COMBINATORICS OF CREMONA MONOMIAL MAPS

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ABSTRACT. We study Cremona monomial maps using linear algebra, lattice theory and linear optimization methods. Among the results is a simple integer matrix theoretic proof that the inverse of a Cremona monomial map is also defined by monomials of fixed degree, and moreover, the set of monomials defining the inverse can be obtained explicitly in terms of the initial data. We present another method to compute the inverse of a Cremona monomial map based on integer programming techniques and the notion of a Hilbert basis. A neat consequence is drawn for the plane Cremona monomial group, in particular, the known result saying that a plane Cremona monomial map and its inverse have the same degree.

1. INTRODUCTION

The expression “birational combinatorics” has been introduced in [11] to mean the combinatorial theory of rational maps $\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{m-1}$ defined by monomials, along with natural integer arithmetic criteria for such maps to be birational onto their image varieties. As claimed there, both the theory and the criteria were intended to be a simple transcription of the initial geometric data. Yet another goal is to write characteristic-free results. Thus, here too one works over an arbitrary field in order that the theory be essentially independent of the nature of the field of coefficients, especially when dealing with square-free monomials.

In this paper, we stick to the case where $m = n$ and deal with Cremona maps. An important step has been silently taken for granted in the background of [11, Section 5.1.2], namely, that the inverse of a Cremona monomial map is also defined by monomials. To be fair this result can be obtained via the method of [11, Section 3] together with the criterion of [9]; however, the latter gives no hint on how to derive explicit data from the given ones. The main results of this paper provide methods for the computation of the inverse of a Cremona monomial map and the related invariants.

Here we add a few steps to the theory, by setting up a direct way to convert geometric results into numeric or combinatorial data regardless of the ground field nature. The conversion allows for an incursion into some of the details of the theory of plane Cremona maps defined by monomials. In particular, it is shown that the group of such maps under composition is completely understood without recurring to the known results about general plane Cremona maps. Thus, one shows that

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this group is generated by two basic monomial quadratic maps, up to reordering of variables in the source and the target. The result is not a trivial consequence of Noether's theorem since the latter requires composing with projective transformations, which is out of the picture here. Moreover, the known proofs of Noether's theorem (see, e.g., [1]) reduce to various special situations, passing through the celebrated de Jonquières maps which are rarely monomial.

The well-known result that a plane Cremona map and its inverse have the same degree is shown here for such monomial maps by an easy numerical counting. The argument for general plane Cremona maps is not difficult but requires quite a bit of geometric insight and preparation (see, e.g., [1, Proposition 2.1.12]).

Cremona monomial maps have been dealt with in [6] and in [7], but the methods and some of the goals are different and have not been drawn upon here. The group structure of Cremona monomial maps is studied in [6] by means of toric algebra with emphasis on the multiplicative structure, while in the present account we take up the, so to say, additive side of the problem by stressing the underlying integer arithmetic. In [7] the group of plane Cremona monomial maps is described by generators and relations and it is shown that this group (except that they allow certain coefficients) is generated by permutations and two degree 2 maps: hyperbolism and the Steiner involution (see Section 4).

2. INTEGER LINEAR ALGEBRA AND CREMONA INVERSES

In this section we use techniques from lattice theory and integer linear algebra to give an algorithm that computes the “Cremona inverse matrix” (see Definition 2.4) of a given d -stochastic square matrix A with entries in $\mathbb{N}$ whose determinant equals $\pm d$. A matrix is called d -stochastic if the sum of the entries of each column is equal to d . The algorithm is included inside the proof of Theorem 2.2.

Recall that if $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, its *support* is defined as $\text{supp}(a) = \{i \mid a_i \neq 0\}$. Note that we can write $a = a^+ - a^-$, where a^+ and a^- are two nonnegative vectors with disjoint support. The vectors a^+ and a^- are called the positive and negative part of a respectively. Following a familiar notation we write $|a| = a_1 + \dots + a_n$. The i th unit vector in $\mathbb{R}^n$ will be denoted by e_i .

We begin with a lemma whose proof uses lattice theory.

Lemma 2.1. *Let $v_1, \dots, v_n$ be a set of vectors in $\mathbb{N}^n$ such that $|v_i| = d \geq 1$ for all i and $\det(A) = \pm d$, where A is the $n \times n$ matrix with column vectors $v_1, \dots, v_n$. Then $A^{-1}(e_i - e_j) \in \mathbb{Z}^n$ for all i, j .*

Proof. Fixing indices i, j , there are $\lambda_1, \dots, \lambda_n$ in $\mathbb{Q}$ such that $A^{-1}(e_i - e_j) = \sum_{k=1}^n \lambda_k e_k$. Notice that $A^{-1}(e_i)$ is the i th column of A^{-1} . Set $\mathbf{1} = (1, \dots, 1)$. Since $\mathbf{1}A = d\mathbf{1}$, we get $\mathbf{1}/d = \mathbf{1}A^{-1}$. Therefore $|A^{-1}(e_i)| = |A^{-1}(e_j)| = 1/d$ and $\sum_k \lambda_k = 0$. Then we can write

$$A^{-1}(e_i - e_j) = \sum_{k=2}^n \lambda_k (e_i - e_1) \implies e_i - e_j = \sum_{k=2}^n \lambda_k (v_k - v_1).$$

