A CONVERGENCE ANALYSIS OF REGULARIZATION BY DISCRETIZATION IN PREIMAGE SPACE

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ABSTRACT. In this paper we investigate the regularizing properties of discretization in preimage space for linear and nonlinear ill-posed operator equations with noisy data. We propose to choose the discretization level, that acts as a regularization parameter in this context, by a discrepancy principle. While general convergence has been shown not to hold, we provide convergence results under appropriate conditions on the exact solution.

1. INTRODUCTION

A wide range of inverse problems can be formulated as nonlinear ill-posed operator equations,

$$(1.1) \quad F(x) = y$$

with a forward operator $F : D(\subseteq X) \rightarrow Y$, mapping between Hilbert spaces X and Y .

Since typically F^{-1} is not continuous and only noisy data y^δ with some noise level δ in

$$(1.2) \quad \|y - y^\delta\| \leq \delta$$

are available, regularization has to be applied (see, e.g., [1, 4, 6, 17, 20, 21, 22, 25, 27, 31, 33, 34]).

Motivated by the fact that on one hand, in practice, very often inverse problems are just discretized and then solved, on the other hand this discretization itself can be expected to have a regularizing effect, we investigate convergence in the sense of a regularization method for discretization in preimage space. For this purpose we consider a sequence $(X_n)_{n \in \mathbb{N}}$ of finite dimensional subspaces of the preimage space $X_n \subseteq X$, along with the corresponding orthogonal projections $P_n := \text{Proj}_{X_n}$, and assume that $\|(I - P_n)x^\dagger\| \rightarrow 0$ as $n \rightarrow \infty$. (The latter is the case, e.g., if $X_{n+1} \subseteq X_n$ and $\bigcup_{n \in \mathbb{N}} X_n = X$.) Therewith we define a regularized approximation as a solution of the finite dimensional minimization problem

$$(1.3) \quad x_n^\delta \in \operatorname{argmin}\{\|F(z_n) - y^\delta\|^2 : z_n \in D \cap X_n\},$$

an approach which is also often called the least squares projection. The discretization level n has to be chosen appropriately in order to balance between the approximation error, that decays as $n \rightarrow \infty$ and the noise propagation, that becomes

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stronger for larger n . Here, we determine n_* by the discrepancy principle

$$(1.4) \quad n_* = \min\{n \in \mathbb{N} : \|F(x_n^\delta) - y^\delta\| \leq \tau\delta\}$$

(with $\tau > 1$ in the linear case, and $\tau > \frac{1+\eta}{1-\eta}$, with $\eta \in (0, 1)$ as in (1.5) below in the nonlinear case). A very efficient approach especially in the context of nonlinear problems is to apply a multilevel strategy for the computation of the regularized solution, i.e., to start on the coarsest level $n = 1$ of projection and successively compute the solution on level n making use of information on the solution on level $n - 1$. When doing so, the discrepancy principle just acts as a stopping rule for this multilevel iteration.

Regularization by discretization in *image space* (also called the dual least squares method or self regularization) has been well investigated in the literature (see, e.g., [26, 28, 32] for the linear case, as well as [10, 12] for the nonlinear case) and can be shown to be generically convergent. However, regularization by discretization in *preimage space* (called least squares method in [26]) is used more often in practice due its often much easier implementation. Additionally, it is of particular interest due to the possibility of using problem adapted ansatz functions for the inverse problem solution, as well as the above mentioned rather straightforward multilevel strategy. Previous results in the literature on this approach for linear ill-posed problems [7, 8, 23], and especially a counterexample by Seidman [30] indicate, that convergence cannot occur for general $x^\dagger \in X$ but only under special assumptions on the solution. Here we formulate a convergence condition that assumes sufficiently fast convergence of the approximation error in image space to compensate for the growth of the norm of the inverse of the projected forward operator. This sufficiently fast convergence of the approximation error in its turn will be implied by a sufficiently strong source condition and hence is satisfied for sufficiently regular solutions, along with appropriate approximation properties of the ansatz spaces X_n .

We wish to mention that the regularizing effect of discretization has been studied in a very general setting with possible discretization of both preimage and image space, e.g., in Chapter 3 of [20] and in [24] for the linear case. The latter also provides results on convergence and convergence rates for noisy data with an a priori and a Lepskii type a posteriori discretization level choice.

Our new results especially pertain to taking into account noisy data by means of a discrepancy principle type a posteriori discretization level choice, as well as to extension to the nonlinear situation.

It is well-known that a convergence analysis for nonlinear ill-posed problems requires some assumption on the structure of the nonlinearity. Here we will assume that the often used Scherzer condition (also called tangential cone condition, cf. [29]) holds:

$$(1.5) \quad \|F(x) - F(\bar{x}) - F'(x)(x - \bar{x})\| \leq \eta \|F(x) - F(\bar{x})\| \quad \forall x, \bar{x} \in D$$

with $\eta \in (0, 1)$, which implies

$$\|F(x) - F(\bar{x})\| \leq \frac{1}{1-\eta} \|F'(x)(x - \bar{x})\|, \quad \|F'(x)(x - \bar{x})\| \leq (1+\eta) \|F(x) - F(\bar{x})\|.$$

Note that for this purpose $F'(x)$ need not necessarily be a Fréchet derivative. It suffices to have a bounded linear operator called $F'(x)$ mapping between X and Y and such that (1.5) holds.

In what follows we will assume that an exact solution $x^\dagger$ of (1.1) exists, i.e., $F(x^\dagger) = y$. Moreover, to simplify the exposition we assume that

$$\forall x \in D : \mathcal{N}(F'(x)) = \{0\},$$

which implies uniqueness of the solution to $F(x) = y$ in the linear case and by (1.5) also in the nonlinear case: From $0 = \|F(x) - F(\tilde{x})\| \geq \frac{1}{1+\eta} \|F'(x)(x - \tilde{x})\|$ it follows that $x - \tilde{x} \in \mathcal{N}(F'(x)) = \{0\}$. Note, however, that most of our results can be generalized to operators with nontrivial nullspaces.

The remainder of this paper is organized as follows. In Section 2, we first provide a convergence analysis for the linear case, whereas the general nonlinear situation is treated in Section 3. For the latter generalization we follow two approaches, namely a variational formulation and an operator equation formulation. Numerical experiments in Section 4 illustrate the results of our convergence analysis.

2. THE LINEAR CASE

In this section we first consider the linear case $F = T \in L(X, Y)$, so that (1.1) becomes

$$(2.1) \quad Tx = y.$$

The first order necessary condition for a minimizer of (1.3), i.e., of

$$(2.2) \quad \min_{z_n \in X_n} \|Tz_n - y^\delta\|^2 = \min_{z_n \in X_n} \|TP_n z_n - y^\delta\|^2$$

implies

$$(2.3) \quad (TP_n)^*(TP_n z_n - y^\delta) = 0,$$

hence

$$TP_n z_n - y^\delta \in \mathcal{N}((TP_n)^*) = \mathcal{R}(TP_n)^\perp = Y_n^\perp,$$

i.e.,

$$(2.4) \quad Q_n(TP_n x_n^\delta - y^\delta) = 0$$

with the orthogonal projections $Q_n := \text{Proj}_{Y_n}$, and the subspaces $Y_n := TX_n \subseteq Y$; see also [20, 21, 24] for a convergence analysis of x_n^δ defined by (2.4) with general Y_n , i.e., not necessarily $Y_n = TX_n$.

