}^{4}f_{3}f_{2} - 2\xi_{3}\xi_{2}f_{6}f_{2}\xi_{1}^{2} - 5\xi_{3}\xi_{2}^{2}f_{6}f_{3}\xi_{1} + 2\xi_{3}\xi_{1}\xi_{2}^{2}f_{5}f_{4}$$

$$- 3\xi_{3}f_{6}\xi_{4}\xi_{2}\xi_{1} - 3\xi_{3}\xi_{2}f_{5}f_{3}\xi_{1}^{2} + 2\xi_{2}^{4}f_{3}f_{6} + \xi_{2}^{3}f_{6}\xi_{4} + 2f_{3}\xi_{4}\xi_{1}^{3} + 2\xi_{2}f_{3}^{2}\xi_{1}^{3}$$

$$+ 2\xi_{2}^{3}f_{5}f_{3}\xi_{1} - \xi_{2}f_{5}f_{1}\xi_{1}^{3} - \xi_{2}^{2}f_{6}f_{1}\xi_{1}^{2} + \xi_{2}^{2}f_{5}\xi_{4}\xi_{1} + \xi_{2}f_{4}\xi_{4}\xi_{1}^{2} + 2\xi_{3}\xi_{2}^{3}f_{6}f_{4}$$

$$+ \xi_{3}^{2}\xi_{2}^{2}f_{6}f_{5} - 4\xi_{3}^{2}\xi_{1}f_{4}f_{6}\xi_{2}$$

$$p_{3} = -2f_{6}^{2}\xi_{1}\xi_{3}^{3} - \xi_{3}^{2}\xi_{1}^{2}f_{5}^{2} + 2\xi_{3}^{2}\xi_{1}^{2}f_{4}f_{6} - \xi_{3}^{2}\xi_{2}f_{6}f_{5}\xi_{1} + 2\xi_{3}^{2}f_{6}^{2}\xi_{2}^{2}$$

$$+\xi_{3}\xi_{1}^{3}f_{5}f_{3} - 2\xi_{3}\xi_{1}^{3}f_{2}f_{6} - \xi_{3}\xi_{1}^{2}\xi_{2}f_{6}f_{3} - 2\xi_{3}\xi_{1}^{2}f_{6}\xi_{4} + 2\xi_{3}\xi_{1}\xi_{2}^{2}f_{5}^{2}$$

$$-4\xi_{3}\xi_{1}f_{4}f_{6}\xi_{2}^{2} + 2\xi_{3}f_{6}f_{5}\xi_{2}^{3} - \xi_{1}^{4}f_{1}f_{5} + 2\xi_{1}^{4}f_{4}f_{2} + 2\xi_{1}^{3}f_{4}\xi_{4} + 2\xi_{1}^{3}\xi_{2}f_{4}f_{3}$$

$$-\xi_{1}^{3}\xi_{2}f_{6}f_{1} + \xi_{1}^{2}\xi_{2}f_{5}\xi_{4} + 2\xi_{1}^{2}f_{4}^{2}\xi_{2}^{2} + \xi_{1}\xi_{2}^{2}f_{6}\xi_{4} + 2\xi_{1}\xi_{2}^{3}f_{5}f_{4} + 2\xi_{2}^{4}f_{6}f_{4}$$

$$p_4 = \xi_3^2 \xi_1^2 f_6 f_5 - 2\xi_3^2 \xi_2 f_6^2 \xi_1 + \xi_3 f_3 f_6 \xi_1^3 - 2\xi_3 \xi_2 f_5^2 \xi_1^2 + 2\xi_3 f_4 f_6 \xi_2 \xi_1^2 - 2\xi_3 \xi_2^2 f_6 f_5 \xi_1 + 2\xi_3 f_6^2 \xi_2^3 - f_6 f_1 \xi_1^4 + 2f_5 f_2 \xi_1^4 + 2f_5 \xi_4 \xi_1^3 + 2\xi_2 f_5 f_3 \xi_1^3 + 2\xi_2^2 f_5 f_4 \xi_1^2 + \xi_2 f_6 \xi_4 \xi_1^2 + 2\xi_2^3 f_5^2 \xi_1 + 2\xi_2^4 f_6 f_5$$

$$p_5 = 2(f_6\xi_1^2\xi_3^2 - \xi_3f_5\xi_2\xi_1^2 - 2\xi_3f_6\xi_2^2\xi_1 + f_2\xi_1^4 + \xi_1^3\xi_4 + \xi_1^3f_3\xi_2 + f_4\xi_2^2\xi_1^2 + \xi_2^3f_5\xi_1 + f_6\xi_2^4)f_6.$$

Let the point $(p_0 : \ldots : p_5)$ be represented by $P(X) = \sum_{i=0}^5 p_i X^i$ and let