Thus there is $0 \neq s \in \mathbb{N}$ such that $s(e_i - e_j)$ belong to $\mathbb{Z}\{v_1 - v_k\}_{k=2}^n$, the subgroup of $\mathbb{Z}^n$ generated by $\{v_1 - v_k\}_{k=2}^n$. By [11, Lemma 2.2 and Theorem 2.6], the quotient group $\mathbb{Z}^n / \mathbb{Z}\{v_1 - v_k\}_{k=2}^n$ is free, in particular, has no nonzero torsion elements. Then we can write

$$e_i - e_j = \eta_2(v_2 - v_1) + \dots + \eta_n(v_n - v_1),$$

for some η_i 's in $\mathbb{Z}$. Since $\mathbb{Z}\{v_1 - v_k\}_{k=2}^n$ is also free (of rank $n - 1$), the vectors $v_2 - v_1, \dots, v_n - v_1$ are linearly independent. Thus $\lambda_k = \eta_k \in \mathbb{Z}$ for all $k \geq 2$, hence ultimately $A^{-1}(e_i - e_j) \in \mathbb{Z}^n$. $\square$

Next we state our main result of an integer linear algebra nature. Its geometric translation to Cremona monomial maps and applications will be given in Section 4.

Theorem 2.2. *Let $v_1, \dots, v_n$ be a set of vectors in $\mathbb{N}^n$ such that $|v_i| = d \geq 1$ for all i and $\det(A) = \pm d$, where A is the $n \times n$ matrix with column vectors $v_1, \dots, v_n$. Then there are unique vectors $\beta_1, \dots, \beta_n, \gamma \in \mathbb{N}^n$ such that the following two conditions hold:*

- (a) $A\beta_i = \gamma + e_i$ for all i , where β_i, γ and e_i are regarded as column vectors;
- (b) The matrix B whose columns are $\beta_1, \dots, \beta_n$ has at least one zero entry in every row.

Moreover, $\det(B) = \pm(|\gamma| + 1)/d = \pm|\beta_i|$ for all i .

Proof. First we show the uniqueness. Assume that $\beta'_1, \dots, \beta'_n, \gamma'$ is a set of vectors in $\mathbb{N}^n$ such that: (a') $A\beta'_i = \gamma' + e_i$ for all i , and (b') The matrix B' whose column vectors are $\beta'_1, \dots, \beta'_n$ has at least one zero entry in every row. Let $\Delta = (\Delta_i)$ and $\Delta' = (\Delta'_i)$ be nonnegative vectors such that $A^{-1}(\gamma - \gamma') = \Delta' - \Delta$. Then from (a) and (a') we get

$$(2.1) \quad \beta_i - \beta'_i = A^{-1}(\gamma - \gamma') = \Delta' - \Delta, \quad \forall i \implies \beta_{ik} - \beta'_{ik} = \Delta'_k - \Delta_k, \quad \forall i, k,$$

where $\beta_i = (\beta_{i1}, \dots, \beta_{in})$ and $\beta'_i = (\beta'_{i1}, \dots, \beta'_{in})$. It suffices to show that $\Delta = \Delta'$. If $\Delta'_k > \Delta_k$ for some k , then, by (2.1), we obtain $\beta_{ik} > 0$ for $i = 1, \dots, n$, which contradicts (b). Similarly, if $\Delta'_k < \Delta_k$ for some k , then, by (2.1), we obtain $\beta'_{ik} > 0$ for $i = 1, \dots, n$, which contradicts (b'). Thus $\Delta_k = \Delta'_k$ for all k , i.e., $\Delta = \Delta'$.

Next we prove the existence of $\beta_1, \dots, \beta_n$ and γ . By Lemma 2.1, for $i \geq 2$ we can write

$$0 \neq \alpha_i = A^{-1}(e_1 - e_i) = \alpha_i^+ - \alpha_i^-$$

where α_i^+ and α_i^- are in $\mathbb{N}^n$. Notice that $\alpha_i^+ \neq 0$ and $\alpha_i^- \neq 0$. Indeed, the sum of the entries of $A^{-1}(e_i)$ is equal to $1/d$. Thus $|\alpha_i| = |\alpha_i^+| - |\alpha_i^-| = 0$, and consequently, the positive and negative part of α_i are both nonzero for $i \geq 2$. The vector α_i^+ can be written as $\alpha_i^+ = (\alpha_{i1}^+, \dots, \alpha_{in}^+)$ for $i \geq 2$. For $1 \leq k \leq n$ consider the integers given by

$$m_k = \max_{2 \leq i \leq n} \{\alpha_{ik}^+\}$$

and set $\beta_1 = (m_1, \dots, m_n)$. Since $\beta_1 \geq \alpha_i^+$, for each $i \geq 2$ there is $\theta_i \in \mathbb{N}^n$ such that $\beta_1 = \theta_i + \alpha_i^+$. Therefore

$$\alpha_i = A^{-1}(e_1 - e_i) = \alpha_i^+ - \alpha_i^- = \beta_1 - (\theta_i + \alpha_i^-).$$

