CORRIGENDA AND ADDENDA TO “FUNDAMENTALITY OF A CUBIC UNIT u FOR $\mathbb{Z}[u]$ ”

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ABSTRACT. Due to a certain ambiguity present in section 3 of [E. Thomas, *Fundamental units for orders in certain cubic number fields*, J. Reine Angew. Math. 310 (1979), 33-55], it became necessary to amend a crucial definition and a few proofs appearing in our article *Fundamentality of a cubic unit u for $\mathbb{Z}[u]$* . Here, the necessary corrections are provided.

Our article *Fundamentality of a cubic unit u for $\mathbb{Z}[u]$* , Math. Comp. **80** (2011), 563–578, depends, in a crucial manner, on a theorem due to E. Thomas as referenced in the article. Recently it has come to our notice that there is an ambiguity in E. Thomas’s paper which appears to result from a typographical error. Namely, Thomas defines a quantity Δ assuming $k := \phi(0) = \pm 1$ whereas the proof of his theorem (3.1) needs $k = -\phi(0) = \pm 1$. At first glance, this might appear inconsequential in view of the fact that replacing $\phi(X)$ by $-\phi(-X)$ affects neither the order under consideration nor the conclusion of the theorem (3.1); however, a careful study of the proof shows the need to have $k = -\phi(0) = \pm 1$. This ambiguity affects a part of our definition of the Thomas number. Although, fortunately, the main theorem of our article remains entirely valid, some definitions and computations need to be rectified. Below we list the necessary corrections. Throughout the following, we employ the same notation and terminology as in the article. Likewise, the section numbers appearing below correspond with the respective sections of the article. Recently, S. R. Louboutin has kindly sent us a preprint of his paper entitled “On the fundamental units of a totally real cubic order generated by a unit” in which he provides a new and shorter proof of our main theorem which does not depend on the theorem of E. Thomas mentioned above.

1.1. The definition of $\theta(a, b)$ should read as follows for the first two cases:

$$\theta(a, b) := \begin{cases} |u - v|(1 + w) & \text{if } \{u, v\} \subset I_+, \\ -|u - v|(1 + w) & \text{if } \{u, v\} \subset I. \end{cases}$$

Likewise, the final “−” should read “+”.

4.3. Append “We postpone the special case of $n = 3$ to be handled separately. From now on, assume $n \geq 4$ unless otherwise specified.” to the end of the first paragraph. Accordingly, all inequalities and equalities in that subsection comparing n to 3 should have “3” replaced with “4”, including those of the form “ $n = k + 3 \geq 3$ ”. $\alpha(m, n)$ should be defined as $(n^2 - n + 1 - m)/n$ and $\beta(m, n)$ should be defined as $n + 13/10$. For h , the calculations should read: “Both critical points”, “ $m = 1, 2, 3, 4, 5, 6, 7, 8$, $h = -2, -\frac{189}{64}, -\frac{27}{8}, -\frac{215}{64}, -3, -\frac{153}{64}, -\frac{13}{8}, -\frac{51}{64}$,”

respectively.”, “At $m = 1$, $h = 2 - n < 0$.”, and “At $m = n^2 - 2n$ and $n = k + 4 \geq 4$,

$$h = \frac{-(k^3 + 12k^2 + 45k + 51)}{(k + 4)^3} < 0.”,$$

respectively. Similarly, for i , the calculations should read “The critical point”, “ $m = 1, 2, 3, 4, 5, 6, 7, 8$,

$$i = -\frac{643}{1000}, -\frac{5943}{1000}, -\frac{11243}{1000}, -\frac{16543}{1000}, -\frac{21843}{1000}, -\frac{27143}{1000}, -\frac{32443}{1000}, -\frac{37743}{1000},$$

respectively.”, “since $m \geq 1$ ” in place of “since m is positive”, “at $m = 1$, $i = (310k + 643)/1000$ and at $m = n^2 - 2n$,

$$i = \frac{-(1000k^3 + 11300k^2 + 39110k + 37743)}{1000} < 0.”,$$

respectively. After $|u - v| > j$, the remainder of the section should read as follows: “We aim to show that $|u - v|(1 + w) > 2$. It suffices to show that $w > (2/j) - 1$. Since w is the lowest root of the cubic $f := f_{(-2n, n^2 - 1 - m)}$ and the other roots are greater than 1, it suffices to show that $(2/j) < 2$ and $f((2/j) - 1) < 0$. The first inequality follows from the fact that if $n = k + 4 \geq 4$,

