EIGENVALUE DECAY OF POSITIVE INTEGRAL OPERATORS ON THE SPHERE

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ABSTRACT. We obtain decay rates for singular values and eigenvalues of integral operators generated by square integrable kernels on the unit sphere in $\mathbb{R}^{m+1}$, $m \geq 2$, under assumptions on both, certain derivatives of the kernel and the integral operators generated by such derivatives. This type of problem is common in the literature but the assumptions are usually defined using standard differentiation in $\mathbb{R}^{m+1}$. In this paper, the assumptions are all defined via the Laplace-Beltrami derivative, a concept first investigated by Rudin in the early fifties and genuinely spherical in nature. The rates we present depend on both, the differentiability order used to define the smoothness conditions and the dimension m . They are shown to be optimal.

1. INTRODUCTION

Let m be a positive integer at least 2 and S^m the unit sphere in $\mathbb{R}^{m+1}$ endowed with the induced Lebesgue measure. In this paper, we will consider integral operators defined by an expression of the form

$$(1.1) \quad \mathcal{K}(f) = \int_{S^m} K(\cdot, y) f(y) d\sigma_m(y),$$

in which the generating kernel $K: S^m \times S^m \rightarrow \mathbb{C}$ is an element of $L^2(S^m \times S^m, \sigma_m \times \sigma_m)$. In this case, (1.1) defines a compact operator on $L^2(S^m, \sigma_m)$. To simplify notation, we will denote the spaces mentioned above by $L^2(S^m \times S^m)$ and $L^2(S^m)$, respectively.

If K is positive definite in the sense that

$$\int_{S^m} \int_{S^m} K(x, y) f(x) \overline{f(y)} d\sigma_m(x) d\sigma_m(y) \geq 0, \quad f \in L^2(S^m),$$

then $\mathcal{K}$ is also self-adjoint and the standard spectral theorem for compact and self-adjoint operators is applicable: it holds that

$$\mathcal{K}(f) = \sum_{n=0}^{\infty} \lambda_n(\mathcal{K}) \langle f, f_n \rangle_2 f_n, \quad f \in L^2(S^m),$$

in which $\{\lambda_n(\mathcal{K})\}$ is a sequence of nonnegative reals (possibly finite) decreasing to 0 and $\{f_n\}$ is an orthonormal basis of $L^2(S^m)$. The numbers $\lambda_n(\mathcal{K})$ are the

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eigenvalues of $\mathcal{K}$ and the sequence $\{\lambda_n(\mathcal{K})\}$ takes into account repetitions implied by the algebraic multiplicity of each eigenvalue. Orthogonality refers to the inner product

$$\langle f, g \rangle_2 := \frac{1}{\sigma_m} \int_{S^m} f(x) \overline{g(x)} d\sigma_m(x), \quad f, g \in L^2(S^m).$$

Here, σ_m stands for the surface measure of S^m . The positive definiteness of $\mathcal{K}$ means nothing but the positivity of the integral operator $\mathcal{K}$. Since it relates to the inner product above, it is a common sense to call it L^2 -positive definiteness.

We observe that the addition of continuity to K implies that $\mathcal{K}$ is also *trace-class* (nuclear) ([4, 7, 8]), that is,

$$\sum_{f \in B} \langle \mathcal{K}^* \mathcal{K}(f), f \rangle_2^{1/2} < \infty,$$

whenever B is an orthonormal basis of $L^2(S^m)$. In particular,

$$\sum_{n=1}^{\infty} \lambda_n(\mathcal{K}) = \int_{S^m} K(x, x) d\sigma_m(x) < \infty,$$

and we can extract the most elementary result on decay rates for the eigenvalues of such operators, namely,

$$\lambda_n(\mathcal{K}) = o(n^{-1}).$$

If the integral operator $\mathcal{K}$ is compact but not self-adjoint, then decay rates for the singular values of the operator become the focus. If T is a compact operator on $L^2(S^m)$, its eigenvalues can be ordered as $|\lambda_1(T)| \geq |\lambda_2(T)| \geq \dots \geq 0$, counting multiplicities ([12]). The singular values of T are, by definition, the eigenvalues of the compact, positive and self-adjoint operator $|T| := (T^*T)^{1/2}$. The sequence $\{s_n(T)\}$ of singular values of T can also be ordered in a decreasing manner, with repetitions being included according to their multiplicities as eigenvalues of $|T|$. That being the case, the classical Weyl's inequality ([8, p. 52])

$$\Pi_{j=1}^n |\lambda_j(T)| \leq \Pi_{j=1}^n s_j(T), \quad n = 1, 2, \dots,$$

provides the convenient bridge between eigenvalues and singular values. We remark that the inequality characterizing the traceability of a compact non self-adjoint operator T on $L^2(S^m)$ reduces itself to

$$\sum_{n=1}^{\infty} s_n(T) < \infty.$$

Classical references on eigenvalue and singular value distribution of compact operators on Banach spaces are [12, 19].

