MATHEMATICS OF COMPUTATION
Volume 81, Number 279, July 2012, Pages 1609–1617
S 0025-5718(2011)02561-5
Article electronically published on November 3, 2011

THE FROBENIUS PROBLEM FOR NUMERICAL SEMIGROUPS WITH EMBEDDING DIMENSION EQUAL TO THREE

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Abstract. Let S be a numerical semigroup with embedding dimension equal to three. Assume that the minimal generators of S are pairwise relatively prime numbers. Under these conditions, we give semi-explicit formulas for the Frobenius number, the genus, and the set of pseudo-Frobenius numbers of S. Moreover, if the multiplicity of S is fixed, then these formulas become explicit.

1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. A *numerical semigroup* is a subset S of $\mathbb{N}$ that is closed under addition, $0 \in S$ and $\mathbb{N} \setminus S$ is finite. The elements of $\mathbb{N} \setminus S$ are the *gaps* of S, and the cardinal of such a set is called the *genus* of S, denoted by $\mathrm{g}(S)$. The *Frobenius number* of S is the largest integer that does not belong to S and it is denoted by $\mathrm{F}(S)$.

If $A \subseteq \mathbb{N}$ is a nonempty set, we denote by $\langle A \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by A, that is,

$$\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\},\ a_1, \ldots, a_n \in A,\ \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}.$$

In [14] it is proved that $\langle A \rangle$ is a numerical semigroup if and only if $\gcd\{A\} = 1$, where gcd means *greatest common divisor*.

It is well known (see [14], for instance) that every numerical semigroup S is finitely generated and, therefore, there exists a finite subset $X \subseteq S$ such that $S = \langle X \rangle$. In addition, if no proper subset of X generates S, then we say that X is a *minimal system of generators* of S. In [14] it is proved that every numerical semigroup admits a unique minimal system of generators $\{n_1 < n_2 < \cdots < n_e\}$. The integers e and n_1 are known as the *embedding dimension* and the *multiplicity* of S, respectively.

The Frobenius problem (see [8]) consists of finding formulas that allow us to compute, in terms of the minimal system of generators of a numerical semigroup, the Frobenius number and the genus of such a numerical semigroup. This problem was solved by Sylvester and Curran Sharp (see [15, 16, 17]) when the embedding dimension is equal to two. In fact, if S is a numerical semigroup with minimal system of generators $\{n_1, n_2\}$, then $\mathrm{F}(S) = n_1 n_2 - n_1 - n_2$ and $\mathrm{g}(S) = \frac{(n_1-1)(n_2-1)}{2}$. At present, the Frobenius problem is open for the case of embedding dimension $e \geq 3$. To be precise, Curtis showed in [2] that it is impossible to find a polynomial formula (that is, a finite set of polynomials) that computes the Frobenius number

Received by the editor October 27, 2010 and, in revised form, March 22, 2011.
2010 *Mathematics Subject Classification.* Primary 11D07, 20M14.
Key words and phrases. Frobenius problem, numerical semigroups, multiplicity.
Both authors were supported by MTM2007-62346 (MEC, Spain), MTM2010-15595 (MICINN, Spain) and FEDER funds.

if $e = 3$. Indeed, in [7] Ramírez-Alfonsín proves that this problem is NP-hard for a dimension e variable.

If $\{n_1, n_2, n_3\}$ is the minimal system of generators of a numerical semigroup S and $d = \gcd\{n_1, n_2\}$, then $\mathrm{F}(S) = d\,\mathrm{F}\left(\langle \frac{n_1}{d}, \frac{n_2}{d}, n_3\rangle\right) + (d-1)n_3$ and $\mathrm{g}(S) = d\,\mathrm{g}\left(\langle \frac{n_1}{d}, \frac{n_2}{d}, n_3\rangle\right) + \frac{(d-1)(n_3-1)}{2}$ (see [6, 10]). Therefore, in order to solve the Frobenius problem for numerical semigroups with embedding dimension equal to three, we focus our attention on numerical semigroups whose three minimal generators are pairwise relatively prime numbers.