2.1.1. Well-definedness.

Lemma 2.1. *The index n_* according to the discrepancy principle (1.4) with $\tau > 1$ is well defined.*

Proof. The residual can be represented as follows:

$$\begin{aligned} Tx_n^\delta - y^\delta &= (I - Q_n)(Tx_n^\delta - y^\delta) = -(I - Q_n)y^\delta \\ &= -(I - Q_n)Tx^\dagger + (I - Q_n)(y - y^\delta) \\ (2.5) \quad &= -(I - Q_n)T(I - P_n)x^\dagger + (I - Q_n)(y - y^\delta) \end{aligned}$$

where we have used $Tx_n^\delta \in Y_n$ in the second equality and $TP_n x^\dagger \in Y_n$ in the fourth equality. From (2.5), and $\|I - Q_n\| \leq 1$ we get

$$\|Tx_n^\delta - y^\delta\| \leq \|T\| \underbrace{\|(I - P_n)x^\dagger\|}_{\rightarrow 0 \text{ as } n \rightarrow \infty} + \delta,$$

hence $\|Tx_n^\delta - y^\delta\| \leq \tau\delta$ for n sufficiently large. $\square$

Denote

$$(2.6) \quad \alpha_n := \|(I - P_n)x^\dagger\|, \quad \beta_n := \|T(I - P_n)x^\dagger\|$$

$$(2.7) \quad \begin{aligned} \gamma_n &:= \inf_{\substack{z_n \in X_n \\ z_n \neq 0}} \frac{\|Tz_n\|}{\|z_n\|} = \inf_{\substack{z_n \in X_n \\ z_n \neq 0}} \frac{\|Tz_n\|}{\|(TP_n)^\dagger TP_n z_n\|} \\ &= \inf_{\substack{w_n \in TX_n \\ w_n \neq 0}} \frac{\|w_n\|}{\|(TP_n)^\dagger w_n\|} = \frac{1}{\|(TP_n)^\dagger\|}, \end{aligned}$$

where we have used the fact that by $\mathcal{N}(T) = \{0\}$ we have

$$(2.8) \quad \mathcal{N}(TP_n)^\perp = \overline{\mathcal{R}(P_n T^*)} = P_n \overline{\mathcal{R}(T^*)} = P_n \mathcal{N}(T)^\perp = P_n X = X_n.$$

In the following proofs we will use the error decomposition

$$(2.9) \quad \begin{aligned} x_n^\delta - x^\dagger &= x_n^\delta - P_n x^\dagger - (I - P_n)x^\dagger \\ &= (TP_n)^\dagger TP_n(x_n^\delta - P_n x^\dagger) - (I - P_n)x^\dagger \\ &= (TP_n)^\dagger(y^\delta - TP_n x^\dagger) - (I - P_n)x^\dagger \\ &= (TP_n)^\dagger(y^\delta - y) + (TP_n)^\dagger T(I - P_n)x^\dagger - (I - P_n)x^\dagger, \end{aligned}$$

where we have used (2.4) and (2.8).

2.1.2. Stability for fixed discretization level.

Proposition 2.2. *Let n be fixed and let $(y^k)_{k \in \mathbb{N}}$ be a sequence converging to y^δ : $\|y^k - y^\delta\| \rightarrow 0$ as $k \rightarrow \infty$ and denote by x_n^k the corresponding regularized solutions according to (2.4) with y^δ replaced by y^k .*

Then

$$\|x_n^k - x_n^\delta\| \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

Proof. From (2.9) we obtain

$$\|x_n^k - x_n^\delta\| = \|(TP_n)^\dagger(y^k - y^\delta)\| \leq \|(TP_n)^\dagger\| \|y^k - y^\delta\| \rightarrow 0$$

as $k \rightarrow \infty$. □

2.1.3. Convergence.

Theorem 2.3. *If*

$$(2.10) \quad \frac{\beta_n}{\gamma_n} \rightarrow 0, \quad \frac{\beta_{n-1}}{\gamma_n} \rightarrow 0, \quad \alpha_n \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

then for $n_ = n_*(\delta)$ according to (1.4),*

$$\|x_{n_*}^\delta(\delta) - x^\dagger\| \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

Proof. From (2.9) we get

$$(2.11) \quad \|x_n^\delta - x^\dagger\| \leq \frac{\delta}{\gamma_n} + \frac{\beta_n}{\gamma_n} + \alpha_n.$$

Let δ_m be an arbitrary sequence converging to zero and y_m a sequence of data satisfying $\|y_m - y\| \leq \delta_m$. We denote by n_m the corresponding index according to the discrepancy principle and by $x_{n_m}^{\delta_m}$ the approximation on level n_m according to (2.4) with y^δ replaced by y_m .

Consider first the case that $n_m \rightarrow \infty$ as $m \rightarrow \infty$. Then according to (2.5) we can estimate as follows:

$$(2.12) \quad \tau \delta_m \leq \|Tx_{n_m}^{\delta_m} - y_m\| \leq \|(I - Q_{n_m-1})T(I - P_{n_m-1})x^\dagger\| + \delta_m,$$

hence

$$(2.13) \quad \frac{\delta_m}{\gamma_{n_m}} \leq \frac{1}{\tau - 1} \frac{\beta_{n_m-1}}{\gamma_{n_m}},$$

so by (2.11) we get

$$\|x_{n_m}^{\delta_m} - x^\dagger\| \leq \frac{1}{\tau - 1} \frac{\beta_{n_m-1}}{\gamma_{n_m}} + \frac{\beta_{n_m}}{\gamma_{n_m}} + \alpha_{n_m} \rightarrow 0$$

as $m \rightarrow \infty$.

In the alternative case that (n_m) has a finite accumulation point there exists an $N \in \mathbb{N}$ and a subsequence $(m_k)_{k \in \mathbb{N}}$ such that

$$\forall k \in \mathbb{N} : \quad n_{m_k} = N.$$

From (2.5) we get

$$\tau \delta_{m_k} \geq \|Tx_{n_{m_k}}^{\delta_{m_k}} - y_{m_k}\| \geq \|(I - Q_N)T(I - P_N)x^\dagger\| - \delta_{m_k},$$

hence

$$(2.14) \quad (\tau + 1)\delta_{m_k} \geq \|(I - Q_N)T(I - P_N)x^\dagger\|$$

which, with $k \rightarrow \infty$, implies

$$(I - Q_N)Tx^\dagger = (I - Q_N)T(I - P_N)x^\dagger = 0,$$

i.e., $Tx^\dagger \in TX_N$, i.e., $\exists z_N \in X_N : Tx^\dagger = Tz_N$, i.e., $\exists z_N \in X_N : x^\dagger - z_N \in \mathcal{N}(T) = \{0\}$, i.e., $x^\dagger \in X_N$, i.e., $(I - P_N)x^\dagger = 0$, hence by (2.11)

$$\|x_{n_{m_k}}^{\delta_{m_k}} - x^\dagger\| \leq \frac{\delta_{m_k}}{\gamma_N} \rightarrow 0$$

as $k \rightarrow \infty$. □

Remark 2.4. Note that the convergence conditions of this theorem imply the convergence criterion $\limsup_{n \rightarrow \infty} \|x_n\| \leq \|x^\dagger\|$ for exact data from [7], since

$$\begin{aligned} \|(TP_n)^\dagger y\| &\leq \|(TP_n)^\dagger TP_n x^\dagger\| + \|(TP_n)^\dagger T(I - P_n)x^\dagger\| \\ &\leq \|Q_n x^\dagger\| + \|(TP_n)^\dagger\| \|T(I - P_n)x^\dagger\|. \end{aligned}$$

The difference to the convergence results from Chapter 3 in [20] is that we consider convergence not for all $x^\dagger \in X$ but for a particular $x^\dagger$ that might have higher regularity than typical elements of X so that divergence to infinity of the factor $\frac{1}{\gamma_n}$ due to unboundedness of $T^\dagger$ might indeed be compensated by sufficiently fast convergence of the approximation error β_n .

