A METRICAL RESULT ON THE APPROXIMATION BY CONTINUED FRACTIONS

H. JAGER

ABSTRACT. Let x be a real irrational number and p_n/q_n , $n = 0, 1, 2, \dots$ its sequence of continued fraction convergents. Define $\theta_n = q_n |q_n x - p_n|$. For almost all x the distribution function of the sequence $|\theta_{n+1} - \theta_{n-1}|$, $n = 0, 1, 2, \dots$ is determined and its expectation calculated.

1. INTRODUCTION

Let x be a real irrational number with continued fraction expansion

$$x = [a_0; a_1, a_2, \dots]$$

and p_n/q_n , $n = 0, 1, 2, \dots$, the corresponding sequence of convergents.

The sequence θ_n , $n = 0, 1, 2, \dots$, of approximation coefficients of x is defined by

$$\theta_n = q_n^2 \left| x - \frac{p_n}{q_n} \right|, \quad n = 0, 1, 2, \dots$$

One of the most important aspects of the approximation by continued fractions is the fact that this is always a sequence in the unit interval.

The sequence has been extensively studied; see for instance the book [1].

In this paper we determine for almost all irrational x the distribution function of the sequence

$$\theta_{n+1} - \theta_{n-1}, \quad n = 1, 2, \dots,$$

and as an application, calculate

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N |\theta_{n+1} - \theta_{n-1}|,$$

for almost all x .

2. SYMMETRY

We omit in our notations the dependence on x . Define t_n and v_n for $n = 1, 2, \dots$ by

$$\begin{aligned} t_n &= [0; a_{n+1}, a_{n+2}, \dots], \\ v_n &= [0; a_n, a_{n-1}, \dots, a_1]. \end{aligned}$$

One has the fundamental equalities

$$(1) \quad \theta_{n-1} = \frac{v_n}{1 + t_n v_n}, \quad \theta_n = \frac{t_n}{1 + t_n v_n};$$

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see [1], Exercise 5.1.11. They occur already in Koksma's book [2] from 1937, page 25.

The next theorem lies at the basis of what follows.

Theorem 1. *The two-dimensional sequence (t_n, v_n) , $n = 1, 2, \dots$, is for almost all irrational x distributed over the unit square in the (t, v) -plane according to the density function μ where*

$$(2) \quad \mu(t, v) = \frac{1}{\log 2} \frac{1}{(1 + tv)^2}.$$

For a proof see [1], Lemma 5.3.11.

In [3] it was shown that the sequence $|\theta_n - \theta_{n-1}|$, $n = 1, 2, \dots$, is distributed over the unit interval with the density function f , where

$$(3) \quad f(z) = \frac{1}{\log 2} \left(\frac{\pi}{2} - 2 \arctan z \right)$$

and as a consequence that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N |\theta_n - \theta_{n-1}| = \frac{4 - \pi}{4 \log 2} = 0.30960 \dots,$$

for almost all x .

From the symmetry in t and v of the two equalities of (1) and of (2) it follows that the density function of the sequence

$$(4) \quad \theta_n - \theta_{n-1}, \quad n = 1, 2, \dots,$$

is an even function on $[-1, 1]$. Its explicit form follows at once from (3).

In the sequel, a will always denote a positive integer. Let R_a be that part of the unit square in the (t, v) -plane where $\frac{1}{a+1} \leq t \leq \frac{1}{a}$. Then $(t_n, v_n) \in R_a$ if and only if $a_{n+1} = a$.

Now consider the subsequence of (4) by imposing the condition $a_{n+1} = a$ on its terms.

The function

$$\frac{t - v}{1 + tv}$$

takes its maximum and minimum on R_a in, respectively, the lower right-hand and upper left-hand vertex. The maximum is $\frac{1}{a}$, the minimum $-\frac{a}{a+2}$. Hence

$$\theta_n - \theta_{n-1} \in \left[-\frac{a_{n+1}}{a_{n+1} + 2}, \frac{1}{a_{n+1}} \right],$$

from which it follows at once that the subsequence cannot have an even density function because its supporting interval is not symmetric around the origin, except when $a = 2$. This case has to be investigated separately.