We set $\beta_i = \theta_i + \alpha_i^-$ for $i \geq 2$. Since we have $A\beta_1 - e_1 = A\beta_i - e_i$ for $i \geq 2$, it follows readily that $A\beta_1 - e_1 \geq 0$ (make $i = 2$ in the last equality and compare entries). Thus, setting $\gamma := A\beta_1 - e_1$, it follows that $\beta_1, \dots, \beta_n$ and γ satisfy (a). If each row of B has some zero entry the proof of the existence is complete. If every entry of a row of B is positive we subtract the vector $\mathbf{1} = (1, \dots, 1)$ from that row and change γ accordingly so that (a) is still satisfied. Applying this argument repeatedly we get a sequence $\beta_1, \dots, \beta_n, \gamma$ satisfying (a) and (b).

We now prove the last part of the assertion. Notice that if β_{ij} denotes the j -entry of β_i , then the equality $A\beta_i = \gamma + e_i$ is equivalent to $\beta_{i1}v_1 + \dots + \beta_{in}v_n =$

$\gamma + e_i$. Thus $|\beta_i|d = |\gamma| + 1$. Note that condition (a) is equivalent to the equality $AB = \Gamma + I$, where Γ is the matrix all of whose columns are equal to γ . Since $\det(B) = \pm \det(\Gamma + I)/d$ it suffices to show that $\det(\Gamma + I) = |\gamma| + 1$. The latter is a classical calculation that can be performed in various ways. One has a more general statement which is given below in Lemma 2.3. In particular, by this lemma, if Γ has rank at most one and D is the identity matrix, one gets $\det(\Gamma + I) = \text{trace}(\Gamma) + 1$, as required. This completes the proof of the theorem. $\square$

Lemma 2.3. *Let $\Gamma = (\gamma_{i,j})$ be an $n \times n$ square matrix over an arbitrary commutative ring, and let $D = \text{diag}(d_1, \dots, d_n)$ be a diagonal matrix over the same ring. Then*

$$\begin{aligned} \det(\Gamma + D) &= \det(\Gamma) \\ &+ \sum_i d_i \Delta_{[n] \setminus \{i\}} + \sum_{1 \leq i_1 < i_2 \leq n} d_{i_1} d_{i_2} \Delta_{[n] \setminus \{i_1, i_2\}} \\ &+ \cdots + \sum_{1 \leq i_1 < \cdots < i_{n-1} \leq n} d_{i_1} \cdots d_{i_{n-1}} \Delta_{[n] \setminus \{i_1, \dots, i_{n-1}\}} \\ &+ \det D, \end{aligned}$$

where $[n] = \{1, \dots, n\}$ and $\Delta_{[n] \setminus \{i_1, \dots, i_k\}}$ denotes the principal $(n-k) \times (n-k)$ -minor of Γ with rows and columns $[n] \setminus \{i_1, \dots, i_k\}$.

Proof. It follows from the multi-linearity of the determinant. $\square$

Definition 2.4. A matrix A that satisfies the hypotheses of Theorem 2.2 is called a *Cremona matrix* and B is called the *Cremona inverse matrix* of A .

Remark 2.5. Bridges and Ryser [2] (cf. [4, Theorem 4.4]) considered a particular class of Cremona matrices. They studied the equation $AB = \Gamma + I$, when A, B are $\{0, 1\}$ matrices and Γ is the matrix with all its entries equal to 1. They show that if equality occurs, then each row and column of A has the same number r of ones, each row and column of B has the same number s of ones with $rs = n + 1$, and $AB = BA$.

As mentioned before the proof of Theorem 2.2 provides an algorithm to compute the vectors $\beta_1, \dots, \beta_n$ and γ (see Example 2.6 and the proof of Proposition 4.6 for specific illustrations of the algorithm). Other means to compute these vectors using linear optimization techniques will be discussed in Section 3.