$$2 - \frac{2}{j} = \frac{20m + 26k + 84}{10m + 23k + 82} > 0.$$

For the second, define

$$\begin{aligned}\phi_1(x) &= 6670x + 32154, \\ \phi_2(x) &= 4900x^2 + 108073x + 245230, \\ \phi_3(x) &= 1000x^3 + 57930x^2 + 627616x + 871656, \\ \phi_4(x) &= 5100x^3 + 212440x^2 + 1540944x + 1402272, \\ \phi_5(x) &= 1000x^4 + 2400x^3 + 236480x^2 + 1331072x + 750976.\end{aligned}$$

Then if $n = k + 4 \geq 4$ and $m = \ell + 1 \geq 1$,

$$f\left(\frac{2}{j} - 1\right) = \frac{-(1587k^5 + \phi_1(\ell)k^4 + \phi_2(\ell)k^3 + \phi_3(\ell)k^2 + \phi_4(\ell)k + \phi_5(\ell))}{(23k + 10\ell + 92)^3} < 0.$$

Finally, it only remains to check the values of $\theta(-2n, n^2 - 1 - m)$ when $n = 3$ and $1 \leq m \leq 3^2 - 2 * 3$. In these three cases, $2 < \theta \approx 2.207, 2.730, 3.112$, respectively.”

4.4. The title of the subsection should read

$$“\theta_{>}(B_4 \setminus \{(-5, 5), (-7, 11), (-9, 19), (-11, 29)\}) \subseteq (2, \infty]”.$$

All inequalities and equalities in that subsection comparing n to 4 or 5 should have “4” replaced with “5” and “5” replaced with “6”, including those of the form “ $n = k + 5 \geq 5$ ”. The definition of α should read:

$$“\alpha(m, n) := \frac{70n^4 - 107n^3 + 37n^2 - 70n - 70 - m(70n^2 - 107n - 70)}{70n^3 - 70n^2 - 70n},”$$

and β should be “ $n + 109/70$ ”. For h , the calculations should read: “None of the six real critical points”, “ $h = -17403/343000 < 0$ when $m = 1$.”, “Since

$m > 3959/4900$ ” in place of “Since $m > 3/4$ ”, “at $m = 1$,

$$h = -\frac{(65870k + 17403)}{343000} < 0$$

and at $m = n^2 - n$,

$$h = -\frac{(k^3 + 19k^2 + 118k + 239)}{(k + 6)^3} < 0.”,$$

respectively. For i , “None of the six real critical points” should read: “The critical point”, and the calculations should read “At $n = 6$, if $m = \ell + 1 \geq 1$, then $i = -(2592100\ell + 321)/343000$. Since $m > 4251/4900$,

$$\lim_{n \rightarrow \infty} i < 0.$$

If $n = k + 6 \geq 6$, then at $m = 1$, $i = -(45430k + 321)/343000 < 0$ and at $m = n^2 - n$,

$$i = -\frac{(343000k^3 + 6365100k^2 + 38505530k + 75171221)}{343000} < 0.”$$

The paragraph from “We know that” to “it follows that $\theta(a, b) > 2$.” should be replaced with the following: “Let

$$j := \beta(m, n) - \alpha(m, n)$$

so that

$$|u - v| > j > 0.$$

We aim to show that $|u - v|(1 + w) > 2$. It suffices to show that $w > (2/j) - 1$. Since w is the lowest root of the cubic $f := f_{(-2n, n^2 - 1 - m)}$ and the other roots are greater than 1, it suffices to show that $(2/j) < 2$ and $f((2/j) - 1) < 0$. The first inequality follows from the fact that if $n = k + 6 \geq 6$,

$$2 - \frac{2}{j} = \left(\frac{152k^3 + (70m + 1222)k^2 + (733m + 6487)k + 1808m + 11416}{146k^3 + (70m + 2412)k^2 + (733m + 13137)k + 1808m + 23596} \right)^2 > 0.$$