The object of study in this paper is the analysis of decay rates for the sequence $\{\lambda_n(\mathcal{K})\}$ (depending on the case, the sequence $\{s_n(\mathcal{K})\}$) under additional assumptions on the kernel K . Results of this very same nature can be found in many references abroad and not necessarily in the context discussed here. In particular, we mention the use of integrated Hölder assumptions on K in [2, 3, 15] and of Lipschitz type in [5, 6]. As a matter of fact, the ideas of some of these cited papers have their origin in [11, 14] where similar kernels have been studied. The intention here is to invest in the very same question analyzed in these references, but to use the Laplace-Beltrami derivative to define the basic assumptions needed. As far as we know, this approach is new and fits more properly since such a derivative is a

concept genuinely spherical, having many interesting properties and applications in connection with Approximation Theory (see references [16, 17] and references therein) and other areas as well. By the way, the Laplace-Beltrami derivative was introduced by Rudin in [20] and further developed by Wherens in [22, 23], but in the case $m = 2$ only. The general case is fully discussed in the survey-like paper [17].

The presentation of the paper is as follows. Section 2 contains basic material about the Laplace-Beltrami derivative and the description of the main results of the paper. In Section 3, we introduce the Laplace-Beltrami integral operator and state its basic properties required in the paper. This is necessary because the approach taken to prove the main results uses a key decomposition of the integral operator, the Laplace-Beltrami integral being one of its components. Section 4 contains proofs for the main results along with other pertinent information. In Section 5 we present examples to show that two of our results are not improvable.

2. STATEMENT OF THE RESULTS

The Laplace-Beltrami derivative is a variation of the usual derivative on S^m when, in the definition of the later, one replaces the usual translation operator with the *spherical shifting operator*, which is defined by the formula

$$T_\epsilon^m(f)(x) := \frac{1}{\sigma_{m-1}(1-\epsilon^2)^{(m-1)/2}} \int_{x \cdot y = \epsilon} f(y) dy, \quad x \in S^m.$$

Here, $\epsilon \in (-1, 1)$, “ $\cdot$ ” is the usual inner product of $\mathbb{R}^{m+1}$ and dy denotes the measure element of the rim $\{y \in S^m : x \cdot y = \epsilon\}$ of the spherical cap $\{y \in S^m : x \cdot y \geq \epsilon\}$. If we write $\Delta_\epsilon := I - T_\epsilon^m$, in which I denotes the identity operator, a function $f \in L^2(S^m)$ is said to be *differentiable in the sense of Laplace-Beltrami* if there exists $\mathcal{D}f \in L^2(S^m)$ such that

$$\lim_{\epsilon \rightarrow 1^-} \|(1-\epsilon)^{-1} \Delta_\epsilon(f) - \mathcal{D}f\|_2 = 0.$$

The symbol $\|\cdot\|_2$ above stands for the usual norm of $L^2(S^m)$. The function $\mathcal{D}f$ is then called the *Laplace-Beltrami derivative* of f . Higher order derivatives are defined by the formulas $\mathcal{D}^1 = \mathcal{D}$ and

$$\mathcal{D}^r := \mathcal{D}^1 \circ \mathcal{D}^{r-1}, \quad r = 2, 3, \dots$$

We now introduce basic Sobolev-type spaces for functions on S^m .

Definition 2.1. The space of all complex functions on S^m which are differentiable, up to order r , in the sense explained above, will be denoted by W_2^r .

The operator $\mathcal{D}^r$ is a multiplier operator in the sense we now explain. If $\mathcal{H}_n^{m+1}$ is the space of all n -th degree spherical harmonics in $m+1$ variables, then $\mathcal{H}_n^{m+1}$ is a subset of W_2^r and

$$\mathcal{D}^r Y = \frac{n^r(n+m-1)^r}{m^r} Y, \quad Y \in \mathcal{H}_n^{m+1}.$$

It acts like a self-adjoint operator on W_2^r , that is,

$$(2.1) \quad \langle \mathcal{D}^r f, \bar{g} \rangle_2 = \langle f, \overline{\mathcal{D}^r g} \rangle_2, \quad f, g \in W_2^r.$$

For more information on the Laplace-Beltrami derivative we refer the reader to [17] and references mentioned there. In particular, one can find explained there a connection among the Laplace-Beltrami derivative, the usual derivative for functions on S^m and the so-called r -th spherical modulus of smoothness.

The action of the Laplace-Beltrami derivative on kernels is done separately: we keep one variable fixed and differentiate with respect to the other. The symbol $\mathcal{D}_y^r K$ will indicate the r -th order derivative of a kernel K with respect to the variable y (we will never differentiate with respect to the first variable x). For $r \in \mathbb{Z}_+$, we find it convenient to introduce the notation,

$$K_{0,r}(x, y) := \mathcal{D}_y^r K(x, y), \quad x, y \in S^m,$$

to abandon the derivative symbols. The integral operator associated with $K_{0,r}$ will be written as $\mathcal{K}_{0,r}$. At this point, it is convenient to introduce Sobolev-type spaces for kernels in a formal way.

Definition 2.2. A kernel $K \in L^2(S^m \times S^m)$ belongs to W_2^r when $K(x, \cdot) \in W_2^r$, $x \in S^m$ a.e.

We are ready to describe the main results of the paper. We emphasize that all the results take for granted the ordering on either the eigenvalues or singular values mentioned before. At first, we will prove a theorem without the L^2 -positive definiteness assumption on K and obtain a decay rate for the sequence of singular values of $\mathcal{K}$.