Let S be a numerical semigroup minimally generated by $\{m_1, m_2, m_3\}$, with $m_1 < m_2 < m_3$. Following the notation of [5], for $\{i, j, k\} = \{1, 2, 3\}$, set

$$c_i = \min\{x \in \mathbb{N} \setminus \{0\} \mid x m_i \in \langle m_j, m_k\rangle\}.$$

Therefore,

$$\begin{cases} c_1 m_1 = r_{12} m_2 + r_{13} m_3, \\ c_2 m_2 = r_{21} m_1 + r_{23} m_3, \\ c_3 m_3 = r_{31} m_1 + r_{32} m_2, \end{cases} \tag{1.1}$$

for certain nonnegative integers $r_{12}, r_{13}, r_{21}, r_{23}, r_{31}, r_{32}$. From now on, we denote by $\mathrm{Six}(S)$ the 6-tuple $(r_{12}, r_{13}, r_{23}, r_{32}, r_{21}, r_{31})$. In [13] it is shown that, if m_1, m_2, m_3 are pairwise relatively prime numbers, then the integers r_{ij} are strictly positive and, moreover, there are known formulas to compute the Frobenius number and the genus of S in terms of $\mathrm{Six}(S)$.

If m is a positive integer, we denote by $\mathfrak{L}(m)$ the set of numerical semigroups with multiplicity equal to m, embedding dimension equal to three, and pairwise relatively prime minimal generators. Our purpose in this paper is to give a formula to compute $\mathrm{Six}(S)$ for each $S \in \mathfrak{L}(m)$.

In order to have an idea of the type of formulas that we will give, we include in this introduction the case $m = 10$. The results that we will prove in this paper show that the formula is correct. Let $S \in \mathfrak{L}(10)$ and let $m_1 = 10 < m_2 < m_3$ be the minimal generators of S. Let us observe that $m_3 \equiv k m_2 \pmod{10}$ for some $k \in \{3, 7, 9\}$. We have that:

1) If $m_3 \equiv 3m_2 \pmod{10}$, then $\mathrm{Six}(S) = \left(1, 3, 1, 2, \frac{3m_2 - m_3}{10}, \frac{2m_3 - m_2}{5}\right)$.
2) If $m_3 \equiv 7m_2 \pmod{10}$ and
   2.1) $\frac{m_2}{m_3} < \frac{1}{2}$, then $\mathrm{Six}(S) = \left(3, 1, 1, 4, \frac{7m_2 - m_3}{10}, \frac{m_3 - 2m_2}{5}\right)$;
   2.2) $\frac{m_2}{m_3} > \frac{1}{2}$, then $\mathrm{Six}(S) = \left(3, 1, 2, 1, \frac{2m_2 - m_3}{5}, \frac{3m_3 - m_2}{10}\right)$.
3) If $m_3 \equiv 9m_2 \pmod{10}$ and
   3.1) $\frac{m_2}{m_3} < \frac{1}{4}$, then $\mathrm{Six}(S) = \left(1, 1, 1, 8, \frac{9m_2 - m_3}{10}, \frac{m_3 - 4m_2}{5}\right)$;
   3.2) $\frac{1}{4} < \frac{m_2}{m_3} < \frac{3}{7}$, then $\mathrm{Six}(S) = \left(1, 1, 2, 7, \frac{4m_2 - m_3}{5}, \frac{3m_3 - 7m_2}{10}\right)$;
   3.3) $\frac{3}{7} < \frac{m_2}{m_3} < \frac{2}{3}$, then $\mathrm{Six}(S) = \left(1, 1, 3, 6, \frac{7m_2 - 3m_3}{10}, \frac{2m_3 - 3m_2}{5}\right)$;
   3.4) $\frac{m_2}{m_3} > \frac{2}{3}$, then $\mathrm{Six}(S) = \left(1, 1, 4, 5, \frac{3m_2 - 2m_3}{5}, \frac{m_3 - m_2}{2}\right)$.

Let us observe that the 6-tuple $\mathrm{Six}(S)$ is not ordered by chance. Because we have taken the set $\mathfrak{L}(m_1)$ with fixed m_1, in each expression of the formula, the values of $r_{12}, r_{13}, r_{23}, r_{32}$ are independent of the m_1, m_2 and m_3, instead of r_{21}, r_{31} that are dependent on them.

In [10] Rødseth showed an algorithm that allows us to determine the Frobenius number and the genus of a numerical semigroup with embedding dimension equal to three and pairwise relatively prime minimal generators. If we analyze such an algorithm, it is not difficult to obtain formulas of $\mathrm{F}(S)$ and $\mathrm{g}(S)$ for $S \in \mathfrak{L}(m)$.