Indeed, note that $\|T(I - P_n)x^\dagger\|$ is likely to go to zero at a faster rate than $\|(I - P_n)x^\dagger\|$, since T typically has a smoothing property. Under an additional source condition,

$$x^\dagger = (T^*T)^\nu w$$

for some index $\nu > 0$ and some $w \in X$ the rate of convergence of β_n to zero will still be improved: If the smoothing property of T can be quantified via boundedness as a mapping in a scale of spaces:

$$\|(T^*T)^\nu\|_{X \rightarrow X^\nu} \leq C_\nu$$

and the approximation property of X_n scales according to

$$\|I - P_{n-1}\|_{X^\nu \rightarrow X} \leq f_\nu(n), \quad \|I - P_{n-1}\|_{X^{1/2} \rightarrow X} \leq f_{1/2}(n),$$

(one might think of X_n as being finite element spaces with mesh sizes $h = \frac{1}{n}$ and of X^ν being Sobolev spaces of order $s = s(\nu)$), then

$$\begin{aligned}
 \|T(I - P_{n-1})x^\dagger\| &= \left\| (T^*T)^{1/2}(I - P_{n-1})^2(T^*T)^\nu w \right\| \\
 &\leq \left\| (I - P_{n-1})(T^*T)^{1/2} \right\| \|(I - P_{n-1})(T^*T)^\nu\| \|w\| \\
 (2.15) \qquad &\leq f_\nu(n)f_{1/2}(n) \|w\|.
 \end{aligned}$$

Typically, for sufficiently good approximation spaces (in the context of finite elements these will have to be of sufficiently high order) the decay of $f_\nu(n)$ will be faster for larger ν . On the other hand, γ_n is independent of the solution, hence the product $f_\nu(n)f_{1/2}(n)\frac{1}{\gamma_n}$ will tend to zero for ν sufficiently large, i.e., for sufficiently smooth solutions.

Note the necessity of considering a particular solution instead of norm estimates in view of the fact that, e.g., in case of nested spaces $X_{n-1} \subseteq X_n$, we have

$$\gamma_n \leq \inf_{\substack{z_n \in X_n \cap X_{n-1}^\perp \\ z_n \neq 0}} \frac{\|Tz_n\|}{\|z_n\|} = \inf_{\substack{z_n \in X_n \cap X_{n-1}^\perp \\ z_n \neq 0}} \frac{\|T(I - P_{n-1})z_n\|}{\|z_n\|} \leq \|T(I - P_{n-1})\|,$$

so estimating $\beta_{n-1} \leq \|T(I - P_{n-1})\| \|x^\dagger\|$ would not enable (2.10).

Expressing a possible rate of convergence in (2.10) in terms of $\tilde{\beta}_n := \|(I - Q_n)T(I - P_n)x^\dagger\|$, we can easily deduce a convergence rate result in terms of δ .

Corollary 2.5. *Let, with some functions $g_1, g_2, g_3 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying*

$$(2.16) \qquad g_i(0) = 0, \quad g_i \text{ monotonically increasing}$$

and

$$(2.17) \qquad \forall C > 1 \exists \bar{C}(C) > 0 \forall \lambda > 0 : g_i(C\lambda) \leq \bar{C}(C)g_i(\lambda)$$

for $i = 1, 2, 3$, the following rates hold in (2.10):

$$(2.18) \quad \frac{\beta_n}{\gamma_n} = O(g_1(\tilde{\beta}_n)), \quad \frac{\tilde{\beta}_{n-1}}{\gamma_n} = O(g_2(\tilde{\beta}_n)), \quad \alpha_n = O(g_3(\tilde{\beta}_n)), \quad \text{as } n \rightarrow \infty.$$

Then

$$\|x_{n_*}^\delta(\delta) - x^\dagger\| = O(\max\{g_1(\delta), g_2(\delta), g_3(\delta)\}) \quad \text{as } \delta \rightarrow 0.$$

Proof. Analogously to (2.13), (2.14) we get

$$\frac{\delta}{\gamma_{n_*}} \leq \frac{1}{\tau - 1} \frac{\tilde{\beta}_{n_*-1}}{\gamma_{n_*}}, \quad (\tau + 1)\delta \geq \tilde{\beta}_{n_*}.$$

Inserting this and the rates assumptions (2.18) into (2.11), we directly get the assertion. $\square$

Remark 2.6. Typical functions satisfying (2.16), (2.17) are $\sigma \mapsto \sigma^\kappa$, $\sigma \mapsto |\ln \sigma|^{-q}$ with $\kappa, q > 0$.

Consider the special case of T being compact and $X_n = \text{span}\{u_1, \dots, u_n\}$, where $(\sigma_j; u_j, v_j)_{j \in \mathbb{N}}$ is a singular system for T (i.e., the method reduces to truncated singular value decomposition). Then a source condition of the form $x^\dagger = f((T^*T))w$,

with an index function f , (i.e., $f(0) = 0$, f monotonically increasing) implies

$$\begin{aligned}\tilde{\beta}_n &= \beta_n = \left(\sum_{j=n+1}^{\infty} \sigma_j^2 f(\sigma_j^2)^2 \langle w, u_j \rangle^2 \right)^{1/2}, \\ \gamma_n &\geq \sigma_n, \\ \alpha_n &= \left(\sum_{j=n+1}^{\infty} f(\sigma_j^2)^2 \langle w, u_j \rangle^2 \right)^{1/2},\end{aligned}$$

so provided the function $\Theta(\lambda) = \sqrt{\lambda}f(\lambda)$ is strictly monotonically increasing (e.g., if $f(\lambda) > 0$ for $\lambda > 0$), we obtain that (2.18) holds with $g_1 = g_2 = g_3 = f \circ \Theta^{-1}$, which yields the usual rates under such source conditions; see, e.g., Corollary 1 in [24].

3. THE NONLINEAR CASE

In the linear case we have seen that the essential quantities for showing convergence are $\alpha_n, \beta_n, \gamma_n$ defined as in (2.6), (2.7). To deal with the nonlinear situation we define $\alpha_n, \beta_n(x), \gamma_n(x)$ as in (2.6), (2.7) with T replaced by $F'(x)$.

In view of the two formulations (2.2), (2.3) we consider two possible generalizations of the linear case, namely a variational one and one based on a projected operator equation.

3.1. Nonlinear case via global minimizer.

3.1.1. *Well-definedness.* First we show well-definedness of x_n^δ according to (1.3).

Proposition 3.1. *Assume that $D \cap X_n \neq \emptyset$, and that F is weakly sequentially closed, i.e., for all sequences $(x_k)_{k \in \mathbb{N}} \subseteq D \cap X_n$,*

$$\left(x_k \rightharpoonup x \wedge F(x_k) \rightharpoonup f \right) \Rightarrow \left(x \in D \wedge F(x) = f \right).$$

Then there exists a minimizer of the cost function $J_n : D \cap X_n \rightarrow \mathbb{R}$, $J(z_n) = \|F(z_n) - y^\delta\|$.

Note that by membership of x_k in the finite dimensional space X_n , continuity of F and closedness of $D \cap X_n$ (see the assumptions of Proposition 3.3 and Theorem 3.4 below) are sufficient for the assumptions of Proposition 3.1.