Theorem 2.3. *Let r be a positive integer at least $(m+1)/2$, K an element of W_2^r and $p \in (m+1, 2r+1]$. If $\mathcal{K}_{0,r}$ is bounded, then*

$$s_n(\mathcal{K}) = o(n^{-1-(2r+1-p)/m}).$$

We observe that the fact that the derivatives $\mathcal{D}_y^r K(x, \cdot)$ exist for $x \in S^m$ a.e. does not imply that $\mathcal{K}_{0,r}$ is a bounded operator. As so, the assumption on $\mathcal{K}_{0,r}$ in Theorem 2.3 is reasonable. Clearly, the smaller the parameter p , the better the estimate.

The next two results incorporate L^2 -positive definiteness as an assumption. As so, they describe decay rates for the eigenvalues of $\mathcal{K}$ under certain hypotheses on either $K_{0,r}$ or $\mathcal{K}_{0,r}$.

Theorem 2.4. *Let K be a L^2 -positive definite kernel in W_2^r . If $K_{0,r}$ belongs to $L^2(S^m \times S^m)$, then*

$$\lambda_n(\mathcal{K}) = o(n^{-1/2-2r/m}).$$

If we replace the basic assumption in Theorem 2.4 with the nuclearity of $\mathcal{K}_{0,r}$, then we can obtain an improvement on the previous decay rate.

Theorem 2.5. *Let K be a L^2 -positive definite kernel in W_2^r . If $\mathcal{K}_{0,r}$ is trace-class, then*

$$\lambda_n(\mathcal{K}) = o(n^{-1-2r/m}).$$

To close the section, we would like to inform the reader that the results above resemble those proved in [7, p. 120] and [9, 10] for the case of an interval. As a matter of fact, one can interpret Theorems 2.4 and 2.5 as spherical versions of some of the results proved in those references.

3. THE LAPLACE-BELTRAMI INTEGRAL

In this section we will introduce an integral operator that acts like an inverse for the Laplace-Beltrami derivative: the Laplace-Beltrami integral operator. Its powers appear quite naturally in decompositions for $\mathcal{K}$ when the generating kernel K satisfies smoothness assumptions defined via the Laplace-Beltrami derivative. For that reason, the Laplace-Beltrami integral operator enters in the proofs of all the main results previously described.

The *Laplace-Beltrami integral operator* is the unique linear mapping $J: L^2(S^m) \rightarrow L^2(S^m)$ defined by the conditions $J1 = 1$ and

$$(3.1) \quad JY = \frac{m}{n(n+m-1)} Y, \quad Y \in \mathcal{H}_n^{m+1}, \quad n = 1, 2, \dots$$

It is a bounded linear operator acting like an inverse of the Laplace-Beltrami derivative in the sense that

$$(3.2) \quad \mathcal{D}Jf = J\mathcal{D}f = f, \quad f \in \bigoplus_{n=1}^{\infty} \mathcal{H}_n^{m+1}.$$

This operator can also be defined via spherical convolution. Indeed, beginning with the function $F: (-1, 1) \rightarrow \mathbb{R}$ given by

$$F(t) = m \int_{-1}^t (1-s^2)^{-m/2} \int_{-1}^s dw_m(u) ds, \quad t \in (-1, 1),$$

in which $dw_m(u) := (1-u^2)^{(m-2)/2} du$, a few calculations reveal that F belongs to $L^1([-1, 1], w_m)$ while the formula

$$\mathcal{L}(t) := F(t) + 1 - \|F\|_{1,m}, \quad t \in (-1, 1),$$

defines a normalized element $\mathcal{L}$ of $L^1([-1, 1], w_m)$ with first Fourier-Legendre coefficient equals 1. Here, the notation $\|\cdot\|_{1,m}$ indicates the usual norm in $L^1([-1, 1], w_m)$. In addition, one can prove that

$$Jf(x) = \int_{S^m} \mathcal{L}(x \cdot y) f(y) d\sigma_m(y), \quad x \in S^m, \quad f \in L^2(S^m).$$

This formula shows that Jf is (a multiple of) the spherical convolution $\mathcal{L} * f$ of $\mathcal{L}$ and f . The powers of J are defined recursively: $J^1 = J$ and $J^r := J \circ J^{r-1}$, $r = 2, 3, \dots$. The easily deduced formula

$$(3.3) \quad \langle J^r f, \overline{g} \rangle_2 = \langle f, \overline{J^r g} \rangle_2, \quad f, g \in L^2(S^m),$$

encompasses the self-adjointness of J^r . All of these facts are proved in [17]. Theorem 3.1 below describes a property we could not find justified anywhere. As so, a proof is included.