These formulas have the same structure as those presented by us. In fact, they depend on an integer parameter k such that $m_3 \equiv km_2 \pmod{m_1}$ and, moreover, that $\frac{m_2}{m_3}$ belongs to a certain interval. In this way, this work can be considered as a continuation of [10], but using different techniques in the proofs. Moreover, we consider that our paper can help to clarify, generalize and simplify the ideas described in [10]. On the other hand, several algorithms that compute the Frobenius number (related closely to the previous one), quasi-formulas, and upper bounds for such a number are collected in [8, Chapters 1-2]. In this way, [9] deserves special attention. In such a paper, Ramírez-Alfonsín and Rødseth improve the ideas of previous papers (see [4, 11]) to obtain an efficient algorithm that computes certain parameters, and then give a semi-explicit formula. In the present paper, we are not interested in the question of the complexity to compute $\mathrm{Six}(S)$. Our aim is to highlight the possibility of obtaining an explicit formula if the multiplicity is fixed. Therefore, an interesting question is: about how long does it take to get this formula? In such a case, we would say that the answer is polynomial time with respect to the multiplicity.

We now summarize the content of this paper. Let S be a numerical semigroup generated by three positive integers $m_1 < m_2 < m_3$ that are pairwise relatively prime numbers. In Section 2 we recall several results and facts of [13] that will be useful in this work. In Section 3 we will solve the Frobenius problem for $\mathfrak{L}(m_1)$. The tool for this purpose will be the concept of a *set of chained solutions* for such a set of numerical semigroups. The existence of sets of chained solution will be shown in Section 4.

2. Preliminaries

We will say that $(a_1, \ldots, a_n)$ is an *integer n-tuple* if $a_1, \ldots, a_n \in \mathbb{Z}$ (where $\mathbb{Z}$ is the set of integers). We will say that the n-tuple $(a_1, \ldots, a_n)$ is *strongly positive* if $a_1, \ldots, a_n \in \mathbb{N} \setminus \{0\}$.

The following result is [13, Theorem 8] and will be fundamental in the development of this paper.

Theorem 2.1. *Let m_1, m_2, m_3 be pairwise relatively prime positive integers. Then the system of equations*

$$\begin{cases} m_1 = x_{12}x_{13} + x_{12}x_{23} + x_{13}x_{32}, \\ m_2 = x_{13}x_{21} + x_{21}x_{23} + x_{23}x_{31}, \\ m_3 = x_{12}x_{31} + x_{21}x_{32} + x_{31}x_{32}, \end{cases} \tag{2.1}$$

has a strongly positive integer solution if and only if $\mathrm{e}(\langle m_1, m_2, m_3\rangle) = 3$. Moreover, if such a solution exists, then it is unique.

In the rest of this paper m_1, m_2, m_3 will be pairwise relatively prime positive integers such that $m_1 < m_2 < m_3$ and $\mathrm{e}(\langle m_1, m_2, m_3\rangle) = 3$. In [13, Lemma 6] it is shown that $(x_{12}, x_{13}, x_{23}, x_{32}, x_{21}, x_{31}) = \mathrm{Six}(\langle m_1, m_2, m_3\rangle)$ is the unique strongly positive integer solution of (2.1). Moreover, in [13, Lemma 3] it is shown that, if $\{i, j, k\} = \{1, 2, 3\}$, then $c_i = r_{ji} + r_{ki}$. From the preceding remarks, together with the next result (which is a consequence of [13, Proposition 15, Proposition 17]), we can assert that, if we know the unique strongly positive integer solution of (2.1), then we have solved the Frobenius problem.

Theorem 2.2. *Under the above conditions, we have*

1) $\mathrm{F}(\langle m_1, m_2, m_3\rangle) = \frac{1}{2}((c_1-2)m_1+(c_2-2)m_2+(c_3-2)m_3+\Delta)$, *where* $\Delta = \sqrt{\left(\sum_{i=1}^{3} c_i m_i\right)^2 - 4(c_1m_1c_2m_2 + c_1m_1c_3m_3 + c_2m_2c_3m_3 - m_1m_2m_3)}$;
2) $\mathrm{g}(\langle m_1, m_2, m_3\rangle) = \frac{1}{2}((c_1-1)m_1+(c_2-1)m_2+(c_3-1)m_3 - c_1c_2c_3 + 1)$.