Example 2.6. Consider the following matrix A and its inverse:

$$A = \begin{pmatrix} d & d-1 & 0 \\ 0 & 1 & d-1 \\ 0 & 0 & 1 \end{pmatrix}; \quad A^{-1} = \frac{1}{d} \begin{pmatrix} 1 & 1-d & (d-1)^2 \\ 0 & d & d(1-d) \\ 0 & 0 & d \end{pmatrix}.$$

To compute the β_i 's and γ we follow the proof of Theorem 2.2. Then $\beta_1 = (2, d, 0)$, $\beta_2 = (1, d+1, 0)$, $\beta_3 = (d, 1, 1)$, $\gamma = (d^2 + d - 1, d, 0)$, and

$$B = \begin{pmatrix} 2 & 1 & d \\ d & d+1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

By subtracting the vector $(1, 1, 1)$ from rows 1 and 2, we get

$$B' = \begin{pmatrix} 1 & 0 & d-1 \\ d-1 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The column vectors $\beta'_1 = (1, d-1, 0)$, $\beta'_2 = (0, d, 0)$, $\beta'_3 = (d-1, 0, 1)$, $\gamma' = (d^2 - d, d-1, 0)$ satisfy (a) and (b).

3. AN INTEGER PROGRAMMING METHOD VIA HILBERT BASES

The proof of Theorem 2.2 provides an algorithm to compute the inverse of a Cremona matrix. In this section we present another algorithm based on integer programming techniques and the notion of a Hilbert basis. From a complexity point of view, the first algorithm based on simple linear algebra is much faster than the second algorithm based on integer programming. The reason is the large number of variables that the second method requires (for small cases, both algorithms work fine). The second approach is quite interesting from a theoretical point of view as shown in [5].

Let $v_1, \dots, v_n$ be a set of vectors in $\mathbb{N}^n$ such that $|v_i| = d \geq 1$ for all i and $\det(A) = \pm d$, where A is the $n \times n$ matrix with column vectors $v_1, \dots, v_n$. Then, by Theorem 2.2, there are unique vectors $\beta_1, \dots, \beta_n, \gamma \in \mathbb{N}^n$ such that the following two conditions hold: (a) $A\beta_i = \gamma + e_i$ for all i , and (b) the matrix B whose columns are $\beta_1, \dots, \beta_n$ has at least one zero entry in every row.

To compute the sequence $\beta_1, \dots, \beta_n, \gamma$ using linear programming we regard the β_i 's and γ as vectors of indeterminates and introduce a new variable τ . Consider the homogeneous system of linear inequalities

$$(3.1) \quad \begin{aligned} A\beta_i &= \gamma + \tau e_i, \quad i = 1, \dots, n, \\ \beta_i &\geq 0, \quad i = 1, \dots, n, \\ \gamma &\geq 0, \quad \tau \geq 0. \end{aligned}$$

This linear system has n^2 equality constraints and $\ell = n^2 + n + 1$ indeterminates. The set C of solutions form a rational pointed polyhedral cone. By [8, Theorem 16.4], there is a unique minimal integral Hilbert basis

$$\mathcal{H} = \{h_1, \dots, h_r\}$$

of C such that $\mathbb{Z}^\ell \cap \mathbb{R}_+ \mathcal{H} = \mathbb{N} \mathcal{H}$ and $C = \mathbb{R}_+ \mathcal{H}$ (minimal relative to taking subsets), where $\mathbb{R}_+ \mathcal{H}$ denotes the cone generated by $\mathcal{H}$ consisting of all linear combinations of $\mathcal{H}$ with nonnegative real coefficients and $\mathbb{N} \mathcal{H}$ denotes the semigroup generated by $\mathcal{H}$ consisting of all linear combinations of $\mathcal{H}$ with coefficients in $\mathbb{N}$. The Hilbert basis of C has the following useful description.

Theorem 3.1 ([8, p. 233]). *$\mathcal{H}$ is the set of all integral vectors $0 \neq h \in C$ such that h is not the sum of two other nonzero integral vectors in C .*

Theorem 3.2. *There is a unique element h of $\mathcal{H}$ with $\tau = 1$ and this element gives the unique sequence $\beta_1, \dots, \beta_n, \gamma$ that satisfies (a) and (b).*

Proof. We set $x_0 = (\beta_1, \dots, \beta_n, \gamma, 1) \in \mathbb{N}^{n^2+n+1}$, where $\beta_1, \dots, \beta_n, \gamma$ is the unique sequence that satisfies (a) and (b). First we show that x_0 is in $\mathcal{H}$. Clearly, x_0 is in C . Thus we may assume that x_0 is written as

$$x_0 = \eta_1 h_1 + \dots + \eta_k h_k, \quad 0 \neq \eta_i \in \mathbb{N} \text{ for } i = 1, \dots, k,$$

where h_k has its last entry equal to 1 and the last entry of h_i is equal to 0 for $i < k$. The vector h_k has the form

$$h_k = (\beta_1^{(k)}, \dots, \beta_n^{(k)}, \gamma^{(k)}, 1),$$

where the $\beta_i^{(k)}$'s and $\gamma^{(k)}$ are in $\mathbb{N}^n$ and satisfy (3.1), i.e., they satisfy (a). Notice that from the first equality one has $x_0 \geq h_k$. Then $\beta_i \geq \beta_i^{(k)}$ for all i and $\gamma \geq \gamma^{(k)}$. Therefore the $\beta_i^{(k)}$'s and $\gamma^{(k)}$ also satisfy (b). Consequently, $x_0 = h_k$ and $x_0 \in \mathcal{H}$, as claimed. Let h_i be any other element in $\mathcal{H}$ whose last entry is equal to 1. Next we show that h_i must be equal to x_0 . The vector h_i has the form