(3.5)
$$g_i(X) = 1 - 2\frac{P_i(X)}{P_i(\theta_i)}, \quad i = 1, \dots, 6$$

where $P_i(X)$ is defined by (3.1). We see that $g_i(\theta_j) = (-1)^{\delta_{ij}}$, so $g_i(X)^2 \equiv 1 \mod F(X)$. There are 6 commuting involutions $\varepsilon^{(i)}$ of \mathcal{S} , defined as follows:

(3.6)
$$\varepsilon^{(i)}(P(X)) = g_i(X)P(X) \mod F(X), \quad i = 1, \dots, 6.$$

In terms of coordinates π_j , one has

(3.7)
$$\varepsilon^{(i)}(\pi_j) = (-1)^{\delta_{ij}} \pi_j.$$

Definition 3.4. We define Inv(S) to be the group of 32 commuting involutions of S generated by the $\varepsilon^{(i)}$.

The $\mathbf{p} = (p_0 : p_1 : 0 : 0 : 0 : 0)$ form a rational line $\Delta_0 \subset \mathcal{S}$. We shall often write $p_0 + p_1 X \in \Delta_0$. Acting on Δ_0 by the involutions gives 31 further lines.

Notation 3.5. We denote:

$$\Delta_i = \varepsilon^{(i)}(\Delta_0), \quad \Delta_{ij} = \varepsilon^{(i)} \circ \varepsilon^{(j)}(\Delta_0) \quad \text{and} \quad \Delta_{ijk} = \varepsilon^{(i)} \circ \varepsilon^{(j)} \circ \varepsilon^{(k)}(\Delta_0).$$

The main result in this section describes how the nodes of \mathcal{K} and \mathcal{K}^* blow up.

Lemma 3.6. The map κ blows up the node $N_0 = (0 : 0 : 0 : 1)$ of \mathcal{K} into the line Δ_0 and the 15 nodes N_{ij} into the lines Δ_{ij} . The tropes T_i and T_{ijk} blow up by κ^* into the lines Δ_i and Δ_{ijk} .

Proof. The node N_0 corresponds to the canonical class, so we consider divisors of the type $\mathfrak{X} = (x, y) + (u, v)$ with u = x + h, h small and $v \approx -y \neq 0$. Then the local behavior of the Kummer coordinates is $\xi_1 = 1$, $\xi_2 = 2x + h \approx 2x$, $\xi_3 = x(x+h) \approx x^2$ and

$$\xi_4 = \frac{F_0(x, x+h) - 2yv}{h^2} \approx \frac{4y^2}{h^2}$$

Replacing this in the formulas for κ and clearing denominators, then taking the limit as $h \to 0$ and dividing by $y^4 \neq 0$, we obtain

$$\begin{split} \kappa(\xi) &\approx (-16xy^4:16y^4:0:0:0:0) \\ &\approx (-x:1:0:0:0:0). \end{split}$$

Note that $\Delta_0 \cap \Delta_i = (-\theta_i : 1 : 0 : 0 : 0 : 0)$, since for $(X - \theta_i) \in \Delta_0$ we have

$$\varepsilon^{(i)}(X-\theta_i) \equiv g_i(X)(X-\theta_i) \equiv (X-\theta_i) \mod F(X).$$

We now show that $\Delta_0 \cap \Delta_{ij} = \emptyset$ for $i \neq j$. Indeed, the intersection point \mathfrak{p} should be invariant by $\varepsilon^{(i)} \circ \varepsilon^{(j)}$. A polynomial P(X) represents such a point iff

$$\alpha P(X) \equiv g_i(X)g_j(X)P(X) \mod F(X) \text{ for some } \alpha \in \bar{k}$$

iff
$$F(X) \mid P(X)(\alpha - g_i(X)g_j(X)).$$

Replacing X by the roots of F(X) one sees that P(X) must have at least two roots among θ_k , so it must be of degree at least 2 and therefore cannot represent a point on Δ_0 . Similarly, $\Delta_0 \cap \Delta_{ijk} = \emptyset$ for $i \neq j \neq k$.

The six Δ_i are strict transforms of the conics Γ_i cut on \mathcal{K} by the tropes T_i . To see this and to define κ for points corresponding to divisors $\mathfrak{X} = \{x, y\} + \{\theta_i, 0\}$ with $y \neq 0$, write $F(X) = f_6(X - \theta_i)P_i(X)$. From this we get formulas for f_k , $k = 0, \ldots, 6$ depending on θ_i and $h_{ij}, j = 0, \ldots, 5$, the coefficients of $P_i(X)$, which we plug into $\xi_4 = (F_0(x, \theta_i)/(x - \theta_i)^2)$. We substitute then $\xi_1 = 1, \xi_2 = x + \theta_i$,