Theorem 3.1. *The operator J^r is compact.*

Proof. Fix r and orthonormal bases $\{Y_{n,k} : k = 1, 2, \dots, N(m, n)\}$ of $\mathcal{H}_n^{m+1}$, $n = 0, 1, \dots$. Consider a function f in $L^2(S^m)$ along with its condensed spherical harmonic expansion $f \sim \sum_{n=0}^{\infty} \Pi_n(f)$, in which Π_n is the $L^2(S^m)$ -orthogonal projection of $L^2(S^m)$ over $\mathcal{H}_n^{m+1}$ given by the formula ([18, p. 35]),

$$\Pi_n(f) = \sum_{k=1}^{N(m,n)} \hat{f}(n, k) Y_{n,k}.$$

Since the Fourier-Legendre coefficient appearing above is computed through the formula

$$\hat{f}(n, k) = \frac{1}{\sigma_m} \int_{S^m} f \overline{Y_{n,k}} d\sigma_m,$$

the invariance property (3.3) and (3.1) lead to

$$\Pi_n(J^r f) = \frac{m^r}{n^r(n+m-1)^r} \Pi_n(f), \quad n = 0, 1, \dots$$

Since the set of compact operators on $L^2(S^m)$ is a closed subset of the space of all bounded linear operators on $L^2(S^m)$ with respect to the operator norm, the proof will be completed as long as we show that the series

$$\sum_{n=0}^{\infty} \frac{m^r}{n^r(n+m-1)^r} \Pi_n(f),$$

converges to $J^r f$ in the $L^2(S^m)$ -norm. A few calculations produce the inequalities

$$\begin{aligned} \left\| J^r f - \sum_{n=0}^l \frac{m^r}{n^r(n+m-1)^r} \Pi_n(f) \right\|_2 &\leq \sum_{n=l+1}^{\infty} \frac{m^r}{n^r(n+m-1)^r} \|\Pi_n(f)\|_2 \\ &\leq \sum_{n=l+1}^{\infty} \frac{m^r}{n^r(n+m-1)^r} \|f\|_2. \end{aligned}$$

Using the obvious inequality $n^{2r} \leq n^r(n+m-1)^r$, we finally see that

$$\left\| J^r f - \sum_{n=0}^l \frac{m^r}{n^r(n+m-1)^r} \Pi_n(f) \right\|_2 \leq m^r \|f\|_2 \sum_{n=l+1}^{\infty} n^{-2r}.$$

Clearly, the very last series above approaches 0 as $l \rightarrow \infty$. $\square$

To finish the section, we need to order the eigenvalues of J^r in accordance with the spectral theorem for compact and self-adjoint operators. In other words, we will assume they are listed in decreasing order counting the repetitions implied by the formulas $J^r 1 = 1$ and

$$J^r Y = \frac{m^r}{n^r(n+m-1)^r} Y, \quad Y \in \mathcal{H}_n^{m+1}, \quad n = 1, 2, \dots$$

The numbers $N(m, n) := \dim \mathcal{H}_n^{m+1}$ are given by $N(m, 0) = 1$, $N(m, 1) = m + 1$ and

$$N(m, n) = \binom{m+n}{m} - \binom{m+n-2}{m}, \quad n \geq 2.$$

As so, we may think the sequence $\{\lambda_n(J^r)\}$ is block ordered in such a way that the first block contains the eigenvalue 1 and the $(n+1)$ -th block ($n \geq 1$) contains $N(m, n)$ entries equal to $m^r n^{-r} (n+m-1)^{-r}$. For future reference, we notice that the first entry in the $(n+1)$ -th block corresponds to the index

$$N(m, 0) + N(m, 1) + \dots + N(m, n-1) + 1 = N(m+1, n-1) + 1.$$

As for the last one, it corresponds to

$$N(m, 0) + N(m, 1) + \dots + N(m, n-1) + N(m, n) = N(m+1, n).$$

4. PROOFS OF THE MAIN RESULTS

This section contains proofs for Theorems 2.3–2.5. They depend upon some general properties of compact operators and their singular values which we now describe in a form adapted to our needs. They can be found in standard references on operator theory such as [7, 8, 12, 19] and depend on the ordering of eigenvalues and singular values as previously mentioned.

Lemma 4.1. *Let T be a compact operator on $L^2(S^m)$. The following assertions hold:*

(i) *If T is self-adjoint, then*

$$s_n(T) = |\lambda_n(T)|, \quad n = 1, 2, \dots$$

(ii) *If A is a bounded operator on $L^2(S^m)$, then both, AT and TA , are compact. In addition,*

$$\max\{s_n(AT), s_n(TA)\} \leq \|A\| s_n(T), \quad n = 1, 2, \dots$$

(iii) *If A is a linear operator on $L^2(S^m)$ of rank at most l , then*

$$s_{n+l}(T) \leq s_n(T + A), \quad n = 1, 2, \dots$$

(iv) *If A is a compact operator on $L^2(S^m)$, then*

$$s_{n+k-1}(AT) \leq s_n(A)s_k(T), \quad n, k = 1, 2, \dots$$

The following additional lemma regarding the singular values of an integral operator generated by a square-integrable kernel is proved in [12, p. 40].

Lemma 4.2. *If $K \in L^2(S^m \times S^m)$, then*

$$\sum_{n=1}^{\infty} s_n^2(K) = \|K\|_2^2.$$

The key idea behind the proof of the main results previously stated resides in the following estimation for the singular values of $\mathcal{K}$, which holds when K is smooth enough.