$$h_i = (\beta_1^{(i)}, \dots, \beta_n^{(i)}, \gamma^{(i)}, 1),$$

where the $\beta_j^{(i)}$'s and $\gamma^{(i)}$ are in $\mathbb{N}^n$ and satisfy (3.1). Since the $\beta_j^{(i)}$'s and $\gamma^{(i)}$ satisfy (a), it suffices to show that they also satisfy (b). We proceed by contradiction. Assume that the ℓ th entry of $\beta_j^{(i)}$ is not zero for $j = 1, \dots, n$. For simplicity of notation assume that $\ell = 1$. Then the vectors

$$h' = (e_1, \dots, e_1, Ae_1, 0) \text{ and } h'' = (\beta_1^{(i)} - e_1, \dots, \beta_n^{(i)} - e_1, \gamma^{(i)} - Ae_1, 1)$$

are integral vectors that satisfy (3.1), i.e., h' and h'' are integral vectors in C and $h = h' + h''$, a contradiction to Theorem 3.1. $\square$

There are computer programs that can be used to find the integral Hilbert basis of polyhedral cones defined by linear systems of the form $x \geq 0; A'x = 0$, where A' is an integral matrix. We have used [3] to compute some specific examples with this procedure:

Example 3.3. Consider the following matrix:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

To compute $\beta_1, \beta_2, \beta_3, \gamma$ we use the following input file for *Normaliz* [3]

```
9
13
2 1 0 0 0 0 0 0 0 -1 0 0 -1
0 1 1 0 0 0 0 0 0 0 -1 0 0
0 0 1 0 0 0 0 0 0 0 0 -1 0
0 0 0 2 1 0 0 0 0 -1 0 0 0
0 0 0 0 1 1 0 0 0 0 -1 0 -1
0 0 0 0 0 1 0 0 0 0 0 -1 0
0 0 0 0 0 0 2 1 0 -1 0 0 0
0 0 0 0 0 0 0 1 1 0 -1 0 0
0 0 0 0 0 0 0 0 1 0 0 -1 -1
5
```

Part of the output file produced by *Normaliz* is:

```
4 generators of integral closure:
0 1 0 0 1 0 0 1 0 1 1 0 0
0 0 1 0 0 1 0 0 1 0 1 1 0
1 0 0 1 0 0 1 0 0 2 0 0 0
1 1 0 0 2 0 1 0 1 2 1 0 1
```

From the last row we get: $\beta_1 = (1, 1, 0), \beta_2 = (0, 2, 0), \beta_3 = (1, 0, 1), \gamma = (2, 1, 0)$.

4. APPLICATION TO CREMONA MAPS

Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field k . Given $\alpha = (a_1, \dots, a_n) \in \mathbb{N}^n$, we set $x^\alpha := x_1^{a_1} \cdots x_n^{a_n}$. In what follows we consider a finite set of distinct monomials $F = \{x^{v_1}, \dots, x^{v_n}\} \subset R$ of the same degree $d \geq 1$ and having no nontrivial common factor. We also assume throughout that every x_i divides at least one member of F , a harmless condition. The set F defines a rational (monomial) map $\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ which will also be denoted F and written as a tuple $F = (x^{v_1}, \dots, x^{v_n})$. This map is said to be a *Cremona map* (or a *Cremona transformation*) if it admits an inverse rational map with source $\mathbb{P}^{n-1}$. Note that a rational monomial map is defined everywhere if and only if the defining monomials are pure powers of the variables, in which case it is a Cremona map if and only if $d = 1$ (the identity map up to permutation of variables or more precisely a Cremona map of the form $F = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for some σ in the symmetric group S_n). Finally, the integer d is often called *degree* of F (not to be confused with its degree as a map).

The *log-matrix* of F , denoted by A , is the $n \times n$ matrix with column vectors $v_1, \dots, v_n$. The next result will be used in several places.

Proposition 4.1 ([10, Proposition 2.1]). *F is a Cremona map if and only if $\det(A) = \pm d$.*

Theorem 4.2. *Let $F : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ stand for a rational map defined by monomials of fixed degree. If F is a Cremona map, then its inverse is also defined by monomials of fixed degree. Moreover, the degree as well as a set of monomials defining the inverse can be obtained explicitly in terms of the given set of monomials defining F .*