 $\xi_3 = x\theta_i$ and ξ_4 in the formulas for κ . On multiplying by $(x - \theta_i)^2/(f_6^2 P_i(x))$ (note that $P_i(x) \neq 0$), we obtain

(3.8)
$$P(X) = 2(x - \theta_i)P_i(X) + P_i(\theta_i)(X - x),$$

that is,

(3.9)
$$p_{0} = 2h_{i0}(x - \theta_{i}) - P_{i}(\theta_{i})x, p_{1} = 2h_{i1}(x - \theta_{i}) + P_{i}(\theta_{i}), p_{j} = 2h_{ij}(x - \theta_{i}) \quad \text{for } 2 \le j \le 5.$$

Equation (2.3) shows that the points $(1 : x + \theta_i : x\theta_i : \xi_4)$ belong to the conic Γ_i . Formulas (3.9) give parametric equations (in x) of the strict transform of Γ_i by κ . To confirm that this is Δ_i , one verifies that

$$P(X) \equiv P_i(\theta_i)g_i(X)(X-x) \mod F(X).$$

Applying the results in Section 4 and especially Corollary 4.12, one concludes that:

1) the tropes T_i considered as singular points of \mathcal{K}^* , blow up by κ^* into Δ_i ;

- 2) each of the fifteen N_{ij} blows up into Δ_{ij} ;
- 3) the tropes T_{ijk} , $i \neq j \neq k$ blow up into Δ_{ijk} ;

4) the ten Δ_{ijk} $(i \neq j \neq k)$ are strict transforms of the ten conics cut on \mathcal{K} by the tropes (planes) not containing N_0 . Each of them intersects six Δ_{ij} since each node is on six tropes.

4. Line complexes

Let $u = (u_1 : u_2 : u_3 : u_4)$ and $v = (v_1 : v_2 : v_3 : v_4)$ be distinct points in \mathbb{P}^3 . Put $p_{ij} = u_i v_j - u_j v_i$. The Grassmann coordinates of the line $\langle u, v \rangle \subset \mathbb{P}^3$ are

$$\mathbf{p} = (p_{43} : p_{24} : p_{41} : p_{21} : p_{31} : p_{32}) = (X_1 : \ldots : X_6).$$

The Grassmannian quadric $\mathcal{G} \subset \mathbb{P}^5$, representing the lines in \mathbb{P}^3 has the equation:

$$G(X_1, \ldots, X_6) = 2X_1X_4 + 2X_2X_5 + 2X_3X_6 = 0.$$

Definition 4.1. A line complex of degree d is a set of lines in \mathbb{P}^3 whose Grassmann coordinates satisfy a homogeneous equation $Q(X_1, \ldots, X_6) = 0$ of degree d.

If d = 1, this is called a linear complex, and if d = 2, it is a quadratic complex.

A line $L \in \mathcal{G}$ parametrizes a pencil of lines in \mathbb{P}^3 . The lines of a pencil L all pass through a point $\mathfrak{f}(L) = u$, called the *focus* of the pencil, and lie in one plane $\mathfrak{h}(L) = \pi_u$, the *plane* of the pencil.

All lines in a linear complex \mathcal{L} passing through a given point u (respectively, lying in a plane π), form a pencil L_u (respectively, L_{π}). Each linear complex \mathcal{L} establishes a *correspondence* between points and planes in \mathbb{P}^3 ,

$$I(u) = \mathfrak{h}(L_u), \quad I(\pi) = \mathfrak{f}(L_\pi), \quad I^2 = 1,$$

which is also defined for lines; if $l \subset \mathbb{P}^3$ is the line $\langle u, u' \rangle$, then $I(l) = I(u) \cap I(u')$. The line I(l) is the *polar* line of l with respect to the given linear complex.

Definition 4.2. Two linear complexes are called apolar if the correspondences they define commute.

Let H be any quadratic form in six variables such that the quadrics G = 0and H = 0 intersect transversely, and denote by $\mathcal{H} = \{x \in \mathbb{P}^5 \mid H(x) = 0\}$. Let $\mathcal{W} = \mathcal{G} \cap \mathcal{H}$ and \mathcal{A} = the set of lines on \mathcal{W} . The points in \mathcal{W} represent the lines in \mathbb{P}^3 whose Grassmann coordinates \mathfrak{p} satisfy $H(\mathfrak{p}) = 0$. A line $L \in \mathcal{A}$ represents a pencil of lines in \mathbb{P}^3 of the quadratic complex defined by H.

Definition 4.3. The Kummer surface $\mathcal{K} \subset \mathbb{P}^3$ associated to the quadratic complex \mathcal{H} is the locus of focuses of such pencils: $\mathcal{K} = \{\mathfrak{f}(L) \mid L \in \mathcal{A}\}.$

Definition 4.4. The dual Kummer surface $\mathcal{K}^* \subset \mathbb{P}^{3^{\checkmark}}$ associated to the quadratic complex \mathcal{H} is the locus of planes of such pencils.