Proof. J is coercive (i.e., boundedness of cost function values implies boundedness of arguments): By (1.5) we have for all $z_n \in D \cap X_n$ and some fixed $x_n \in D \cap X_n$,

$$\begin{aligned}\|F(z_n) - y^\delta\| &= \|F(z_n) - F(x_n) + F(x_n) - y^\delta\| \\ &\geq \frac{1}{1+\eta} \|F'(z_n)P_n(z_n - x_n)\| - \|F(x_n) - y^\delta\|,\end{aligned}$$

hence by (2.8) (with T replaced by $F'(x_n)$) we get

$$\begin{aligned}\|z_n - x_n\| &= \|(F'(x_n)P_n)^\dagger F'(x_n)P_n(z_n - x_n)\| \\ &\leq \frac{1}{\gamma_n(x_n)} ((1+\eta)J(z_n) + \|F(x_n) - y^\delta\|).\end{aligned}$$

Moreover, J_n is weakly lower semicontinuous due to weak sequential closedness of F and weak lower semicontinuity of the norm.

By a standard argument we therefore get existence of a minimizer. $\square$

In the rest of this subsection we assume that $x_n^\delta \in X_n$ is defined as a (not necessarily unique) global minimizer of (1.3) according to Proposition 3.1 and that

$$(3.1) \quad P_n x^\dagger \in D, \quad n \in \mathbb{N}.$$

Lemma 3.2. *Let F be continuous, $P_n x^\dagger \rightarrow x^\dagger$ as $n \rightarrow \infty$, and (3.1) hold, and let for all n x_n^δ be defined by (1.3).*

Then the index n_ according to the discrepancy principle (1.4) with $\tau > 1$ is well-defined.*

Proof. By minimality of x_n^δ and (3.1) we get the following estimate of the residual

$$(3.2) \quad \|F(x_n^\delta) - y^\delta\| \leq \|F(P_n x^\dagger) - y^\delta\| \leq \|F(P_n x^\dagger) - F(x^\dagger)\| + \delta.$$

Hence, by continuity of F and $P_n x^\dagger \rightarrow x^\dagger$ as $n \rightarrow \infty$ we get $\|F(x_n^\delta) - y^\delta\| \leq \tau\delta$ for n sufficiently large. $\square$

The following estimate of the error will be useful in our stability and convergence proofs: By (1.5) we have

$$(3.3) \quad \begin{aligned} & \|F'(x_n^\delta)P_n(x_n^\delta - x^\dagger)\| - \|F'(x_n^\delta)(I - P_n)x^\dagger\| \\ & \leq \|F'(x_n^\delta)(x_n^\delta - x^\dagger)\| \leq (1 + \eta)(\|F(x_n^\delta) - y^\delta\| + \delta), \end{aligned}$$

hence by (2.7) with T replaced by $F'(x_n^\delta)$, (2.8), and (3.2)

$$(3.4) \quad \begin{aligned} \|x_n^\delta - x^\dagger\| & \leq \|x_n^\delta - P_n x^\dagger\| + \|(I - P_n)x^\dagger\| \\ & \leq \frac{1}{\gamma_n(x_n^\delta)} (\|F'(x_n^\delta)(I - P_n)x^\dagger\| + (1 + \eta)(\|F(x_n^\delta) - y^\delta\| + \delta)) \\ & \quad + \|(I - P_n)x^\dagger\|. \end{aligned}$$

3.1.2. Stability for fixed discretization level.

Proposition 3.3. *Let F be continuous and satisfy (1.5), fix n , and let $(y^k)_{k \in \mathbb{N}}$ be a sequence converging to y^δ : $\|y^k - y^\delta\| \rightarrow 0$ as $k \rightarrow \infty$, denote by x_n^k the corresponding regularized solutions according to (1.3) with y^δ replaced by y^k and assume that $D \cap X_n$ is closed and nonempty.*

Then $(x_n^k)_{k \in \mathbb{N}}$ has a convergent subsequence and the limit of any convergent subsequence of $(x_n^k)_{k \in \mathbb{N}}$ is a global minimizer of (1.3). If this global minimizer is unique, then

$$\|x_n^k - x_n^\delta\| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Proof. On one hand, analogously to (3.2), we get by minimality of x_n^k and x_n^δ ,

$$\begin{aligned} \|F(x_n^k) - y^k\| & \leq \|F(x_n^\delta) - y^k\| \leq \|F(x_n^\delta) - y^\delta\| + \|y^k - y^\delta\|, \\ \|F(x_n^\delta) - y^\delta\| & \leq \|F(x_n^k) - y^\delta\| \leq \|F(x_n^k) - y^k\| + \|y^k - y^\delta\|, \end{aligned}$$

hence

$$(3.5) \quad \|\|F(x_n^k) - y^k\| - \|F(x_n^\delta) - y^\delta\|\| \leq \|y^k - y^\delta\| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

On the other hand, for some fixed $x_n \in D \cap X_n$, we get by minimality,

$$\begin{aligned} \|F'(x_n^\delta)(x_n^k - x_n^\delta)\| &\leq (1 + \eta) \|F(x_n^k) - F(x_n^\delta)\| \\ &\leq (1 + \eta) \left(\|F(x_n^k) - y^k\| + \|F(x_n^\delta) - y^\delta\| + \|y^k - y^\delta\| \right) \\ &\leq (1 + \eta) \left(\|F(x_n) - y^k\| + \|F(x_n) - y^\delta\| + \|y^k - y^\delta\| \right) \\ &\leq 2(1 + \eta) \left(\|F(x_n) - y^\delta\| + c \right) \end{aligned}$$

where c is such that (by convergence) $\|y^k - y^\delta\| \leq c$, hence by (2.7) with T replaced by $F'(x_n^\delta)$ and (2.8),

$$\|x_n^k - x_n^\delta\| \leq \frac{1 + \eta}{\gamma_n(x_n^\delta)} (2\|F(x_n) - y^\delta\| + 2c) =: r,$$

so the sequence x_n^k lies in the bounded closed and finite dimensional, hence compact, set $\overline{B_r(x_n^\delta)} \cap X_n$ and therefore has a convergent subsequence $(x_n^{k_m})_{m \in \mathbb{N}}$. For any convergent subsequence $(x_n^{k_m})_{m \in \mathbb{N}}$ of $(x_n^k)_{k \in \mathbb{N}}$ with limit $\tilde{x}$, by (3.5) and a standard argument (see [3]) from $\|y^k - y^\delta\| \rightarrow 0$, continuity of F and closedness of $D \cap X_n$ it follows that $\tilde{x}$ lies in $D \cap X_n$ and is a global minimizer of $\|F(\cdot) - y^\delta\|$ over $D \cap X_n$. Namely, we have by minimality of x_n^k ,

$$\begin{aligned} \forall x_n \in D \cap X_n : \|F(x_n) - y^\delta\| &\geq \limsup_{k \rightarrow \infty} \left(\|F(x_n) - y^k\| - \|y^k - y^\delta\| \right) \\ &\geq \limsup_{k \rightarrow \infty} \left(\|F(x_n^k) - y^k\| - \|y^k - y^\delta\| \right) \\ &\geq \|F(\tilde{x}) - y^\delta\|. \end{aligned}$$

In case of uniqueness of this global minimizer, a subsequence subsequence argument yields convergence. $\square$

3.1.3. Convergence of global minimizers.

Theorem 3.4. *Let F be continuous and satisfy (1.5), assume that for all $n \in \mathbb{N}$, $D \cap X_n$ is closed and nonempty, that (3.1) holds and that x_n^δ is defined as a solution to (1.3).*