Proof. Let F be a Cremona map defined by a set of monomials $f_1, \dots, f_n$ of the same degree $d \geq 1$. We set $f_i = x^{v_i}$ for $i = 1, \dots, n$. By Proposition 4.1 the matrix of exponents of these monomials (i.e., their log-matrix) A has determinant $\pm d$. Therefore Theorem 2.2 implies the existence of an $n \times n$ matrix B such that $AB = \Gamma + I$, where Γ is a matrix with repeated column γ throughout. Let $g_1, \dots, g_n$ denote the monomials whose log-matrix is B and call G the corresponding rational monomial map. Letting x^γ denote the monomial whose exponents are the coordinates of γ , the above matrix equality translates into the equality

$$(f_1(g_1, \dots, g_n), \dots, f_n(g_1, \dots, g_n)) = (x^\gamma \cdot x_1, \dots, x^\gamma \cdot x_n).$$

Thus the left-hand side is proportional to the vector $(x_1, \dots, x_n)$ which means that the composite map $F \circ G$ is the identity map wherever the two are defined (see [9, proof of Proposition 2.1]). On the other hand, since B is the log-matrix of $g_1, \dots, g_n$, Theorem 2.2 applied in the opposite direction says that G is also a Cremona map. Therefore G has to be the inverse of F , as required. Finally, notice that the proof of Theorem 2.2 provides an algorithm to compute B and γ . The input for this algorithm is the log-matrix A of $f_1, \dots, f_n$. $\square$

We will call a Cremona map as above a *Cremona monomial map*. The theorem allows us to introduce the following group.

Definition 4.3. The *Cremona monomial group* of order $n - 1$ is the subgroup of the Cremona group of $\mathbb{P}^{n-1}$ whose elements are Cremona monomial maps.

Here we will not distinguish between rational maps $F, G : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{m-1}$ defined, respectively, by forms $f_1, \dots, f_m$ of the same degree and their multiples $g_1 = gf_1, \dots, g_m = gf_m$ by a fixed form g of arbitrary degree.

There is a potential confusion in this terminology. For instance, in [7] one allows for more general maps by considering the free product with certain Klein groups. Since our goal is combinatorial we will stick to the above definition of the Cremona monomial group.

As a matter of notation, the composite of two Cremona maps F, G will be indicated by FG (first G , then F). Likewise, the power composite $F \cdots F$ of m factors will be denoted F^m .

We shift our attention to *plane* Cremona monomial maps, i.e., we will study the structure of the Cremona monomial group for $n = 3$. Consider the maps $H = (x_1^2, x_1x_2, x_2x_3)$ and $S = (x_1x_2, x_1x_3, x_2x_3)$.

- For any $d \geq 1$, one has $H^{d-1} = (x_1^d, x_1^{d-1}x_2, x_2^{d-1}x_3)$.

This is a straightforward composite calculation: by induction, we assume that H^{d-2} is defined by $x_1^{d-1}, x_1^{d-2}x_2, x_2^{d-2}x_3$, hence $H^{d-1} = H^{d-2}H$ is defined by $x_1^{2(d-1)}, x_1^{2d-3}x_2, x_1^{d-2}x_2^{d-1}x_3$ which is the same rational map as the one defined by $x_1^d, x_1^{d-1}x_2, x_2^{d-1}x_3$ (by canceling the gcd x_1^{d-2}).

- For any $d \geq 1$, up to permutation of the variables both in the source and the target, H^{d-1} is involutive, i.e., coincides with its own inverse; in [11] this was called “ p -involutive”.

Namely, consider the Cremona map $G = (x_1x_2^{d-1}, x_2^d, x_1^{d-1}x_3)$. Again, one finds that $H^{d-1}G$ is defined by $x_1^d x_2^{d(d-1)}, x_1^{d-1} x_2^{(d-1)^2+d} x_3^d, x_1^{d-1} x_2^{d(d-1)} x_3$ straightforwardly; canceling the gcd $x_1^{d-1} x_2^{d(d-1)}$ yields the identity map (x_1, x_2, x_3) .

- The monomial maps S and H “nearly” commute; actually they are conjugate by a transposition.

This is once again straightforward: the general composites SH^{d-1} and $H^{d-1}S$ are defined, respectively, by $x_1^d, x_1x_2^{d-2}x_3, x_2^{d-1}x_3$ and $x_1x_2^{d-1}, x_1x_2^{d-2}x_3, x_3^d$ and these are the same up to transposing x_1 and x_3 and the extreme terms. This means that they are conjugate by a transposition. It has as a consequence that the subgroup of the Cremona group of $\mathbb{P}^2$ generated by S and H is Abelian up to free product with the symmetric group S_3 (cf. [7, p. 1672]).

Lemma 4.4. *Let F be a plane Cremona map of degree d of the form $F = (x_1^{a_1}x_2^{a_2}, x_2^{b_2}x_3^{b_3}, x_1^{c_1}x_3^{c_3})$. Then, up to permutation of the source (variables) and the target (monomials), F is one of the following two kinds:*

$$F = (x_1x_2, x_2x_3, x_1x_3) \quad \text{or} \quad F = (x_1^d, x_2x_3^{d-1}, x_1^{d-1}x_3).$$

Proof. Let A be the log-matrix of F . The determinant of A is equal to $\pm d$ by Proposition 4.1. Since $\det(A) = a_1b_2c_3 + c_1a_2b_3$, we get that $\det(A) = d$. Hence, one has

$$(4.1) \quad d = a_1 + a_2 = b_2 + b_3 = c_1 + c_3 = a_1b_2c_3 + c_1a_2b_3 \quad \text{and} \\ a_1(b_2c_3 - 1) = a_2(1 - c_1b_3).$$

The cases below can be readily verified using these equations.