From now on we suppose $f_6 = 1$.

Lemma 4.5. For any curve C of genus 2, the Kummer surface belonging to the curve C given by (2.1) coincides with the Kummer surface just defined, if one takes the quadratic complex H to be given by

$$H = -4X_1X_5 - 4X_2X_6 - X_3^2 + 2f_5X_3X_6 + 4f_0X_4^2 + 4f_1X_4X_5 + 4f_2X_5^2 + 4f_3X_5X_6 + (4f_4 - f_5^2)X_6^2.$$

Proof. See [3], Lemma 17.3.1 and pages 182–183.

If a point $\xi \in \mathbb{P}^3$ is the focus of the pencil corresponding to the line $L_{\xi} \in \mathcal{A}$, then L_{ξ} lies in the plane $\Pi_{\xi} \subset \mathcal{G}$ corresponding to lines in \mathbb{P}^3 passing through ξ . But then the conic $\Pi_{\xi} \cap \mathcal{H}$ contains L_{ξ} , so it is degenerate; Π_{ξ} is tangent to \mathcal{H} and $\Pi_{\xi} \cap \mathcal{H} = L_{\xi} \cup L'_{\xi}$. The lines of the quadratic complex passing through ξ are in the two pencils L_{ξ} and L'_{ξ} , each with focus ξ , lying in the planes π_{ξ} and π'_{ξ} in \mathbb{P}^3 . The line $l_{\xi} = \pi_{\xi} \cap \pi'_{\xi}$ is represented on \mathcal{G} by the point $\mathfrak{p}_{\xi} = L_{\xi} \cap L'_{\xi}$ and is called a singular line of the quadratic complex.

If $L_{\xi} \neq L'_{\xi}$ the pencils are distinct and ξ is a simple point of the Kummer; there is a one-to-one correspondence $\xi \leftrightarrow \mathfrak{p}_{\xi}$. However, if $L_{\xi} = L'_{\xi}$, then $\pi_{\xi} = \pi'_{\xi}$ and all the lines in L_{ξ} are singular lines. The point ξ is a singular point of the Kummer, because the map $\mathfrak{f} : \mathcal{A} \longrightarrow \mathcal{K}$ is algebraic. Therefore, the variety Σ parametrizing singular lines is a desingularization of the Kummer.

Definition 4.6. The birational map $\kappa_1 : \mathcal{K} \dashrightarrow \Sigma$ is defined by $\kappa_1(\xi) = \mathfrak{p}_{\xi}$.

Definition 4.7. The birational map $\kappa_1^* : \mathcal{K}^* \dashrightarrow \Sigma$ associates to a plane π tangent to \mathcal{K} the intersection point of the lines in \mathcal{A} parametrizing the two pencils in \mathcal{H} contained in π .

The maps κ_1^{-1} and κ_1^{*-1} extend to minimal desingularizations $\kappa_1^{-1} : \Sigma \longrightarrow \mathcal{K}$ and $\kappa_1^{*-1} : \Sigma \longrightarrow \mathcal{K}^*$ (see [3] and [6]). The following is proved in [3], page 181:

Lemma 4.8. The surface \mathcal{K}^* is the projective dual of \mathcal{K} ; that is, if $\xi = \mathfrak{f}(L) \in \mathcal{K}$, then $\eta = \mathfrak{h}(L) \in \mathcal{K}^*$ is the tangent plane of \mathcal{K} at ξ . Therefore $\kappa_1(\xi) = \kappa_1^*(\eta)$.

Denote by $G(\vec{X}, \vec{Y})$ the bilinear form associated to the Grassmannian G. Make the change of coordinates

(4.1)
$$\zeta_i = \frac{G(\overline{X}, \overrightarrow{v}(\theta_i))}{\sqrt{\omega_i}},$$

with vectors $\overrightarrow{v}(\theta_i)$ as in [3] formula (17.4.3). Let \mathcal{K} be the Kummer surface associated to the quadratic complex H of Lemma 4.5. By [6], Section 31 or [3] formula (17.4.2), a minimal desingularization of \mathcal{K} is the K3 surface in \mathbb{P}^5 :

$$\Sigma = \Sigma_0 \cap \Sigma_1 \cap \Sigma_2$$
, where Σ_i : $\sum_j \theta_j^i \zeta_j^2 = 0$ for $i = 0, 1, 2$.

Proposition 4.9. There is an explicit isomorphism $\Theta : \Sigma \longrightarrow S$.