Let S be a numerical semigroup. We say that $x \in \mathbb{Z} \setminus S$ is a *pseudo-Frobenius number* of S (see [12]) if $x + s \in S$ for all $s \in S \setminus \{0\}$. We denote by $\mathrm{PF}(S)$ the set of pseudo-Frobenius numbers of S. From the definition it follows that $\mathrm{F}(S) = \max\{\mathrm{PF}(S)\}$. The cardinal of $\mathrm{PF}(S)$ is an important invariant of S which is called the *type* of S (see [1]). In [3] it is shown that the type of $\langle m_1, m_2, m_3\rangle$ is equal to two. The following result (which is part of [13, Proposition 15]) shows how we can compute the pseudo-Frobenius numbers of $\langle m_1, m_2, m_3\rangle$ if we know $\mathrm{Six}(\langle m_1, m_2, m_3\rangle)$.

Theorem 2.3. *Let* Δ *be as in Theorem* 2.2. *Then*

$$\mathrm{PF}(\langle m_1, m_2, m_3\rangle) = \left\{\frac{1}{2}((c_1-2)m_1+(c_2-2)m_2+(c_3-2)m_3+\Delta)\ ,\right.$$
$$\left.\frac{1}{2}((c_1-2)m_1+(c_2-2)m_2+(c_3-2)m_3-\Delta)\right\}.$$

We finish these preliminaries by pointing out that the knowledge of the set $\mathrm{Six}(\langle m_1, m_2, m_3\rangle)$ is also important in other areas of semigroup theory and of ring theory. In fact, in [5] it is shown that (1.1) gives a minimal presentation by generators and relations of $\langle m_1, m_2, m_3\rangle$. Moreover, in the same paper, such a presentation is interpreted in terms of the semigroup ring associated to $\langle m_1, m_2, m_3\rangle$.

3. Chained solutions

First of all, let us observe that, because m_1, m_2, m_3 are pairwise relatively prime positive integers, there exists $k \in \{1, \ldots, m_1 - 1\}$ such that $m_3 \equiv km_2 \pmod{m_1}$. Moreover, since $\gcd\{m_1, m_3\} = 1$, we have that $\gcd\{k, m_1\} = 1$. In the sequel we assume both conditions.

Proposition 3.1. *Let* $(x_{12}, x_{13}, x_{23}, x_{32}) = (a_{12}, a_{13}, a_{23}, a_{32})$ *be a strongly positive integer solution of* $m_1 = x_{12}x_{13} + x_{12}x_{23} + x_{13}x_{32}$. *Then there exist* a_{21}, a_{31} *positive integers such that* $(a_{12}, a_{13}, a_{23}, a_{32}, a_{21}, a_{31})$ *is a strongly positive integer solution of* (2.1) *if and only if the following conditions are fulfilled:*

1) $a_{12} + a_{32} - ka_{23} \equiv 0 \pmod{m_1}$;
2) $ka_{13} + ka_{23} - a_{32} \equiv 0 \pmod{m_1}$;
3) $\frac{a_{23}}{a_{12}+a_{32}} < \frac{m_2}{m_3} < \frac{a_{13}+a_{23}}{a_{32}}$.

Proof (Necessity). From the hypotheses, we deduce that the system

$$\begin{cases} m_2 = (a_{13}+a_{23})x_{21} + a_{23}x_{31}, \\ m_3 = a_{32}x_{21} + (a_{12}+a_{32})x_{31}, \end{cases}$$

has a strongly positive integer solution. If we solve this system, we have that $x_{21} = \frac{1}{m_1}((a_{12}+a_{32})m_2 - a_{23}m_3)$ and $x_{31} = \frac{1}{m_1}((a_{13}+a_{23})m_3 - a_{32}m_2)$. Since x_{21}, x_{31} are integers, then $(a_{12}+a_{32})m_2 - a_{23}m_3 \equiv 0 \pmod{m_1}$ and $(a_{13}+a_{23})m_3 - a_{32}m_2 \equiv 0 \pmod{m_1}$. Having in mind that $m_3 \equiv km_2 \pmod{m_1}$ and that $\gcd\{m_1, m_2\} = 1$,

we get conditions 1) and 2). Finally, from the positivity of x_{21} and x_{31}, we obtain condition 3).