Case (I): $a_1 \geq 1, a_2 = 0$. Then $a_1 = d, b_2 = c_3 = 1$, and $F = (x_1^d, x_2x_3^{d-1}, x_1^{d-1}x_3)$.

Case (II): $a_1 = 0, a_2 \geq 1$. Then $a_2 = d, b_3 = c_1 = 1$, and $F = (x_2^d, x_2^{d-1}x_3, x_1x_3^{d-1})$.

Case (III)(a): $a_1 \geq 1, a_2 \geq 1, b_2c_3 = 0, b_2 = 0$. Then $F = (x_1^{d-1}x_2, x_3^d, x_1x_3^{d-1})$.

Case (III)(b): $a_1 \geq 1, a_2 \geq 1, b_2c_3 = 0, c_3 = 0$. Then $F = (x_1^{d-1}x_2, x_2^{d-1}x_3, x_1^d)$.
 Case (III)(c): $a_1 \geq 1, a_2 \geq 1, c_1b_3 = 0, c_1 = 0$. Then $F = (x_1x_2^{d-1}, x_2x_3^{d-1}, x_3^d)$.
 Case (III)(d): $a_1 \geq 1, a_2 \geq 1, c_1b_3 = 0, b_3 = 0$. Then $F = (x_1x_2^{d-1}, x_2^d, x_1^{d-1}x_3)$.
 Case (III)(e): $a_1 \geq 1, a_2 \geq 1, b_2c_3 - 1 \geq 0, 1 - c_1b_3 \leq 0$. From the last equality in (4.1) we get $b_2c_3 = 1$ and $c_1b_3 = 1$. Then $F = (x_1x_2, x_2x_3, x_1x_3)$. $\square$

We next give a purely integer matrix theoretic proof of the following result which was essentially proved in [7, Section 3].

Proposition 4.5. *If $n = 3$, then up to permutation of the variables (both in the source and the target), the plane Cremona monomial group is generated by the maps S, H , where S is the quadratic map of the first kind defined by x_1x_2, x_1x_3, x_2x_3 (Steiner involution) and H is the quadratic map of the second kind defined by x_1^2, x_1x_2, x_2x_3 ("hyperbolism").*

Proof. Let $F = (x_1^{a_1}x_2^{a_2}x_3^{a_3}, x_1^{b_1}x_2^{b_2}x_3^{b_3}, x_1^{c_1}x_2^{c_2}x_3^{c_3})$ be a plane Cremona map of degree d . We will prove that, up to a permutation of the variables and the monomials, every plane Cremona monomial map is of the form $F \cdots SH^{d_i}SH^{d_{i+1}} \cdots G$, where F, G are in $\{S, H^d \mid d \geq 1\}$. The proof is by induction on the degree. Consider the log-matrix of F :

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

Up to permutation of variables and monomials, there are essentially two cases to consider:

$$A = \begin{pmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & 0 & d \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} a_1 & 0 & c_1 \\ a_2 & b_2 & 0 \\ 0 & b_3 & c_3 \end{pmatrix}.$$

The determinant of A is $\pm d$ by Proposition 4.1. By Lemma 4.4 we may assume that A is the matrix on the left, i.e., $F = (x_1^{a_1}x_2^{a_2}x_3^{a_3}, x_1^{b_1}x_2^{b_2}, x_3^d)$.

Case (I): $a_1 \geq b_1$. From $a_1 + a_2 + a_3 = b_1 + b_2 = d$, we get that $b_2 \geq a_2$. Consider the Steiner involution given by $S = (x_1x_2, x_2x_3, x_1x_3)$. Then

$$\begin{aligned} SF &= (x_1^{a_1+b_1}x_2^{a_2+b_2}x_3^{a_3}, x_1^{b_1}x_2^{b_2}x_3^d, x_1^{a_1}x_2^{a_2}x_3^{d+a_3}) \\ &= x_1^{b_1}x_2^{a_2}x_3^{a_3}(x_1^{a_1}x_2^{b_2}, x_2^{b_2-a_2}x_3^{d-a_3}, x_1^{a_1-b_1}x_3^d). \end{aligned}$$

Thus by Lemma 4.4 we get that $SF = H^{a_1+b_2-1}$ for some hyperbolism H or SF is a Steiner involution. Thus multiplying by the inverse of S , we obtain that F has the required form.