Lemma 4.3. *Let K be an element of W_2^r . If $\mathcal{K}_{0,r}$ is bounded then*

$$s_{n+1}(\mathcal{K}) \leq s_n(\mathcal{K}_{0,r}J^r), \quad n = 1, 2, \dots$$

Proof. Consider the orthogonal projection Q of $L^2(S^m)$ onto $\bigoplus_{\ell=1}^{\infty} \mathcal{H}_{\ell}^{m+1}$. Since $I - Q$ is a projection onto the orthogonal complement of $\bigoplus_{\ell=1}^{\infty} \mathcal{H}_{\ell}^{m+1}$, then $\mathcal{K} - \mathcal{K}Q$ is an operator on $L^2(S^m)$ of rank at most 1. Using Lemma 4.1(iii), we may deduce that

$$(4.1) \quad s_{n+1}(\mathcal{K}) \leq s_n(\mathcal{K} - \mathcal{K}(I - Q)) = s_n(\mathcal{K}Q), \quad n = 1, 2, \dots$$

To proceed, we need a convenient decomposition for $\mathcal{K}Q$. Looking at the action of $\mathcal{K}Q$ on a generic element f from $L^2(S^m)$ and using (3.2) we see that

$$\mathcal{K}Q(f) = \int_{S^m} K(\cdot, y)Qf(y) d\sigma_m(y) = \int_{S^m} K(\cdot, y)\mathcal{D}^r J^r Qf(y) d\sigma_m(y).$$

Since $K \in W_2^r$, we employ (2.1) to obtain

$$\mathcal{K}Q(f) = \int_{S^m} K_{0,r}(\cdot, y)J^r(Qf)(y) d\sigma_m(y) = \mathcal{K}_{0,r}J^rQ(f),$$

that is, $\mathcal{K}Q = \mathcal{K}_{0,r}J^rQ$. Now, assuming $\mathcal{K}_{0,r}$ is bounded, we can apply (4.1) and Lemma 4.1(ii) to see that

$$s_{n+1}(\mathcal{K}) \leq s_n(\mathcal{K}Q) \leq \|Q\|s_n(\mathcal{K}_{0,r}J^r) \leq s_n(\mathcal{K}_{0,r}J^r), \quad n = 1, 2, \dots$$

The proof is complete. $\square$

In the next three lemmas, we detach technical inequalities to be used in the proofs ahead. The first one includes a refinement to the fact that $N(m, n) = O(n^{m-1})$ ([18]).

Lemma 4.4. *There exists an integer $\beta(m) \geq 1$ such that*

$$N(m+1, n) \leq 2n^m, \quad n \geq \beta(m).$$

Proof. It is a consequence of the well-known formula

$$\lim_{n \rightarrow \infty} \frac{N(m, n)}{n^{m-1}} = \frac{2}{(m-1)!}, \quad m \geq 2,$$

which can be easily seen from the definition of $N(m, n)$ (see Chapter 5 in [1] for example). $\square$

Lemma 4.5. *If m is an integer at least 2, then*

$$(n+1)^m - (n^m + 1) + 1 \leq m2^{m-1}n^{m-1}, \quad n \geq 1.$$

Proof. It suffices to apply the mean value theorem to the function x^m on the interval $[n, n+1]$ and estimate the resulting formula conveniently. $\square$

Lemma 4.6. *Let m be an integer at least 2, r a nonnegative integer and $\epsilon \in (0, 1)$. Then there exists $\delta = \delta(m, \epsilon) \geq 1$ so that*

$$(1 - \epsilon)n^{2r} \leq (n-1)^r(n+m-2)^r, \quad n \geq \delta.$$

Proof. In the cases $m = 2, 3$, it suffices to observe that $(n-1)^r(n+m-2)^rn^{-2r}$ approaches 1 from the left when $n \rightarrow \infty$. As for the case $m > 3$, $(n-1)^r(n+m-2)^rn^{-2r}$ approaches 1 from the right when $n \rightarrow \infty$. So, the property follows from the obvious inequality $1 - \epsilon < 1$. $\square$

The following technical result is borrowed from [13].

Lemma 4.7. *Let $\{a_n\}$ be a decreasing sequence of positive real numbers. If the series $\sum_{n=1}^{\infty} n^{\alpha} a_n^{\beta}$ is convergent for some positive constants α and β , then $a_n = o(n^{-(\alpha+1)/\beta})$.*

We now proceed to the proofs of the main results in the paper.

Proof of Theorem 2.3. We assume $\mathcal{K}_{0,r}$ is bounded and show that

$$(4.2) \quad \sum_{n=1}^{\infty} n^{(2r+1-p)/m} s_n(\mathcal{K}) < \infty.$$

Lemma 4.7 takes care of the rest. In the first half of the proof we intend to derive the convergence of the series

$$(4.3) \quad \sum_{n=1}^{\infty} n^{2r+m-p} s_n^m(\mathcal{K}).$$

An application of Lemma 4.1(ii) in the inequality provided by Lemma 4.3 leads to

$$s_{n+1}(\mathcal{K}) \leq \|\mathcal{K}_{0,r}\| s_n(J^r), \quad n = 1, 2, \dots$$

Since J^r is self-adjoint, Lemma 4.1(i) implies that

$$s_{n+1}(\mathcal{K}) \leq \lambda_n(J^r) \|\mathcal{K}_{0,r}\|, \quad n = 1, 2, \dots$$

Keeping in mind the information provided at the end of the previous section, we can deduce that

$$(n-1)^r (n+m-2)^r \sum_{k=N(m+1,n-2)+1}^{N(m+1,n-1)} s_{k+1}(\mathcal{K}) \leq m^r \|\mathcal{K}_{0,r}\| N(m, n-1), \quad n = 2, 3, \dots$$