If

$$(3.6) \quad \sup_{x \in D \cap X_n} \frac{\beta_n(x)}{\gamma_n(x)} \rightarrow 0, \quad \sup_{\tilde{x}, x \in D \cap X_n} \frac{\beta_{n-1}(\tilde{x})}{\gamma_n(x)} \rightarrow 0, \quad \alpha_n \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

then for $n_* = n_*(\delta)$ according to (1.4),

$$\|x_{n_*(\delta)}^\delta - x^\dagger\| \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

Remark 3.5. Instead of (3.6) we actually need

$$\frac{\beta_n(x_n^\delta)}{\gamma_n(x_n^\delta)} \rightarrow 0, \quad \frac{\beta_{n-1}(P_{n-1}x^\dagger)}{\gamma_n(x_n^\delta)} \rightarrow 0, \quad \alpha_n \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Proof. Analogously to the linear case we distinguish between two cases:

If $n_m \rightarrow \infty$ as $m \rightarrow \infty$, then we use (3.2) to obtain

$$\begin{aligned} (3.7) \quad \tau \delta_m &\leq \|F(x_{n_m-1}^{\delta_m}) - y_m\| \leq \|F(P_{n_m-1}x^\dagger) - F(x^\dagger)\| + \delta_m \\ &\leq \frac{1}{1 - \eta} \|F'(P_{n_m-1}x^\dagger)(P_{n_m-1}x^\dagger - x^\dagger)\| + \delta_m, \end{aligned}$$

hence

$$\frac{\delta_m}{\gamma_{n_m}(x_{n_m}^{\delta_m})} \leq \frac{1}{\tau-1} \frac{1}{1-\eta} \frac{\beta_{n_m-1}(P_{n_m-1}x^\dagger)}{\gamma_{n_m}(x_{n_m}^{\delta_m})},$$

so by (3.4) we get

$$\|x_{n_m}^{\delta_m} - x^\dagger\| \leq \frac{\beta_{n_m}(x_{n_m}^{\delta_m})}{\gamma_{n_m}(x_{n_m}^{\delta_m})} + \frac{1+\eta}{1-\eta} \frac{\tau+1}{\tau-1} \frac{\beta_{n_m-1}(P_{n_m-1}x^\dagger)}{\gamma_{n_m}(x_{n_m}^{\delta_m})} + \alpha_{n_m} \rightarrow 0$$

as $m \rightarrow \infty$.

If there exists an $N \in \mathbb{N}$ and a subsequence $(m_k)_{k \in \mathbb{N}}$ such that $n_{m_k} = N$ for all $k \in \mathbb{N}$, we get

$$\tau \delta_{m_k} \geq \|F(x_{n_{m_k}}^{\delta_{m_k}}) - y_{m_k}\| = \|F(x_N^{\delta_{m_k}}) - y_{m_k}\|,$$

hence taking the limit on both sides and using stability for fixed N (see Proposition 3.3) we get that $x_N^{\delta_{m_k}}$ has a convergent subsequence and the limit $\tilde{x}$ of any convergent subsequence of $x_N^{\delta_{m_k}}$ satisfies $F(\tilde{x}) = y$, so $\tilde{x}$ is a solution.

By uniqueness of this solution, a subsequence argument yields convergence. $\square$

3.2. Nonlinear case via Euler equation.

3.2.1. Well-definedness. If the global minimizer according to Proposition 3.1 lies in the interior of D , it will also satisfy the Euler equation

$$(3.8) \quad (F'(x_n^\delta)P_n)^*(F(x_n^\delta) - y^\delta) = 0,$$

which implies

$$(3.9) \quad Q_n(x_n^\delta)(F(x_n^\delta) - y^\delta) = 0 \quad \text{with } Q_n(x) = \text{Proj}_{Y_n(x)}, \quad Y_n(x) := F'(x)X_n.$$

Our following investigations on the formulation (3.8) are also motivated by the fact that a numerical optimization method will usually not yield a global minimizer but a stationary point.

In this subsection we assume that x_n^δ is defined (not necessarily uniquely) by (3.9), which is possible, e.g., under the assumptions of Proposition 3.1 provided the minimizer lies in the interior of D , or alternatively by a fixed point argument analogously to Lemma 2 in [12].

Again we start with a result on the discrepancy principle.

Lemma 3.6. *Let (1.5) be satisfied, $F'(x)$ be uniformly bounded on D , let $P_n x^\dagger \rightarrow x^\dagger$ as $n \rightarrow \infty$, and for all n let x_n^δ be defined by (3.9).*

Then the index n_ according to the discrepancy principle (1.4) with $\tau > \frac{1+\eta}{1-\eta}$ is well-defined.*

Proof.

$$\begin{aligned} F(x_n^\delta) - y^\delta &= (I - Q_n(x_n^\delta))(F(x_n^\delta) - y^\delta) \\ &= (I - Q_n(x_n^\delta)) \cdot \\ &\quad \left(F'(x_n^\delta)(x_n^\delta - x^\dagger) + F(x_n^\delta) - F(x^\dagger) - F'(x_n^\delta)(x_n^\delta - x^\dagger) + y - y^\delta \right) \\ &= (I - Q_n(x_n^\delta)) \cdot \\ (3.10) \quad &\left(F'(x_n^\delta)(P_n x^\dagger - x^\dagger) + F(x_n^\delta) - F(x^\dagger) - F'(x_n^\delta)(x_n^\delta - x^\dagger) + y - y^\delta \right) \end{aligned}$$

since $(I - Q_n(x_n^\delta))F'(x_n^\delta)x_n^\delta = 0 = (I - Q_n(x_n^\delta))F'(x_n^\delta)P_nx^\dagger$, hence, due to (1.5)

$$(3.11) \quad \|F(x_n^\delta) - y^\delta\| \leq \frac{1}{1-\eta} \left(\|(I - Q_n(x_n^\delta))F'(x_n^\delta)(I - P_n)x^\dagger\| + (1+\eta)\delta \right)$$

and therewith $\|F(x_n^\delta) - y^\delta\| \leq \tau\delta$ for n sufficiently large. $\square$

The error decomposition (2.9) in the nonlinear case becomes

$$(3.12) \quad \begin{aligned} x_n^\delta - x^\dagger &= (F'(x_n^\delta)P_n)^\dagger(y^\delta - y) + (F'(x_n^\delta)P_n)^\dagger F'(x_n^\delta)(I - P_n)x^\dagger - (I - P_n)x^\dagger \\ &\quad + (F'(x_n^\delta)P_n)^\dagger(F(x^\dagger) - F(x_n^\delta) - F'(x_n^\delta)(x^\dagger - x_n^\delta)), \end{aligned}$$

where we have used the identity

$$(F'(x_n^\delta)P_n)^\dagger(y^\delta - F(x_n^\delta)) = (F'(x_n^\delta)P_n)^\dagger Q_n(x_n^\delta)(y^\delta - F(x_n^\delta)) = 0$$

and (2.8).

3.2.2. Stability for fixed discretization level.

Proposition 3.7. *Let n be fixed and let $(y^k)_{k \in \mathbb{N}}$ be a sequence converging to y^δ : $\|y^k - y^\delta\| \rightarrow 0$ as $k \rightarrow \infty$ and denote by x_n^k the corresponding regularized solutions according to (3.9) with y^δ replaced by y^k . Additionally, assume that F , F' are continuous, satisfy (1.5), that $D \cap X_n$ is closed, and that*

$$(3.13) \quad \exists \underline{\gamma}_n > 0 \quad \forall x \in D \cap X_n : \gamma_n(x) \geq \underline{\gamma}_n > 0$$

as well as

$$(3.14) \quad \exists \overline{\beta}_n > 0 \quad \forall x \in D \cap X_n : \beta_n(x) \leq \overline{\beta}_n$$

holds.