Case (II)(a): $b_1 > a_1$ and $\det(A) = d$. By Lemma 4.4 we may assume $a_1 \geq 1$. From the equality $d = \det(A) = d(a_1b_2 - a_2b_1)$, we get that $b_2 \geq a_2$. Consider the hyperbolism $H = (x_1^2, x_1x_2, x_2x_3)$. In this case we have

$$\begin{aligned} HF &= (x_1^{2a_1}x_2^{2a_2}x_3^{2a_3}, x_1^{a_1+b_1}x_2^{a_2+b_2}x_3^{a_3}, x_1^{b_1}x_2^{b_2}x_3^d) \\ &= x_1^{a_1+1}x_2^{a_2}x_3^{a_3}(x_1^{a_1-1}x_2^{a_2}x_3^{a_3}, x_1^{b_1-1}x_2^{b_2}, x_1^{b_1-a_1-1}x_2^{b_2-a_2}x_3^{d-a_3}) \\ &= x_1^{a_1+1}x_2^{a_2}x_3^{a_3}F_1. \end{aligned}$$

Since F_1 has degree at most $d - 1$, we have lowered the degree of F . Thus by induction F has the required form.

Case (II)(b): $b_1 > a_1$ and $\det(A) = -d$. We may assume that $a_2 \geq b_2$, otherwise we may proceed as in Case (II)(a). By Lemma 4.4 we may also assume that $a_1 \geq 1$.

We claim that F has the form $F = (x_1^{d-2}x_2x_3, x_1^{d-1}x_2, x_3^d)$. The condition on the determinant is equivalent to the equality $a_2b_1 - a_1b_2 = 1$. Thus $a_2 \geq 1$. Hence using that $b_1 \geq a_1 + 1$ one has

$$a_2b_1 \geq a_2a_1 + a_2 \geq b_2a_1 + 1 = a_2b_1.$$

Consequently, $a_2 = b_2 = 1$. The condition $\det(A) = -d$, becomes $a_1 = b_1 - 1 = d - 2$. Therefore F has the asserted form. Consider the hyperbolism $H = (x_1^2, x_1x_2, x_2x_3)$. It is easy to see that HF is equal to $x^\gamma(x_1^{d-3}x_2x_3, x_1^{d-2}x_2, x_3^{d-1})$ for some monomial x^γ . Thus we have lowered the degree of F and we may apply induction. $\square$

The next result is classically well known. We give a simple direct proof in the case of plane Cremona monomial maps. The proof shows, moreover, explicit simple formulae for the Cremona inverse in the case of 3 variables.

Proposition 4.6. *If $n = 3$, then a plane Cremona monomial map and its inverse have the same degree.*

Proof. The proof is based on the method employed in the proof of Theorem 2.2. Let A be the log-matrix of a plane Cremona map F of degree d :

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

By Proposition 4.1 we may assume that $\det(A) = d$, the case $\det(A) = -d$ can be shown similarly. Up to permutation of variables, as in the proof of Proposition 4.5, there are essentially two cases to consider:

$$A = \begin{pmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & 0 & d \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} a_1 & 0 & c_1 \\ a_2 & b_2 & 0 \\ 0 & b_3 & c_3 \end{pmatrix}.$$

It is readily seen that the inverse of A is given by

$$A^{-1} = \begin{pmatrix} b_2 & -b_1 & 0 \\ -a_2 & a_1 & 0 \\ -a_3b_2/d & b_1a_3/d & 1/d \end{pmatrix} \quad \text{or} \quad A^{-1} = \frac{1}{d} \begin{pmatrix} b_2c_3 & b_3c_1 & -b_2c_1 \\ -a_2c_3 & a_1c_3 & a_2c_1 \\ a_2b_3 & -a_1b_3 & a_1b_2 \end{pmatrix},$$

respectively. By the argument given in the proof of Theorem 2.2 we get that

$$\beta_1 = (d, 0, 0), \quad \gamma = (da_1 - 1, da_2, da_3)$$

or

$$\beta_1 = (b_2, 0, b_3), \quad \gamma = (a_1b_2 + b_3c_1 - 1, a_2b_2, b_3c_3),$$

respectively. Therefore

$$B = \begin{pmatrix} d & 0 & b_1 \\ 0 & a_1 + a_2 & a_2 \\ 0 & a_3 & (a_3b_2 + 1)/d \end{pmatrix} \quad \text{or} \quad B = \begin{pmatrix} b_2 & c_1 & 0 \\ 0 & c_3 & a_2 \\ b_3 & 0 & a_1 \end{pmatrix},$$

respectively. Using that the sum of the entries in every column of A is equal to d and $\det(A) = d$, it is seen that the sum of the entries in each column of B is equal to d and $\det(B) = d$. $\square$

One of the peculiarities of the theory is that even if the given monomials are square-free to start with, the inverse map is generally defined by nonsquare-free monomials. This makes classification in high degrees, if not the structure of the Cremona monomial group itself, a difficult task (see [5, 6]). A complete classification of monomial Cremona maps of degree 2 in any number of variables is given in [5].

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