(Sufficiency). It is enough to observe that $(a_{12}, a_{13}, a_{23}, a_{32}, a_{21}, a_{31})$ is a strongly positive integer solution of (2.1) if $a_{21} = \frac{1}{m_1}((a_{12} + a_{32})m_2 - a_{23}m_3)$ and $a_{31} = \frac{1}{m_1}((a_{13} + a_{23})m_3 - a_{32}m_2)$. □

Let us observe that Proposition 3.1 shows that the unique strongly positive integer solution of (2.1) is also a solution of the system

$$\begin{cases} x_{12}x_{13} + x_{12}x_{23} + x_{13}x_{32} & = & m_1, \\ x_{12} + x_{32} - kx_{23} & \equiv & 0 \pmod{m_1}, \\ kx_{13} + kx_{23} - x_{32} & \equiv & 0 \pmod{m_1}. \end{cases} \tag{3.1}$$

If α, β are real numbers, we denote by $]\alpha, \beta[$ the open interval with ends α and β, that is, $]\alpha, \beta[= \{x \in \mathbb{R} \mid \alpha < x < \beta\}$. We will say that α is the *initial end* and β is the *final end* of such an interval.

If $(a_{12}, a_{13}, a_{23}, a_{32})$ is a strongly positive integer solution of (3.1), we denote by $\mathrm{I}(a_{12}, a_{13}, a_{23}, a_{32})$ the open interval $]\frac{a_{23}}{a_{12}+a_{32}}, \frac{a_{13}+a_{23}}{a_{32}}[$. Let us observe that, since $m_1 = a_{12}a_{13} + a_{12}a_{23} + a_{13}a_{32}$, then we have $\max\{a_{23}, a_{12} + a_{32}, a_{13} + a_{23}, a_{32}\} \leq m_1 - 1$.

Lemma 3.2. *Let $X = \{s_1, \ldots, s_n\}$ be a set of strongly positive integer solutions of* (3.1) *such that:*

1) *the initial end of $\mathrm{I}(s_1)$ is $\frac{1}{k}$;*
2) *for each $i \in \{1, \ldots, n-1\}$, the final end of $\mathrm{I}(s_i)$ is equal to the initial end of $\mathrm{I}(s_{i+1})$;*
3) *the final end of $\mathrm{I}(s_1)$ is greater than or equal to one.*

Then there exists a unique $i \in \{1, \ldots, n\}$ such that $\frac{m_2}{m_3} \in \mathrm{I}(s_i)$.

Proof. Since $m_3 \equiv km_2 \pmod{m_1}$ and $m_3 \notin \langle m_1, m_2\rangle$, then $m_3 < km_2$ and, therefore, $\frac{1}{k} < \frac{m_2}{m_3}$. Moreover, since $m_2 < m_3$, then $\frac{m_2}{m_3} < 1$. In order to complete the proof, we have to show that $\frac{m_2}{m_3}$ is not any of the ends of $\mathrm{I}(s_i)$, for all $i \in \{1, \ldots, n\}$. Otherwise, we have $\frac{m_2}{m_3} = \frac{a}{b}$ with a, b positive integers that are less than m_1. Then $am_3 = bm_2$ in contradiction to the assumption that $\gcd\{m_2, m_3\} = 1$. □

Let X be a set of strongly positive integer solutions of (3.1) such that it satisfies conditions 1), 2) and 3) of the previous lemma. We will say that X is a *set of chained solutions* of (3.1).

As an immediate consequence of Proposition 3.1 and its proof, we obtain the next result, which is the key in the development of this paper.

Theorem 3.3. *Let $X = \{s_1, \ldots, s_n\}$ be a set of chained solutions of* (3.1). *If $\frac{m_2}{m_3} \in \mathrm{I}(s_i)$ and $s_i = (a_{12}, a_{13}, a_{23}, a_{32})$, then $\mathrm{Six}(\langle m_1, m_2, m_3\rangle)$ is equal to*

$$(a_{12}, a_{13}, a_{23}, a_{32}, \frac{1}{m_1}((a_{12} + a_{32})m_2 - a_{23}m_3), \frac{1}{m_1}((a_{13} + a_{23})m_3 - a_{32}m_2)).$$

As an application of Theorem 3.3, we are going to show that the formula given in the introduction for computing $\mathrm{Six}(S)$, for all $S \in \mathfrak{L}(10)$, is true. Let us observe that, since $\gcd\{10, k\} = 1$, then $k \in \{3, 7, 9\}$. We analyze each one of these cases separately.