Then $(x_n^k)_{k \in \mathbb{N}}$ has a convergent subsequence and the limit of any convergent subsequence of $(x_n^k)_{k \in \mathbb{N}}$ solves (3.9). If this solution is unique, then

$$\|x_n^k - x_n^\delta\| \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Proof. From (3.11), (3.12), which remain valid under the present assumptions, we obtain

$$\begin{aligned} \|x_n^k - P_n x^\dagger\| &\leq \frac{1}{\gamma_n(x_n^k)} \left(\|y^k - y\| + \beta_n(x_n^k) + \eta \|F(x_n^k) - F(x^\dagger)\| \right) \\ &\leq \frac{1}{\gamma_n(x_n^k)} \left(\delta_k + \beta_n(x_n^k) + \eta(\delta_k + \frac{1}{1-\eta}(\beta_n(x_n^k) + (1+\eta)\delta_k)) \right) \end{aligned}$$

with $\delta_k := \|y^k - y^\delta\| + \delta$, which by (3.13), (3.14) implies boundedness of x_n^k . Hence by compactness of balls in the finite dimensional subspace X_n we get existence of a convergent subsequence of x_n^k . By continuity of F and F' , as well as $(F'(x_n^k)P_n)^*(F(x_n^k) - y_n^k) = 0$, the limit $\tilde{x}$ of any convergent subsequence of x_n^k satisfies $(F'(\tilde{x})P_n)^*(F(\tilde{x}) - y^\delta) = 0$. $\square$

3.2.3. Convergence of solutions to Euler equations.

Theorem 3.8. Assume that x_n^δ is defined as a solution to (3.9), that F, F' are continuous with $F'(x)$ uniformly bounded and satisfy (1.5), that $D \cap X_n$ is closed, and that

$$(3.15) \quad \sup_{x \in D \cap X_n} \frac{\beta_n(x)}{\gamma_n(x)} \rightarrow 0, \quad \sup_{\tilde{x}, x \in D \cap X_n} \frac{\beta_{n-1}(\tilde{x})}{\gamma_n(x)} \rightarrow 0, \quad \alpha_n \rightarrow 0, \quad \text{as } n \rightarrow \infty$$

holds.

Then for $n_* = n_*(\delta)$ according to (1.4),

$$\|x_{n_*(\delta)}^\delta - x^\dagger\| \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

Remark 3.9. More precisely, we actually need

$$\frac{\beta_n(x_n^\delta)}{\gamma_n(x_n^\delta)} \rightarrow 0, \quad \frac{\beta_{n-1}(x_{n-1}^\delta)}{\gamma_n(x_n^\delta)} \rightarrow 0, \quad \alpha_n \rightarrow 0, \quad \text{as } n \rightarrow \infty$$

in place of (3.15).

Proof. With (3.12), by (1.5) we get

$$(3.16) \quad \|x_n^\delta - x^\dagger\| \leq \frac{\delta}{\gamma_n(x_n^\delta)} + \frac{\beta_n(x_n^\delta)}{\gamma_n(x_n^\delta)} + \alpha_n + \frac{\eta \|F(x_n^\delta) - F(x^\dagger)\|}{\gamma_n(x_n^\delta)}$$

$$(3.17) \quad \leq \frac{1}{1-\eta} \frac{\delta}{\gamma_n(x_n^\delta)} + \frac{1}{1-\eta} \frac{\beta_n(x_n^\delta)}{\gamma_n(x_n^\delta)} + \alpha_n$$

where analogously to (3.10), (3.11) we have

$$\begin{aligned} F(x_n^\delta) - F(x^\dagger) &= (I - Q_n(x_n^\delta)) \left(F'(x_n^\delta)(P_n x^\dagger - x^\dagger) + F(x_n^\delta) \right. \\ &\quad \left. - F(x^\dagger) - F'(x_n^\delta)(x_n - x^\dagger) \right) + Q_n(y^\delta - y), \end{aligned}$$

hence, using once more (1.5),

$$\|F(x_n^\delta) - F(x^\dagger)\| \leq \frac{1}{1-\eta} \left(\|(I - Q_n(x_n^\delta))F'(x_n^\delta)(P_n x^\dagger - x^\dagger)\| + \delta \right).$$

The rest of the proof goes analogously to the linear case:

In case of $n_m \rightarrow \infty$, in place of (2.12), we get by (3.11),

$$\tau \delta_m \leq \|F(x_{n_m-1}^{\delta_m}) - y_m\| \leq \frac{1}{1-\eta} \left(\|(F'(x_{n_m-1}^{\delta_m})(I - P_{n_m-1})x^\dagger\| + (1+\eta)\delta_m \right),$$

and therewith, instead of (2.13),

$$(3.18) \quad \frac{\delta_m}{\gamma_{n_m}(x_{n_m}^{\delta_m})} \leq \frac{1}{\tau - \frac{1+\eta}{1-\eta}} \frac{\beta_{n_m-1}(x_{n_m-1}^{\delta_m})}{\gamma_{n_m}(x_{n_m}^{\delta_m})}.$$

For the case with a finite accumulation point N of the discretization levels n_m , we estimate, using (3.10),

$$\tau \delta_{m_k} \geq \|F(x_{n_{m_k}}^{\delta_{m_k}}) - y_{m_k}\| \geq \frac{1}{1+\eta} \|(I - Q_N(x_N^{\delta_{m_k}}))F'(x_N^{\delta_{m_k}})(I - P_N)x^\dagger\| - (1+\eta)\delta_{m_k},$$

hence

$$(\tau + \eta + 1)(1 + \eta)\delta_{m_k} \geq \|(I - Q_N(x_N^{\delta_{m_k}}))F'(x_N^{\delta_{m_k}})(I - P_N)x^\dagger\|.$$

Taking on both sides the limit along a subsequence k_l for which $x_N^{\delta_{m_{k_l}}}$ converges to some x_N^0 according to Proposition 3.7 and arguing similar to the stability proof, we get

$$0 = \|(I - Q_N(x_N^0))F'(x_N^0)(I - P_N)x^\dagger\|,$$

which analogously to the linear case implies $(I - P_N)x^\dagger = 0$, hence convergence. $\square$

Remark 3.10. By inspection of the proof we see that instead of solving (3.8) exactly, it suffices to find x_n^δ such that

$$(3.19) \quad \|(F'(x_n^\delta)P_n)^*(F(x_n^\delta) - y^\delta)\| \leq c\gamma_n(x_n^\delta)\delta,$$

with $0 < c < \tau(1 - \eta) - (1 + \eta)$. In our computations we use the heuristic tolerance

$$(3.20) \quad \|(F'(x_n^\delta)P_n)^*(F(x_n^\delta) - y^\delta)\| \leq c\delta^2,$$

which cannot be shown to yield convergence, since in place of (3.18) we get an expression that is quadratic in $\frac{\delta}{\gamma_n(x_n^\delta)}$, namely

$$\frac{\delta_m}{\gamma_{n_m}(x_{n_m}^{\delta_m})} \leq \frac{1}{\tau - \frac{1+\eta}{1-\eta}} \frac{\beta_{n_m-1}(x_{n_m-1}^{\delta_m})}{\gamma_{n_m}(x_{n_m}^{\delta_m})} + c \left(\frac{\delta_m}{\gamma_{n_m}(x_{n_m}^{\delta_m})} \right)^2,$$

which does not exclude large values of $\frac{\delta}{\gamma_n(x_n^\delta)}$.