Using the fact that the sequence $\{s_n(\mathcal{K})\}$ is decreasing, we can estimate in the previous inequality to reach

$$(n-1)^r (n+m-2)^r s_{N(m+1,n-1)+1}(\mathcal{K}) \leq m^r \|\mathcal{K}_{0,r}\|, \quad n = 2, 3, \dots$$

Invoking Lemma 4.4 to select $\beta = \beta(m) \geq 1$ so that

$$N(m+1, n-1) + 1 \leq 2(n-1)^m + 1 \leq (2n)^m, \quad n \geq \beta,$$

the previous inequality can be reduced to

$$(n-1)^r (n+m-2)^r s_{(2n)^m}(\mathcal{K}) \leq m^r \|\mathcal{K}_{0,r}\|, \quad n \geq \beta.$$

This is the point where we choose $\epsilon \in (0, 1)$ and pick δ as described in Lemma 4.6 to write

$$(1-\epsilon)n^{2r} s_{(2n)^m}(\mathcal{K}) \leq m^r \|\mathcal{K}_{0,r}\|, \quad n \geq \max\{\delta, \beta\}.$$

It is now clear that

$$\sum_{n \geq \max\{\delta, \beta\}} n^{2r+m-p} s_{(2n)^m}(\mathcal{K}) \leq \frac{m^r}{1-\epsilon} \|\mathcal{K}_{0,r}\| \sum_{n \geq \max\{\delta, \beta\}} n^{m-p} < \infty,$$

due to the fact that $p-m > 1$. Consequently,

$$\sum_{n \geq \max\{\delta, \beta\}} (2n)^{2r+m-p} s_{(2n)^m}(\mathcal{K}) \leq 2^{2r+m} \sum_{n \geq \max\{\delta, \beta\}} n^{2r+m-p} s_{(2n)^m}(\mathcal{K}) < \infty$$

and

$$\sum_{n \geq \max\{\delta, \beta\}} (2n+1)^{2r+m-p} s_{(2n+1)^m}(\mathcal{K}) \leq 4^{2r+m} \sum_{n \geq \max\{\delta, \beta\}} n^{2r+m-p} s_{(2n)^m}(\mathcal{K}) < \infty.$$

The convergence of the series in (4.3) follows. To close the proof, we will use this convergence to show (4.2) holds. To do that, it suffices to show that the following re-ordering of (4.2),

$$\sum_{n=1}^{\infty} \sum_{k=0}^{(n+1)^m - (n^m+1)} (n^m + k)^{(2r+1-p)/m} s_{n^m+k}(\mathcal{K})$$

converges. Call the inner sum in the double sum above $S(n)$ and observe that

$$(4.4) \quad S(n) \leq [(2n)^m]^{(2r+1-p)/m} \sum_{k=0}^{(n+1)^m - (n^m+1)} s_{n^m+k}(\mathcal{K}).$$

The sequence $\{s_n(\mathcal{K})\}$ being decreasing, (4.4) can be reduced to

$$S(n) \leq (2n)^{2r+1-p} s_{n^m}(\mathcal{K}) [(n+1)^m - (n^m+1) + 1].$$

Invoking Lemma 4.5, we now see that

$$S(n) \leq 4^r m 2^{m-1} n^{2r+m-p} s_{n^m}(\mathcal{K}).$$

It follows that

$$\sum_{n=1}^{\infty} S(n) \leq 4^r m 2^{m-1} \sum_{n=1}^{\infty} n^{2r+m-p} s_{n^m}(\mathcal{K}) < \infty.$$

The proof is complete. $\square$

Proof of Theorem 2.4. We proceed as in the proof of Theorem 2.3. We assume $K_{0,r} \in L^2(S^m \times S^m)$ and show that

$$\sum_{n=1}^{\infty} n^{4r/m} \lambda_n^2(\mathcal{K}) < \infty.$$

Combining Lemma 4.3 with Lemma 4.1(iv) we can deduce the inequalities

$$s_{n+k}(\mathcal{K}) \leq s_{n+k-1}(\mathcal{K}_{0,r} J^r) \leq s_k(\mathcal{K}_{0,r}) s_n(J^r), \quad n, k = 1, 2, \dots,$$

while the setting in Theorem 2.4 allows us to write

$$(4.5) \quad \lambda_{n+k}(\mathcal{K}) \leq s_k(\mathcal{K}_{0,r}) \lambda_n(J^r), \quad n, k = 1, 2, \dots$$

Next, we square both sides of (4.5) and sum in k , letting k run inside the $(n+1)$ -th block of the sequence of the eigenvalues of J^r :

$$n^{2r} (n+m-1)^{2r} \sum_{k=N(m+1, n-1)+1}^{N(m+1, n)} \lambda_{N(m+1, n)+k}^2(\mathcal{K}) \leq m^{2r} \sum_{k=N(m+1, n-1)+1}^{N(m+1, n)} s_k^2(\mathcal{K}_{0,r}).$$

Estimating on the left-hand side leads to

$$n^{4r} \sum_{k=N(m+1, n-1)+1}^{N(m+1, n)} \lambda_{N(m+1, n)+k}^2(\mathcal{K}) \leq m^{2r} \sum_{k=N(m+1, n-1)+1}^{N(m+1, n)} s_k^2(\mathcal{K}_{0,r}), \quad n = 1, 2, \dots$$