1) If $k = 3$, then $X = \{(1, 3, 1, 2)\}$ is a set of chained solutions of (3.1) because $\mathrm{I}(1, 3, 1, 2) =]\frac{1}{3}, \frac{2}{1}[$. From Theorem 3.3, we deduce that, if $\frac{m_2}{m_3} \in]\frac{1}{3}, 2[$, then $\mathrm{Six}(S) = (1, 3, 1, 2, \frac{3m_2-m_3}{10}, \frac{2m_3-m_2}{5})$.

   Let us observe that $\frac{1}{k} < \frac{m_2}{m_3} < 1$ always. Therefore, in this case the condition $\frac{m_2}{m_3} \in]\frac{1}{3}, 2[$ is redundant. Consequently, we can assert that, if $m_3 \equiv 3m_2 \pmod{10}$, then $\mathrm{Six}(S) = (1, 3, 1, 2, \frac{3m_2-m_3}{10}, \frac{2m_3-m_2}{5})$.
2) If $k = 7$, then $X = \{(3, 1, 1, 4), (3, 1, 2, 1)\}$ is a set of chained solutions of (3.1) because $\mathrm{I}(3, 1, 1, 4) =]\frac{1}{7}, \frac{1}{2}[$ and $\mathrm{I}(3, 1, 2, 1) =]\frac{1}{2}, \frac{3}{1}[$. From Theorem 3.3, we deduce that:
   2.1) if $\frac{m_2}{m_3} < \frac{1}{2}$, then $\mathrm{Six}(S) = (3, 1, 1, 4, \frac{7m_2-m_3}{10}, \frac{m_3-2m_2}{5})$;
   2.2) if $\frac{m_2}{m_3} > \frac{1}{2}$, then $\mathrm{Six}(S) = (3, 1, 2, 1, \frac{2m_2-m_3}{5}, \frac{3m_3-m_2}{10})$.
3) If $k = 9$, then $X = \{(1, 1, 1, 8), (1, 1, 2, 7), (1, 1, 3, 6), (1, 1, 4, 5)\}$ is a set of chained solutions of (3.1) because $\mathrm{I}(1, 1, 1, 8) =]\frac{1}{9}, \frac{1}{4}[$, $\mathrm{I}(1, 1, 2, 7) =]\frac{1}{4}, \frac{3}{7}[$, $\mathrm{I}(1, 1, 3, 6) =]\frac{3}{7}, \frac{2}{3}[$, and $\mathrm{I}(1, 1, 4, 5) =]\frac{2}{3}, \frac{1}{1}[$. From Theorem 3.3 again, we deduce that:
   3.1) if $\frac{m_2}{m_3} < \frac{1}{4}$, then $\mathrm{Six}(S) = \big(1, 1, 1, 8, \frac{9m_2-m_3}{10}, \frac{m_3-4m_2}{5}\big)$;
   3.2) if $\frac{1}{4} < \frac{m_2}{m_3} < \frac{3}{7}$, then $\mathrm{Six}(S) = \big(1, 1, 2, 7, \frac{4m_2-m_3}{5}, \frac{3m_3-7m_2}{10}\big)$;
   3.3) if $\frac{3}{7} < \frac{m_2}{m_3} < \frac{2}{3}$, then $\mathrm{Six}(S) = \big(1, 1, 3, 6, \frac{7m_2-3m_3}{10}, \frac{2m_3-3m_2}{5}\big)$;
   3.4) if $\frac{m_2}{m_3} > \frac{2}{3}$, then $\mathrm{Six}(S) = \big(1, 1, 4, 5, \frac{3m_2-2m_3}{5}, \frac{m_3-m_2}{2}\big)$.

4. Existence of sets of chained solutions

Our aim in this section will be to show that there always exists a set of chained solutions of (3.1). Moreover, the proof is constructive and, therefore, we can compute such a set in an easy and quick way.

As usual, if x is a real number, we set $\lfloor x \rfloor = \max\{z \in \mathbb{Z} \mid z \leq x\}$. Moreover, if a, b are integers such that $b \neq 0$, we denote by $a \bmod b$ the remainder of the division of a by b, that is, the number $a - \lfloor \frac{a}{b} \rfloor b$.