Proof. Let $\Theta: \Sigma \longrightarrow \mathcal{S}$ be defined by

(4.2)
$$\Theta(\zeta_1:\ldots:\zeta_6) = \left(\frac{\zeta_1}{\sqrt{\omega_1}}:\ldots:\frac{\zeta_6}{\sqrt{\omega_6}}\right) = (\pi_1:\cdots:\pi_6).$$

To pass to variables X_j and p_j , recall that $P(X) = \sum_{j=0}^{5} p_j X^j$. By (3.2) and (4.1):

$$\frac{P(\theta_i)}{\omega_i} = \pi_i = \frac{\zeta_i}{\sqrt{\omega_i}} = \frac{G(\vec{X}, \vec{v}(\theta_i))}{\omega_i}.$$

Now, as polynomials in X, we have $G(\vec{X}, \vec{v}(X)) = P(X)$, because both have degree 5 and agree on the six θ_i . Explicit formulas for Θ are:

$$p_0 = X_1 + f_1 X_4 \qquad p_2 = X_3 + 2f_4 X_5 + 2f_3 X_4 + f_5 X_6 \qquad p_4 = 2f_5 X_4 + 2X_5 \\ p_1 = X_2 + 2f_2 X_4 + f_3 X_5 \qquad p_3 = 2f_4 X_4 + 2f_5 X_5 + 2X_6 \qquad p_5 = 2X_4. \qquad \Box$$

Proposition 4.10. Denoting by κ^{-1} and κ_1^{-1} the blow-downs from S, respectively, Σ to \mathcal{K} one has $\kappa_1^{-1} = \kappa^{-1} \circ \Theta$.

Proof. Pick a point $\xi \in \mathbb{P}^3$ and write the equations of the plane $\Pi_{\xi} \subset \mathcal{G}$ of lines through ξ (see equations (4.7) of [8]). Take \mathcal{H} to be defined as in Lemma 4.5. Now, Π_{ξ} is tangent to \mathcal{H} iff the intersection consists of two lines: $\Pi_{\xi} \cap \mathcal{H} = L_{\xi} \cup L'_{\xi}$. Computing in terms of ξ the coordinates of $\mathfrak{p}_{\xi} = L_{\xi} \cap L'_{\xi}$, we find homogeneous formulas for X_i in ξ_i of degree 4: $\mathfrak{p}_{\xi} = (X_1(\xi) : \ldots : X_6(\xi)) = \kappa_1(\xi)$. Comparing $\kappa(\xi) = (p_0(\xi) : \ldots : p_5(\xi))$ from Lemma 3.3 with

$$\Theta \circ \kappa_1(\xi) = (\hat{p}_0(\xi) : \dots : \hat{p}_5(\xi)) : \mathcal{K} \longrightarrow \mathcal{S}$$

yields $\hat{p}_i p_5 - \hat{p}_5 p_i = \delta_i K$, with K given by (2.2), for δ_i a homogeneous polynomial in ξ .

Associated with a quadratic complex $\mathcal{H} : H = 0$ there is a set of 6 mutually apolar linear complexes \mathcal{L}_k , such that the polar of any line in \mathcal{H} with respect to \mathcal{L}_k is in \mathcal{H} . If G and H are written in diagonal form, these complexes are $\mathcal{L}_k : \zeta_k = 0$ for $k = 1, \ldots, 6$. The action of the correspondences I_k on lines in \mathbb{P}^3 translates in coordinates $\zeta = (\zeta_1 : \ldots : \zeta_6)$ by

$$(4.3) I_k(\zeta_i) = (-1)^{\delta_{ik}} \zeta_i$$

which restricts to Σ . The Kummer surface is determined by \mathcal{H} , so it is invariant under the transformation I_k . The set of nodes and tropes is invariant (see [6], Section 30).

Proposition 4.11. Let W_k be as in Lemma 2.4. For any k, the map I_k is the unique automorphism of Σ such that $\kappa_1^{*-1} \circ I_k = W_k \circ \kappa_1^{-1}$.

Proof. Let $\xi \in \mathcal{K}$ be a simple point and denote $\mathfrak{p}_{\xi} = \kappa_1(\xi)$. For a subset $V \subset \mathcal{G}$, put $I_k(V) = \{I_k(l) \in \mathcal{G} \mid l \in V\}$. The pencils $I_k(L_{\xi})$ and $I_k(L'_{\xi})$ are both contained in the polar plane of ξ with respect to \mathcal{L}_k , which by Lemma 4.13 is $W_k(\xi)$. The plane in \mathbb{P}^5 parametrizing lines in $W_k(\xi)$ is therefore tangent to \mathcal{H} at $I_k(L_{\xi}) \cap I_k(L'_{\xi}) = I_k(L_{\xi} \cap L'_{\xi}) = I_k(\mathfrak{p}_{\xi}) = I_k \circ \kappa_1(\xi)$. By definition of κ_1^* we have $\kappa_1^*(W_k(\xi)) = I_k \circ \kappa_1(\xi)$.

The following corollary illustrates how the projective duality (over $k(\theta_k)$) between \mathcal{K} and \mathcal{K}^* lifts to \mathcal{S} .