Condition (3.19) (or (3.20)) can be used as a stopping criterion in the iterative solution of the Euler equation.

4. NUMERICAL RESULTS

4.1. Test problem. For illustrating the theoretical results from above, we consider the test problem of identifying the source term q in the PDE

$$(4.1) \quad \begin{aligned} -\Delta u + qu &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

on the unit square $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ from measurements u in the whole domain Ω . Defining F as the parameter-to-solution map for the PDE (4.1) leads to a nonlinear inverse problem

$$(4.2) \quad F(q) = u,$$

where $q \in X = L^2(\Omega)$, $u \in Y = L^2(\Omega)$, and $F : X \rightarrow Y$, whose properties have been studied, e.g., in [3], [9], where also the Scherzer condition (1.5) has been verified. This problem has been used as a model problem for testing numerical methods for nonlinear inverse problems in many papers. In order to achieve a nonempty interior of the domain, similarly to [9], we set

$$(4.3) \quad D = \{q \in L^2(\Omega) \mid \|q - \hat{q}\|_{L^2(\Omega)} \leq \beta \text{ for some } \hat{q} \in L^\infty(\Omega) \text{ with } \hat{q} \geq 0 \text{ a.e.}\},$$

where $\frac{1}{\beta}$ is larger than the norm of the continuous embedding $H_0^1(\Omega) \rightarrow L^4(\Omega)$.

Obviously, the range of the forward operator F , mapping a parameter q to the exact solution of the PDE problem (4.1) is actually $H^2(\Omega) \cap H_0^1(\Omega)$ due to convexity of Ω (cf., e.g., [5]). On the other hand, only values but not derivatives of u can be measured, so the natural choice of Y as a Hilbert space is $L^2(\Omega)$. This discrepancy of smoothness yields an ill-posedness of degree two of the inverse problem, which can also be seen directly from the fact that we consider q and u in the same space, but q is determined from u by application of the Laplace operator.

We consider three test cases for the parameter q , which in the first two cases is defined as a Gaussian density function:

$$q^\dagger(x, y) = \frac{1}{2\pi\|\Sigma\|} e^{\frac{(x-\mu)^T \Sigma^{-1} (y-\mu)}{2}}$$

with $\Sigma = 0.005Id$, and $\Sigma = 0.03Id$, respectively, where in both cases $\mu = (\frac{1}{2}, \frac{1}{2})$. The third test parameter is piecewise constant and supported on three nonintersecting circles

$$q^\dagger(x, y) = \begin{cases} 2 & \text{if } (x, y) \in K_1 \cup K_2 \cup K_3, \\ 0 & \text{else,} \end{cases}$$

where K_1, K_2, K_3 are three circles with radius $r = 0.05$ and midpoints $M_1(\frac{1}{4}/\frac{1}{4})$, $M_2(\frac{1}{4}/\frac{3}{4})$ and $M_3(\frac{3}{4}/\frac{3}{4})$. See Figure 1 for a plot of the three test parameters on a very fine grid, with about 16.000 triangles. The corresponding solutions u of (4.1) are shown in Figure 2.

FIGURE 1. Exact parameter on fine grid.

FIGURE 2. Exact solutions to the corresponding source parameter on fine grid.

4.2. Numerical Implementation. For generating synthetic data and for solving the forward problem (4.1) we use the Finite Element Method as provided in a Matlab routine, that is capable of generating different meshes (p_n, t_n) , consisting of the set of points p_n and the set of triangles t_n . Since the discretizations of the parameter q are defined on the triangles and those of the PDE solutions on the set of points, the natural choice of X_n is the space of piecewise constant functions with possible jumps over the triangle edges, whereas by choosing a very fine mesh for u we get — up to a small approximation error — an exact PDE solution. Considering L levels of discretization, labeling by 1 the coarsest and by L the finest one, we carry out the PDE solution for data generation on level L , whereas all other PDE solutions are computed on level $L - 1$ in order to avoid an inverse crime (see e.g., [11, 21]). For the projection spaces X_n the level n runs from 1 to $L - 1$.

Hence the discretized forward operator consists of two components: interpolation from mesh (p_n, t_n) to mesh (p_{L-1}, t_{L-1}) and solving the underlying boundary value problem on the finest mesh.

In order to be able to transfer information on the parameter q over different grid levels in an exact manner, we generate nested grids, i.e., such that

$$(4.4) \quad p_{n-1} \subseteq p_n \text{ and } \forall \omega \in t_n \exists \tilde{\omega} \in t_{n-1} : \omega \subseteq \tilde{\omega}, \quad \text{for } n = 2, \dots, L - 1$$

where $\omega, \tilde{\omega}$ denote triangles. Again, in order to avoid an inverse crime, the finest mesh for data generation is designed such that it is not part of the hierarchical mesh structure. For this purpose we initialize our computations by generating a hierarchical sequence of grids and one additional very fine grid for data generation, as well as by computing a PDE solution on the finest grid, which is then corrupted with random noise of a given percentage of the data norm as listed in the tables below. Then we solve (4.2) with q_n element of different subspaces X_n (given by respective meshes) as described in the previous section. Summarizing, we have implemented the following procedure:

- (1) Generate L finite element meshes $(p_1, t_1), \dots, (p_L, t_L)$ satisfying (4.4).
- (2) Solve (4.1) with given $q^\dagger$ on mesh L to get data u .
- (3) Interpolate u to the $L - 1$ -th mesh to avoid an inverse crime.
- (4) Add random noise to u according to the desired noise level δ in order to get u^δ .
- (5) Solve (1.3) with X_n defined by t_n , $n = 1, \dots, L - 1$.

For validating the regularizing property of discretization in preimage space, we solve (4.2) in a least-squares sense, i.e., such that $\|F(q) - u^\delta\|^2$ is minimal, which is the method described and analyzed in the previous sections of this paper. For the minimization of the least squares functional we used a Quasi-Newton algorithm with BFGS update, as implemented in the Matlab function *fminunc*, where we provided the gradient $(F'(x_n^\delta)P_n)^*(F(x_n^\delta) - y^\delta)$, which is computed using the adjoint technique. The algorithm stops if

$$\|\nabla J(x)\| = \|(F'(x_n^\delta)P_n)^*(F(x_n^\delta) - y^\delta)\| \leq c\delta^2$$

according to equation (3.20) with $c = 8 \cdot 10^{-3}$.

FIGURE 3. The $L = 6$ different meshes for the Gaussian parameter together with their corresponding maximum edge length.

4.3. Results. For each of the test examples, we generated a sequence of $L = 6$ meshes (see Figure 3) and interpolated the data to the fifth mesh, with ≈ 4000 triangles for the circle parameter and ≈ 9000 triangles in case of gaussian parameters. We corrupted the exact data with white Gaussian noise according to different noise levels $\delta = 0.5\%$, 1% , 2% , 4% , 8% . Then we solved the inverse problem on each subspace X_n , $n = 1, \dots, L$ in the least squares sense (1.3). For comparison of the different solutions we computed the relative error:

$$e_r = \frac{\|q - q^\dagger\|}{\|q^\dagger\|}$$

with $q^\dagger$ given on (p_5, t_5) . For computing this error, we interpolated q_k , $k = 1, \dots, 4$ to mesh 5, such that every small triangle from the fine mesh obtains the value from its surrounding triangle of the coarser mesh. This is possible because of the hierarchical structure of the mesh sequence.