It is easy to check the next lemma.

Lemma 4.1. *Under the stated conditions,*

$$(x_{12}, x_{13}, x_{23}, x_{32}) = (m_1 \bmod k, \lfloor \frac{m_1}{k} \rfloor, 1, k - m_1 \bmod k)$$

is a strongly positive integer solution of (3.1).

The following result allows us to obtain new strongly positive integer solutions from the solution given by Lemma 4.1. Once again, the proof is easy and, therefore, we omit it.

Lemma 4.2. *Let* $(x_{12}, x_{13}, x_{23}, x_{32}) = (a_{12}, a_{13}, a_{23}, a_{32})$ *be a strongly positive integer solution of* (3.1) *such that* $a_{32} > a_{12}$. *Then*

$$(x_{12}, x_{13}, x_{23}, x_{32}) = (a_{12}, a_{13}, a_{13} + a_{23}, a_{32} - a_{12})$$

is another strongly positive integer solution of (3.1).

Let us observe that $k - (i+1)(m_1 \bmod k) \geq 1$ if and only if $i \leq \lfloor \frac{k-1}{m_1 \bmod k} \rfloor - 1$. Lemmas 4.1 and 4.2, together with this observation, immediately lead to the next result.

Proposition 4.3. *Let us have* $s = \lfloor \frac{k-1}{m_1 \bmod k} \rfloor - 1$ *and* $X = \{(m_1 \bmod k, \lfloor \frac{m_1}{k} \rfloor, i\lfloor \frac{m_1}{k} \rfloor + 1, k - (i+1)(m_1 \bmod k)) \mid i \in \{0, \ldots, s\}\}$. *Then* X *is a set of strongly positive integer solutions of* (3.1) *that satisfies conditions 1) and 2) of Lemma* 3.2.

As we can see in the following example, the set X given by Proposition 4.3 does not satisfy, in general, condition 3) of Lemma 3.2.

Example 4.4. If we take $m_1 = 36$, $k = 23$, then $s = 0$, $X = \{(13, 1, 1, 10)\}$, and $\mathrm{I}(13, 1, 1, 10) =]\frac{1}{23}, \frac{1}{5}[$.

To obviate this problem, we are going to show another construction of solutions in such a way that we can continue to produce chaining solutions. An easy computation proves the following result.

Lemma 4.5. *Let* $(x_{12}, x_{13}, x_{23}, x_{32}) = (a_{12}, a_{13}, a_{23}, a_{32})$ *be an integer solution of* (3.1). *Let* t *be an integer. Then*

$$(x_{12}, x_{13}, x_{23}, x_{32}) = (a_{12} - (t-1)a_{32}, ta_{13} + (t-1)a_{23}, a_{13} + a_{23}, ta_{32} - a_{12})$$

is another integer solution of (3.1).

Let us observe that, in the previous lemma, we do not impose the condition that the solutions have to be strongly positive.

Lemma 4.6. *Let* $(x_{12}, x_{13}, x_{23}, x_{32}) = (a_{12}, a_{13}, a_{23}, a_{32})$ *be a strongly positive integer solution of* (3.1). *Then:*

1) $\gcd\{a_{12}, a_{32}\} = 1$.
2) *If* $a_{32} = 1$, *then the final end of* $\mathrm{I}(a_{12}, a_{13}, a_{23}, a_{32})$ *is greater than* 1.

Proof.

1) If $\gcd\{a_{12}, a_{32}\} \neq 1$, then we deduce that $\gcd\{m_1, m_3\} \neq 1$ from (2.1), and this is a contradiction to the stated hypotheses in this paper.
2) If $a_{32} = 1$, then the final end of $\mathrm{I}(a_{12}, a_{13}, a_{23}, a_{32})$ is equal to $a_{13} + a_{23}$ and the conclusion is obvious. □

Proposition 4.7. *Let* $(x_{12}, x_{13}, x_{23}, x_{32}) = (a_{12}, a_{13}, a_{23}, a_{32})$ *be a strongly positive integer solution of* (3.1) *such that* $a_{12} \geq a_{32}$ *and the final end of* $\mathrm{I}(a_{12}, a_{13}, a_{23}, a_{32})$ *is less than one. Let us have* $t = \lfloor \frac{a_{12}}{a_{32}} \rfloor + 1$. *Then*

$$(x_{12}, x_{13}, x_{23}, x_{32}) = (a_{12} - (t-1)a_{32}, ta_{13} + (t-1)a_{23}, a_{13} + a_{23}, ta_{32} - a_{12})$$

is another strongly positive integer solution of (3.1).