Corollary 4.12. For any k, the map $\varepsilon^{(k)}$ is the unique automorphism of S such that $\kappa^{*^{-1}} \circ \varepsilon^{(k)} = W_k \circ \kappa^{-1}$.

Proof. Let $\xi \in \mathcal{K}$ and $\eta \in \mathcal{K}^*$ be dual. We have:

(4.4)
$$\Theta \circ \kappa_1^*(\eta) \stackrel{(4.8)}{=} \Theta \circ \kappa_1(\xi) \stackrel{(4.10)}{=} \kappa(\xi) \stackrel{(3.2)}{=} \kappa^*(\eta).$$

Note that $\Theta \circ I_k \circ \Theta^{-1} = \varepsilon^{(k)}$, by (4.2), (4.3) and (3.7). Therefore,

$$\kappa^{*-1} \circ \varepsilon^{(k)} \stackrel{(4.4)}{=} \kappa_1^{*-1} \circ \Theta^{-1} \circ \Theta \circ I_k \circ \Theta^{-1} \stackrel{(4.11)}{=} W_k \circ \kappa_1^{-1} \circ \Theta^{-1} \stackrel{(4.10)}{=} W_k \circ \kappa^{-1}.$$

This is summarized in the following diagram, where the arrows to \mathcal{K} and \mathcal{K}^* are the minimal desingularizations:

Now Corollary 4.12 is useful for finding explicit formulas for κ^* , because

$$\kappa^* = \kappa^* \circ W_i \circ \kappa^{-1} \circ \kappa \circ W_i^{-1} = \varepsilon^{(i)} \circ \kappa \circ W_i^{-1}$$

on an open dense set in \mathcal{K}^* . We find formulas for W_i in [8], Lemma 4.13. The resulting formulas for κ^* are huge and not listed in this paper, since on a given example it is easier to apply successively each map involved.

Lemma 4.13. For any point $\xi \in \mathbb{P}^3$ the plane with dual coordinates $W_i(\xi)$ is the polar plane of ξ with respect to \mathcal{L}_i .

Proof. See [8], Lemma 4.14.

5. Linear automorphisms of S

Keeping Notation 3.5, we let

(5.1)
$$p_{i} = \Delta_{0} \cap \Delta_{i},$$
$$p_{ij} = \Delta_{i} \cap \Delta_{ij} = \varepsilon^{(i)}(p_{j}),$$
$$p_{ijk} = \Delta_{ij} \cap \Delta_{ijk} = \varepsilon^{(i)}(p_{jk}).$$

There are no other lines on \mathcal{S} ([5], page 775), so this is the whole structure of line intersections on \mathcal{S} . Now let $GL(\mathcal{S})$ be the group of linear automorphisms of \mathcal{S} .

Lemma 5.1. Let $A, B \in GL(S)$ such that $A_{|\Delta_0} = B_{|\Delta_0}$. Then A = B.

1158

Proof. Let $I \in GL(\mathcal{S})$ be the identity. If $A \in GL(\mathcal{S})$ and $A_{|\Delta_0} = I_{|\Delta_0}$, then A fixes the \mathfrak{p}_i , so invaries the Δ_i . But then by A invaries also Δ_{ij} , the unique line other than Δ_0 which meets Δ_i and Δ_j , so A fixes \mathfrak{p}_{ij} , $j = 1, \ldots, 6$. Hence $A_{|\Delta_i} = I_{|\Delta_i}$. Similarly, one sees that A is the identity on any of the 32 lines on \mathcal{S} , so A = I. \Box

Let $A \in \operatorname{GL}(\mathcal{S})$. Since $A(\Delta_0)$ is a line, by Lemma 5.1 there exists a unique involution $\varepsilon \in \operatorname{Inv}(\mathcal{S})$ such that $\varepsilon \circ A(\Delta_0) = \Delta_0$. We associate to A the permutation $\sigma \in S_6$ such that

(5.2)
$$\varepsilon \circ A(\mathfrak{p}_i) = \mathfrak{p}_{\sigma(i)} \quad \text{for } i = 1, \dots, 6.$$

Note that $\sigma = \operatorname{id} \operatorname{iff} \varepsilon \circ A_{|\Delta_0} = I_{|\Delta_0} \operatorname{iff} \varepsilon \circ A = I$ (by Lemma 5.1) iff $A \in \operatorname{Inv}(\mathcal{S})$.

Definition 5.2. $\operatorname{GL}_0(\mathcal{S})$ is the subgroup of $\operatorname{GL}(\mathcal{S})$ of linear automorphisms A such that $A(\Delta_0) = \Delta_0$.