In addition, for every solution we computed the corresponding ratio between discrepancy and noise level, to see if we can numerically validate the theoretically investigated discrepancy principle:

$$\tilde{\tau} = \frac{\|F(q) - u^\delta\|}{\delta}.$$

In Table 1 we list the relative error e_r and the associated ratio $\tilde{\tau}$ for the five different meshes and the different noise levels for the Gaussian parameter with the least squares solution. Additionally, we mention the computation time t for the minimization process on the domain and the number of iterations the Quasi Newton solver carried out. There are three different features to mention.

First, if we look at a fixed noise level, e.g., $\delta = 1\%$, we see the well-known error behavior of regularization methods for ill-posed problems (see [4]): For finer grids the error gets smaller but when the mesh size gets too small the error grows again. Thus there is an optimal discretization level, for which the error is minimal.

Second, it can be seen, that we are able to validate the discrepancy principle also numerically. If we look at the smallest error (the bold numbers) we see that the corresponding ratio $\tilde{\tau} \approx 1$, which confirms our convergence analysis. Also for smaller noise level the best n determining the projection space X_n gets larger, e.g., for the Gaussian parameter with $\Sigma = 0.03Id$ and 0.5% noise, we achieved the smallest error on the finest mesh.

Additionally, we find that for $\delta \rightarrow 0$ the estimated parameter q tends towards the exact parameter $q^\dagger$; see the bold error numbers e.g. in Table 2

In case of the circle parameter (see Table 3), errors are higher than in the Gaussian case, even in the case of $\delta = 0.25\%$ noise, which is probably due to the nonsmoothness of the parameter.

TABLE 1. Gaussian Parameter with $\Sigma = 0.005Id$

Mesh		1	2	3	4	5
Edge Length		0.25	0.125	0.0625	0.03125	0.016
Noise Level						
8 %	e_r	0.8570	0.7935	0.9413	0.9999	0.9999
	$\tilde{\tau}$	1.0721	1.0742	1.8609	2.1985	2.1985
	t	1.3220	1.3203	1.2083	0.1461	0.1659
	$\#It$	3	3	1	0	0
4 %	e_r	0.9533	0.7809	0.8143	0.8778	0.9999
	$\tilde{\tau}$	1.0783	1.1862	1.5464	2.4434	4.0388
	t	2.8897	1.4680	1.3593	1.4594	0.1679
	$\#It$	15	4	2	1	0
2 %	e_r	0.9568	0.5297	0.7710	0.8014	0.9659
	$\tilde{\tau}$	1.2571	1.0665	1.5834	2.1592	6.9880
	t	3.0220	2.7704	1.6703	1.9957	2.5002
	$\#It$	16	14	4	2	1
1 %	e_r	1.1083	0.5224	0.4721	0.7759	0.8174
	$\tilde{\tau}$	1.7815	1.2184	1.2824	2.8203	5.1237
	t	6.7834	3.0331	3.0807	2.4399	8.3380
	$\#It$	45	16	13	3	2
0.5 %	e_r	1.2973	0.4612	0.4492	0.4475	0.7684
	$\tilde{\tau}$	3.0110	1.1056	1.6886	1.8145	5.2024
	t	11.4500	6.6803	3.4147	7.1335	18.6847
	$\#It$	80	43	15	13	4

TABLE 2. Gaussian Parameter with $\Sigma = 0.03Id$

Mesh		1	2	3	4	5
Edge Length		0.25	0.125	0.0625	0.03125	0.016
Noise Level						
8.000 %	e_r	0.3072	0.2823	0.9994	0.9994	0.9994
	$\tilde{\tau}$	1.0006	1.0241	1.5076	1.5076	1.5076
	t	1.3217	1.1925	0.1383	0.1454	0.1669
	#It	3	2	0	0	0
4.000 %	e_r	0.3012	0.2119	0.2497	0.9994	0.9994
	$\tilde{\tau}$	1.0050	1.0083	1.0797	2.4754	2.4754
	t	1.3407	1.3273	1.3917	0.1467	0.1655
	#It	3	3	2	0	0
2.000 %	e_r	0.3088	0.2159	0.1896	0.2176	0.9994
	$\tilde{\tau}$	1.0053	1.0073	1.0211	1.1550	4.6261
	t	1.4493	1.4552	1.5153	1.9793	0.1683
	#It	4	4	3	2	0
1.000 %	e_r	0.3017	0.2181	0.1900	0.1848	0.2607
	$\tilde{\tau}$	1.0022	1.0170	1.0195	1.0358	2.0177
	t	1.9879	1.4568	1.6819	2.4417	8.3386
	#It	8	4	4	3	2
0.500 %	e_r	0.2862	0.2034	0.1887	0.1838	0.1816
	$\tilde{\tau}$	1.0142	1.0259	1.1019	1.1643	1.3474
	t	4.6533	2.1385	1.6900	2.4584	13.4880
	#It	28	9	4	3	3

TABLE 3. Circle Parameter

Mesh		1	2	3	4	5
Edge Length		0.500	0.250	0.125	0.062	0.031
Noise Level						
8.000 %	e_r	1.0165	0.7526	0.9180	0.8370	0.9519
	$\tilde{\tau}$	1.0123	1.0081	1.0745	1.0292	1.0953
	t	0.4930	0.4989	0.5019	0.6032	0.8584
	#It	2	2	1	1	1
4.000 %	e_r	1.0182	0.7734	0.9251	0.8501	0.9561
	$\tilde{\tau}$	1.0287	1.0190	1.2275	1.0863	1.2891
	t	0.5498	0.4992	0.5027	0.6034	0.8595
	#It	3	2	1	1	1
2.000 %	e_r	1.3231	0.6318	0.7663	0.8462	0.9553
	$\tilde{\tau}$	1.0854	1.0312	1.1211	1.3619	1.9426
	t	1.0880	0.7182	0.5602	0.6040	0.8599
	#It	13	6	2	1	1
1.000 %	e_r	1.2857	0.5813	0.6211	0.7685	0.7739
	$\tilde{\tau}$	1.2855	1.0110	1.1025	1.3473	1.3788
	t	1.1296	1.0419	0.8014	0.7344	2.0487
	#It	14	12	6	2	2
0.500 %	e_r	1.2617	0.5591	0.5157	0.6126	0.7751
	$\tilde{\tau}$	1.9447	1.0184	1.0481	1.3632	2.1241
	t	1.2188	1.4464	1.1634	1.2503	2.0698
	#It	15	19	12	6	2
0.250 %	e_r	1.2706	0.5562	0.4845	0.5097	0.6230
	$\tilde{\tau}$	3.3979	1.0105	1.0822	1.1901	2.1400
	t	1.9215	2.9264	1.5109	1.9494	6.3133
	#It	28	46	18	12	6

5. CONCLUSION AND REMARKS

Motivated by the fact that in practice many ill-posed problems are regularized by discretization in preimage space and aware of the fact that this does not yield convergence in general, we give here some new results on conditional convergence and convergence rates for this approach, including noisy data and nonlinear problems. Future research will be devoted to an investigation of different strategies for choosing the regularization parameter and on a posteriori selecting the sequence of spaces X_n . One of them might be the refinement and coarsening algorithm proposed in [2], [18]. Moreover, we will investigate multilevel and multigrid approaches in the context of regularization by discretization in preimage space; see, e.g., [13, 19, 14, 15, 16] for the case of regularization by discretization in image space.

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