For the second, first note that the fourteen real critical points of $f((2/j) - 1)$ all have $n < 6$. Therefore, it suffices to check the boundaries: $n = 6$, $n = \infty$, $m = 1$ and $m = n^2 - n$. If $n = 6$, then $f((2/j) - 1)$ reduces to a function which is negative when $m = 1$ and whose 2 real roots are outside the range $[1, 6^2 - 6]$. Furthermore,

$$\lim_{n \rightarrow \infty} f\left(\frac{2}{j} - 1\right) < 0.$$

At $m = 1$, if $n = k + 6 \geq 6$,

$$f\left(\frac{2}{j} - 1\right) = -\frac{(15987k^2 + 209145k + 275018)}{389017} < 0.$$

At $m = n^2 - n$, $f((2/j) - 1)$ reduces to a function which is negative at $n = 6$ and whose three real roots are all less than 6.” The final part of the subsection starting with “The following four...” should be replaced with “The following eight

3-column tables exhibit our computation of these values:

	n	m	$\theta \approx$		n	m	$\theta \approx$		n	m	$\theta \approx$
	3	1	1.775		4	1	1.913		4	7	4.875
	3	2	2.508		4	2	2.678		4	8	5.176
n	2	1	1.487	m	3	3	3.047	$\theta \approx$	4	9	5.448
	2	2	2.104		3	4	3.476		4	10	5.692
					3	5	3.825		4	11	5.907
					3	6	4.107		4	12	6.092
n	5	1	1.991	m	5	6	4.708	$\theta \approx$	5	16	7.428
	5	2	2.766		5	7	5.071		5	17	7.614
	5	3	3.364		5	8	5.407		5	18	7.786
	5	4	3.868		5	9	5.719		5	19	7.941
	5	5	4.310		5	10	6.011		5	20	8.080

Evidently, $\theta(-2n-1, n^2+n-m) > 2$ in each of these cases except the four exceptions $(-2n-1, n^2+n-m) = (-5, 5), (-7, 11), (-9, 19), (-11, 29)$

5. Before the final paragraph, the following paragraph should be inserted: “Next, we must deal with the 4 exceptional cases from B_4 . For $(a, b) = (-5, 5)$, $\{u-2, u^2-2u\}$ is a complete set of fundamental units of the corresponding maximal order, and hence also of R , according to Sage. Observe that $u = (u^2-2u)/(u-2)$ and hence u is indeed a fundamental unit of R . For $(a, b) = (-7, 11)$, $\{u, u^2-3u+1\}$ is a complete set of fundamental units of the corresponding maximal order, and hence also of R , according to Sage; thus u is a fundamental unit. For $(a, b) = (-9, 19)$, $\{(1/2)u^2-2u+1/2, (3/2)u^2-8u-1/2\}$ is a complete set of fundamental units of the corresponding maximal order according to Sage. It is straightforward to verify that u is the product of these units. It follows that u is a fundamental unit of R . For $(a, b) = (-11, 29)$, $\{u^2-7u+3, u\}$ is a complete set of fundamental units of the corresponding maximal order, and hence also of R , according to Sage; thus u is a fundamental unit.” To the first sentence of the final paragraph, “aside from the four exceptions from B_4 ” should be appended.

6. The code should be replaced with:

```
case[a_, b_] :=
Module[{roots=x/.NSolve[x^3+a*x^2+b*x+1==0,x]},
  u=roots[[1]];v=roots[[2]];w=roots[[3]];
  If[!((a-b)*(a+b+2)!=0&&4*(a^3+b^3)+27-a*b*(18+a*b)<0),
    If[w>1&&1>v>0&&u<0,{0},
      If[w>1&&v>1,{1,u,v,w},
        If[1>w>0&&1>v>0,{2,u,v,w},
          If[0>w>-1&&0>v>-1,{3,u,v,w},{4,u,v,w}]]],{-1}]]];
theta[a_, b_] :=
Module[{c=case[a,b]},
  Switch[c[[1]],-1,"Undefined",0,Infinity,
    1,(c[[4]]-c[[3]])*(1+c[[2]]),
    2,(c[[4]]-c[[3]])*(-1-c[[2]]),
    3,(c[[4]]-c[[3]])*(-1-c[[2]]),
    4,(c[[3]]-c[[2]])*(1+c[[4]])]
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