Lemma 5.3. Let $A \in GL(S)$ and $\sigma \in S_6$ be the permutation associated to A by (5.2). Then, for any $1 \le i \le 6$ we have:

(5.3)
$$A \circ \varepsilon^{(i)} = \varepsilon^{(\sigma(i))} \circ A.$$

Proof. Let $B = \varepsilon \circ A$. Then $B(\Delta_0) = \Delta_0$ and $B(\mathfrak{p}_i) = \mathfrak{p}_{\sigma(i)}$, so $B(\Delta_i) = \Delta_{\sigma(i)}$. The unique line cutting $\Delta_{\sigma(i)}$ and $\Delta_{\sigma(j)}$ is $\Delta_{\sigma(i)\sigma(j)}$, hence $B(\Delta_{ij}) = \Delta_{\sigma(i)\sigma(j)}$. Then

$$B(\mathfrak{p}_{ij}) = B(\Delta_i \cap \Delta_{ij}) = B(\Delta_i) \cap B(\Delta_{ij}) = \Delta_{\sigma(i)} \cap \Delta_{\sigma(i)\sigma(j)} = \mathfrak{p}_{\sigma(i)\sigma(j)}$$

Now one sees that $(\varepsilon \circ A)^{-1} \circ \varepsilon^{(\sigma(i))} \circ (\varepsilon \circ A)$ acts like $\varepsilon^{(i)}$ on \mathfrak{p}_j . By Lemma 5.1 and knowing that $\operatorname{Inv}(\mathcal{S})$ is commutative, we conclude $A \circ \varepsilon^{(i)} = \varepsilon^{(\sigma(i))} \circ A$. \Box

Proposition 5.4. Let ψ : $GL(S) \longrightarrow GL_0(S)$ be the map $A \mapsto \varepsilon \circ A$ defined by formula (5.2). We have an exact sequence of groups

$$1 \longrightarrow \operatorname{Inv}(\mathcal{S}) \longrightarrow GL(\mathcal{S}) \xrightarrow{\psi} GL_0(\mathcal{S}) \longrightarrow 1.$$

By Proposition 5.4, $Inv(S) = ker(\psi)$ is a normal subgroup of GL(S).

Corollary 5.5. For any linear automorphism A of S not in Inv(S), the centralizer of A in Inv(S) is not equal to Inv(S).

We now show that $\operatorname{GL}_0(\mathcal{S})$ is in bijection with the group of linear automorphisms of Δ_0 which invariate the set $\{p_i, i = 1, \ldots, 6\}$. Lemma 5.1 and Proposition 5.6 below give necessary and sufficient conditions for the existence of nontrivial elements of $\operatorname{GL}_0(\mathcal{S})$. For the existence of noncommuting involutions of \mathcal{S} , see [8], Section 6.

Proposition 5.6. Let $\sigma \in S_6$ and let $B : \Delta_0 \longrightarrow \Delta_0$ be a linear automorphism of Δ_0 such that for $1 \leq i \leq 6$, we have $B(\mathfrak{p}_i) = \mathfrak{p}_{\sigma(i)}$. Then there exists a unique $A \in GL_0(\mathcal{S})$ such that $A_{|\Delta_0} = B$.

Proof. Suppose σ and B given. If A exists, it is unique by Lemma 5.1 and σ is the permutation associated to A defined by (5.2). Let \widetilde{A} be the linear operator of \mathcal{P}_5 (polynomials of degree ≤ 5) associated to A. Let $a, b, c, d \in \overline{k}$ such that

$$A(1) = aX + b \quad \text{and} \quad A(X) = cX + d.$$

After some linear algebra and using (5.3), we find that the image of a point $\mathfrak{p} \in \mathcal{S}$ represented by $P(X) = \sum_j \pi_j P_j(X)$ is

(5.4)
$$\widetilde{A}(P(X)) = \sum_{j} \underbrace{\left(\pi_{j} \frac{\omega_{j}}{\omega_{\sigma(j)}} (a\theta_{\sigma(j)} + b)\right)}_{\pi'_{\sigma(j)}} P_{\sigma(j)}(X)$$

We have to prove that the point $(\pi'_1 : \ldots : \pi'_6)$ satisfies equations (3.3). We show that $k_{\sigma(j)}\omega_{\sigma(j)}{\pi'}_{\sigma(j)}^2 = \alpha_j\omega_j\pi_j^2$ for a quadratic polynomial α_j in θ_j . We have:

$$k_{\sigma(j)}\omega_{\sigma(j)}{\pi'}_{\sigma(j)}^2 = k_{\sigma(j)}(a\theta_{\sigma(j)} + b)^2 \frac{\omega_j}{\omega_{\sigma(j)}}\omega_j\pi_j^2,$$

and then

$$\alpha_j = k_{\sigma(j)} (a\theta_{\sigma(j)} + b)^2 \frac{\omega_j}{\omega_{\sigma(j)}}$$

Write $\widetilde{A}(X - \theta_i)$ in two ways, using the fact that $A(\mathfrak{p}_i) = \mathfrak{p}_{\sigma(i)}$ or linearity of \widetilde{A} :

$$\mu_j(X - \theta_{\sigma(j)}) = \widetilde{A}(X - \theta_j) = cX + d - \theta_j(aX + b) \quad \text{where } \mu_j \in \overline{k} \,.$$

Replacing $X = \theta_{\sigma(j)}$, we obtain the formula

(5.5)
$$\theta_j = \frac{c\theta_{\sigma(j)} + d}{a\theta_{\sigma(j)} + b},$$

which gives the relations between the roots of F(X) necessary for the existence of the linear automorphism B.