Due to Lemma 4.2, it is now clear that

$$(4.6) \quad \sum_{n=1}^{\infty} n^{4r} \sum_{k=N(m+1, n-1)+1}^{N(m+1, n)} \lambda_{N(m+1, n)+k}^2(\mathcal{K}) \leq m^{2r} \|K_{0,r}\|_2^2 < \infty.$$

To proceed, we apply Lemma 4.4 to select a constant $\beta(m) \geq 1$ so that

$$2N(m+1, n) \leq 2n^m \leq (2n)^m, \quad n \geq \beta(m).$$

Since $n^{m-1} = O(N(m, n))$ and $\{\lambda_n(\mathcal{K})\}$ decreases, we now see that

$$\begin{aligned} \sum_{n=\beta(m)}^{\infty} (2n)^{4r+m-1} \lambda_{(2n)^m}^2(\mathcal{K}) &\leq C \sum_{n=\beta(m)}^{\infty} (2n)^{4r} N(m, n) \lambda_{(2n)^m}^2(\mathcal{K}) \\ &\leq C 2^{4r} \sum_{n=1}^{\infty} n^{4r} \sum_{k=N(m+1, n-1)+1}^{N(m+1, n)} \lambda_{N(m+1, n)+k}^2(\mathcal{K}), \end{aligned}$$

in which C is a constant depending on m only. Hence,

$$\sum_{n=\beta(m)}^{\infty} (2n)^{4r+m-1} \lambda_{(2n)^m}^2(\mathcal{K}) < \infty,$$

due to (4.6). On the other hand,

$$\begin{aligned} \sum_{n=\beta(m)}^{\infty} (2n+1)^{4r+m-1} \lambda_{(2n+1)^m}^2(\mathcal{K}) &\leq 2^{4r+m} \sum_{n=\beta(m)}^{\infty} (2n)^{4r+m-1} \lambda_{(2n+1)^m}^2(\mathcal{K}) \\ &\leq 2^{4r+m} \sum_{n=\beta(m)}^{\infty} (2n)^{4r+m-1} \lambda_{(2n)^m}^2(\mathcal{K}) < \infty, \end{aligned}$$

due to the previous calculations. Hence, we may infer that

$$\sum_{n=1}^{\infty} n^{4r+m-1} \lambda_{n^m}^2(\mathcal{K}) < \infty.$$

Repeating the same trick used in the second half of the proof of Theorem 2.3 leads to

$$\sum_{n=1}^{\infty} (n+1)^{m-(n^m+1)} \sum_{k=0}^{n^m+1} (n^m+k)^{4r/m} \lambda_{n^m+k}^2(\mathcal{K}) < \infty.$$

Thus,

$$\sum_{n=1}^{\infty} n^{4r/m} \lambda_n^2(\mathcal{K}) < \infty,$$

and an application of Lemma 4.7 closes the proof. $\square$

Proof of Theorem 2.5. We assume $\mathcal{K}_{0,r}$ is trace-class and show that

$$\sum_{n=1}^{\infty} n^{2r/m} \lambda_n(\mathcal{K}) < \infty. \quad (4.42)$$

The proof is identical to the proof of Theorem 2.4 until (4.5). From there, we can write

$$n^{2r} \lambda_{N(m+1,n)+k}(\mathcal{K}) \leq n^r (n+m-1)^r \lambda_{N(m+1,n)+k}(\mathcal{K}) \leq m^r s_k(\mathcal{K}_{0,r}), \quad n, k = 1, 2, \dots$$

By adding on k and n leads to

$$\sum_{n=1}^{\infty} n^{2r} \sum_{k=N(m+1,n-1)+1}^{N(m+1,n)} \lambda_{N(m+1,n)+k}(\mathcal{K}) \leq m^r \sum_{n=1}^{\infty} \sum_{k=N(m+1,n-1)+1}^{N(m+1,n)} s_k(\mathcal{K}_{0,r}).$$

The expression on the right-hand side of the inequality above is at most the trace of $\mathcal{K}_{0,r}$, hence finite. Proceeding as in the proof of the previous theorem, we deduce that

$$\sum_{n=1}^{\infty} n^{2r+m-1} \lambda_{n^m}(\mathcal{K}) < \infty.$$

Repeating the tricks used in the second half of that same proof we reach the announced convergence. $\square$

5. OPTIMALITY

In this section we construct examples to show that the decay rates presented in Theorems 2.4 and 2.5 are optimal.

Theorem 5.1. *Let $\varepsilon > 0$ be fixed. If $r \geq 0$, then there exists an L^2 -positive definite kernel K possessing the following features:*

- (i) K belongs to W_2^r .
- (ii) $\mathcal{K}_{0,r}$ is trace-class.
- (iii) $\lambda_n(K) = o(n^{-1-2r/m})$.
- (iv) If $\lim_{n \rightarrow \infty} n^{\varepsilon+1+2r/m} \lambda_n(K)$ exists, then it is positive.