Moreover, the final end of $\mathrm{I}(a_{12}, a_{13}, a_{23}, a_{32})$ *is equal to the initial end of* $\mathrm{I}(a_{12} - (t-1)a_{32}, ta_{13} + (t-1)a_{23}, a_{13} + a_{23}, ta_{32} - a_{12})$.

Proof. By Lemma 4.5, we know that $(a_{12} - (t-1)a_{32}, ta_{13} + (t-1)a_{23}, a_{13} + a_{23}, ta_{32} - a_{12})$ is an integer solution of (3.1). This solution is strongly positive if and only if $a_{12} - (t-1)a_{32} > 0$ and $ta_{32} - a_{12} > 0$, which is equivalent to $\frac{a_{12}}{a_{32}} < t < \frac{a_{12}}{a_{32}} + 1$. But the last chain of inequalities is true because, otherwise, we would have that $a_{32} | a_{12}$, and then $a_{32} = 1$ (applying item 1) of Lemma 4.6). From this equality and item 2) of Lemma 4.6, we conclude that the final end of $\mathrm{I}(a_{12}, a_{13}, a_{23}, a_{32})$ is greater than 1, which is a contradiction to the hypotheses. Finally, a simple computation shows that the final end and the initial end of both solutions are equal. □

Now we are in position to prove the result announced at the beginning of this section.

Theorem 4.8. *There exists a set of chained solutions of* (3.1).

Proof. Let us take the solution given by Lemma 4.1 as the starting point, that is, we take $(x_{12}, x_{13}, x_{23}, x_{32}) = (m_1 \bmod k, \lfloor \frac{m_1}{k} \rfloor, 1, k - m_1 \bmod k)$. Applying Lemma 4.2 to this solution, we get new solutions of (3.1). If, for one of the intervals associated with these solutions, the final end is greater than or equal to one, then the theorem is proved. In another case, we have that the last possible given solution $(a_{12}, a_{13}, a_{23}, a_{32})$ satisfies the hypotheses of Proposition 4.7. Therefore, by applying this proposition, we get another new solution. If the final end for this solution is at least 1, then the proof is finished. Otherwise, new solutions are obtained via Lemma 4.2 or, if this is not possible, via Proposition 4.7. It is clear that, in a finite number of steps, we have a set of chained solutions of (3.1). In fact, it suffices to observe that, for each interval associated to each solution, we have

$$\frac{a_{13} + a_{23}}{a_{32}} - \frac{a_{23}}{a_{12} + a_{32}} = \frac{m_1}{a_{32}(a_{12} + a_{32})} \geq \frac{1}{m_1}. \qquad \square$$

We end the paper with an example illustrating this theorem.

Example 4.9. Let us have $m_1 = 36$, $k = 23$. We are going to obtain a set of chained solutions of (3.1). First, as we saw in Example 4.4, $X_1 = \{(13, 1, 1, 10)\}$ is a set of strongly positive integer solutions of (3.1) that satisfies conditions 1) and 2) of Lemma 3.2. But, since X_1 does not satisfy condition 3) of Lemma 3.2, we can apply Proposition 4.7, taking $t = \lfloor \frac{13}{10} \rfloor + 1 = 2$, and we get $(3, 3, 2, 7)$ as a new solution. Therefore, $X_2 = \{(13, 1, 1, 10), (3, 3, 2, 7)\}$ is a new set of strongly positive integer solutions of (3.1) that, once again, satisfies conditions 1) and 2) of Lemma 3.2, but not condition 3). Now, by applying Lemma 4.2 (instead of Proposition 4.7 on this occasion), we get the solution $(3, 3, 5, 4)$. Finally, it is obvious that $X = \{(13, 1, 1, 10), (3, 3, 2, 7), (3, 3, 5, 4)\}$ is a set of chained solutions of (3.1).

Acknowledgement

The authors would like to thank the referees for their constructive and helpful comments and suggestions.

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