Now, we calculate ω_j replacing each θ_j by the formula (5.5):

$$\omega_{j} = \prod_{i \neq j} (\theta_{i} - \theta_{j}) = \prod_{i \neq j} \left(\frac{c\theta_{\sigma(i)} + d}{a\theta_{\sigma(i)} + b} - \frac{c\theta_{\sigma(j)} + d}{a\theta_{\sigma(j)} + b} \right)$$
$$= \frac{1}{(a\theta_{\sigma(j)} + b)^{4}} \underbrace{\frac{1}{\prod_{i} (a\theta_{\sigma(i)} + b)}}_{\text{constant}} \prod_{i \neq j} \left((\theta_{\sigma(i)} - \theta_{\sigma(j)}) \underbrace{(bc - ad)}_{\text{constant}} \right)$$

Call γ the constant part of the equation:

(5.6)
$$\frac{\omega_j}{\omega_{\sigma(j)}} = \frac{\gamma}{\left(a\theta_{\sigma(j)} + b\right)^4}$$

Replacing (5.6) in α_j , we have:

$$\alpha_j = k_{\sigma(j)} (a\theta_{\sigma(j)} + b)^2 \frac{\gamma}{\left(a\theta_{\sigma(j)} + b\right)^4} = \gamma \frac{k_{\sigma(j)}}{\left(a\theta_{\sigma(j)} + b\right)^2}$$

To see that α_j is quadratic in θ_j (for each k_j), we use formula (5.5) to obtain:

$$a\theta_j - c = \frac{ad - bc}{a\theta_{\sigma(j)} + b} \quad \text{which gives the result for } k_{\sigma(j)} = 1;$$

$$a^2\theta_j^2 - c^2 = \frac{2ac(ad - bc)\theta_{\sigma(j)} + a^2d^2 - b^2c^2}{(a\theta_{\sigma(j)} + b)^2}$$
which gives the result for $k_{\sigma(j)} = \theta_{\sigma(j)};$

$$b\theta_j - d = \frac{(bc - ad)\theta_{\sigma(j)}}{a\theta_{\sigma(j)} + b}$$
 which gives the result for $k_{\sigma(j)} = \theta_{\sigma(j)}^2$. \Box

DUAL KUMMER SURFACE

References

- [1] A. Beauville, Surfaces algébriques complexes. Soc. Math. France, 1978. MR0485887 (58:5686)
- [2] P. Corn, Tate-Shafarevich Groups and K3-Surfaces, Math. Comp. 79 (2010), 563-581. MR2552241 (2011d:11139)
- J. W. S. Cassels and E. V. Flynn, Prolegomena to a Middlebrow Arithmetic of Curves of Genus 2. London Math. Soc. Lecture Note Series 230, Cambridge, 1996. MR1406090 (97i:11071)
- [4] E. V. Flynn, The Jacobian and formal group of a curve of genus 2 over an arbitrary ground field, Math. Proc. Cambridge Phil. Soc. 107 (1990), 425-441. MR1041476 (91b:14025)
- [5] Griffiths and Harris, Principles of Algebraic Geometry. John Wiley and Sons, New York, 1978. MR507725 (80b:14001)
- [6] R.W.H.T. Hudson, Kummer's Quartic Surface. Cambridge Univ. Press, 1990. MR1097176 (92e:14033)
- [7] A. Logan and R. van Luijk, Nontrivial elements of Sha explained through K3 surfaces, Math. Comp. 78 (2009), 441-483. MR2448716 (2010a:14048)
- [8] V.G. Lopez-Neumann and C. Manoil, Explicit computations on the desingularized Kummer surface, arXiv:0906.0790v1.

FACULDADE DE MATEMÁTICA, UNIVERSIDADE FEDERAL DE UBERLÂNDIA, MG, BRAZIL E-mail address: gonzalo@famat.ufu.br

Section de Mathématiques, Université de Genève, CP 64, 1211 Geneva 4, Switzerland $E\text{-mail}\ address:\ \texttt{constantin.manoil@edu.ge.ch}$

Current address: Collège Sismondi, 3 Chemin Rigot, 1202 Genève (Geneva), Switzerland