Proof. Let K be a kernel having a condensed spherical harmonic expansion in the form

$$(5.1) \quad K(x, y) \sim 1 + \sum_{n=1}^{\infty} \frac{N(m, n)}{n^{m(1+\varepsilon)+2r}} P_n^m(x \cdot y), \quad x, y \in S^m.$$

Here, P_n^m is the Legendre polynomial of degree n associated with the integer m . The spherical harmonic expansion of K is easily obtained with the help of the well-known addition formula ([18, 21])

$$\sum_{k=1}^{N(m, n)} Y_{nk}(x) \overline{Y_{nk}(y)} = N(m, n) P_n^m(x \cdot y), \quad n = 0, 1, \dots, \quad x, y \in S^m.$$

Since

$$1 + \sum_{n=1}^{\infty} \frac{N(m, n)}{n^{m(1+\varepsilon)+2r}} \leq 1 + C \sum_{n=1}^{\infty} \frac{1}{n^{2r+1+m\varepsilon}} < \infty,$$

for some constant C depending on m , the expansion in (5.1) converges uniformly to $K(x, y)$. As so, the kernel K is continuous and, consequently, K is L^2 -positive definite. The kernel $K_{0,r}$ has a condensed spherical harmonic expansion in the form

$$K_{0,r}(x, y) \sim \frac{1}{m^r} \sum_{n=1}^{\infty} \frac{N(m, n) n^r (n + m - 1)^r}{n^{m(1+\varepsilon)+2r}} P_n^m(x \cdot y), \quad x, y \in S^m.$$

Since

$$\sum_{n=1}^{\infty} \frac{N(m, n) n^r (n + m - 1)^r}{n^{m(1+\varepsilon)+2r}} \leq C_1 \sum_{n=1}^{\infty} \frac{1}{n^{1+m\varepsilon}} < \infty,$$

for some constant C_1 depending on m and r , $K_{0,r}$ is continuous. It follows that $K \in W_2^r$. Using the orthonormality of the $Y_{n,k}$, we can deduce that

$$\begin{aligned} \int_{S^m} K_{0,r}(x, x) d\sigma_m(x) &= \frac{1}{m^r} \sum_{n=1}^{\infty} \frac{n^r (n + m - 1)^r}{n^{m(1+\varepsilon)+2r}} \sum_{k=1}^{N(m, n)} \int_{S^m} |Y_{n,k}(x)|^2 d\sigma_m(x) \\ &\leq \frac{\sigma_m C_1}{m^r} \sum_{n=1}^{\infty} \frac{1}{n^{1+m\varepsilon}} < \infty. \end{aligned}$$

In particular, $\mathcal{K}_{0,r}$ is trace-class. Finally, let us move to the decay rate of the sequence $\lambda_n(K)$. It is composed by blocks, the first one having the entry 1 and the $(n + 1)$ -th block ($n \geq 1$) having $N(m, n)$ entries equal to $n^{-(m(\varepsilon+1)+2r)}$. A quick

analysis reveals that in order to ratify the decay rate provided by Theorem 2.5, it suffices to verify that

$$\lim_{n \rightarrow \infty} \frac{(1 + N(m, 1) + \cdots + N(m, n))^{(m+2r)/m}}{n^{m(1+\varepsilon)+2r}} = 0.$$

But this equality is easily verified due to the fact that the quotient in the limit can be bounded above by

$$\frac{N(m+1, n)^{(m+2r)/m}}{n^{m(1+\varepsilon)+2r}} \leq C_2 \frac{1}{n^{m\varepsilon}},$$

for some positive constant C_2 . In order to complete the proof, we will assume that the limit

$$\lim_{n \rightarrow \infty} n^{\varepsilon+1+2r/m} \lambda_n(\mathcal{K})$$

exists and will consider the subsequence $\{s_n\}$ of $\{n^{\varepsilon+1+2r/m} \lambda_n(\mathcal{K})\}$ given by

$$s_n := \frac{(2 + N(m, 1) + \cdots + N(m, n-1))^{\varepsilon+1+2r/m}}{n^{m(1+\varepsilon)+2r}}.$$

Clearly,

$$s_n \geq \frac{N(m+1, n-1)^{\varepsilon+1+2r/m}}{n^{m(1+\varepsilon)+2r}} \geq C_3 \frac{(n-1)^{m(1+\varepsilon)+2r}}{n^{m(1+\varepsilon)+2r}},$$

for some positive constant C_3 depending on m and r , so that

$$\lim_{n \rightarrow \infty} s_n \geq C_3 > 0.$$

The proof is complete. $\square$

In a similar way one can prove the following result.

Theorem 5.2. *Let $\varepsilon > 0$ be fixed. If $r \geq m/4$, then there exists a L^2 -positive definite kernel K possessing the following features:*

- (i) K belongs to W_2^r .
- (ii) $K_{0,r}$ belongs to $L^2(S^m \times S^m)$.
- (iii) $\lambda_n(\mathcal{K}) = o(n^{-(m+4r)/2m})$.
- (iv) If $\lim_{n \rightarrow \infty} n^{\varepsilon+(m+4r)/2m} \lambda_n(\mathcal{K})$ exists, then it is positive.

Proof. It suffices to consider the kernel K having a condensed spherical harmonic expansion in the form

$$K(x, y) \sim 1 + \sum_{n=1}^{\infty} \frac{N(m, n)}{n^{m(\varepsilon+1/2)+2r}} P_n^m(x \cdot y), \quad x, y \in S^m$$

and proceed as in the proof of Theorem 5.1. $\square$

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