In this paper we study the same questions for the sequence

$$(5) \quad \theta_{n+1} - \theta_{n-1}, \quad n = 1, 2, \dots,$$

also a sequence in $[-1, 1]$. Instead of (1) we now need

$$(6) \quad \theta_{n-1} = \frac{v_n}{1 + t_n v_n}, \quad \theta_{n+1} = \frac{(1 - a_{n+1} t_n)(a_{n+1} + v_n)}{1 + t_n v_n};$$

see [1], Proposition 5.3.6.

This suggests the map Ψ_a from R_a onto the (α, β) -plane given by

$$\Psi_a(t, v) = \left(\frac{v}{1+tv}, \frac{(1-at)(a+v)}{1+tv} \right).$$

It is easy to see that $\Psi_a(R_a)$ is a quadrangle with vertices

$$(7) \quad (0, 0), \left(\frac{a}{a+1}, 0 \right), \left(\frac{a+1}{a+2}, \frac{a+1}{a+2} \right) \text{ and } \left(0, \frac{a}{a+1} \right),$$

from which it follows that

$$\theta_{n+1} - \theta_{n-1} \in \left[-\frac{a_{n+1}}{a_{n+1}+1}, \frac{a_{n+1}}{a_{n+1}+1} \right].$$

We call the quadrangle from (7) Q_a .

Denote the Jacobean of Ψ_a by J_a . Then

$$|\det J_a| = \frac{a(a+2v-atv)}{(1+tv)^3}.$$

The sequence $\Psi_a(t_n, v_n) = (\theta_{n-1}, \theta_{n+1})$ is distributed over Q_a according to the density function

$$\frac{1}{\log 2} \frac{1}{(1+tv)^2} \frac{1}{|\det J_a|},$$

which in terms of α and β can be written as

$$\frac{1}{\log 2} \frac{1}{a\sqrt{a^2+4\alpha\beta}}.$$

We call this expression $m_a(\alpha, \beta)$ and the measure connected with it m_a . Note the symmetry in α and β of Q_a and of m_a . This shows that the density function of the sequence $\theta_{n+1} - \theta_{n-1}$, $n = 1, 2, \dots$, under the condition $a_{n+1} = a$, is for all a and for almost all x an even function. Summing all these even functions over a gives an even density function on $[-1, 1]$ for the sequence from (5).

In the next section we will determine this function.

3. THE DISTRIBUTION OF $\theta_{n+1} - \theta_{n-1}$

Let $0 \leq z \leq \frac{a}{a+1}$. Denote by $Q_{a,z}$ that part of Q_a that lies to the right of the line $\beta = \alpha - z$. In order to determine the distribution of the sequence $|\theta_{n+1} - \theta_{n-1}|$, $n = 1, 2, \dots$, we calculate

$$m_a(Q_{a,z}) = \frac{1}{\log 2} \iint_{Q_{a,z}} \frac{d\alpha d\beta}{a\sqrt{a^2+4\alpha\beta}}.$$

First we integrate over α :

$$\int_{\beta+z}^{\frac{\beta}{(a+1)^2} + \frac{a}{a+1}} \frac{d\alpha}{a\sqrt{a^2+4\alpha\beta}} = \frac{1}{2\beta} + \frac{1}{a(a+1)} - \frac{1}{2a\beta} \sqrt{a^2+4\beta^2+4\beta z}.$$

Then we integrate over β , from $\beta = 0$ to $\beta = \frac{a+1}{a+2} - z \frac{(a+1)^2}{a(a+2)}$, the last expression being the β -coordinate of the intersection point of the lines $\beta = \alpha - z$ and $\beta =$

$(a+1)^2\alpha - a(a+1)$. For this we need¹

$$\begin{aligned} \frac{1}{2a} \int \frac{\sqrt{a^2 + 4\beta^2 + 4\beta z}}{\beta} d\beta &= \frac{1}{2} \log \beta + \frac{1}{2a} \sqrt{a^2 + 4\beta^2 + 4\beta z} \\ &+ \frac{z}{2a} \log(z + 2\beta + \sqrt{a^2 + 4\beta^2 + 4\beta z}) - \frac{1}{2} \log(a^2 + 2\beta z + a\sqrt{a^2 + 4\beta^2 + 4\beta z}). \end{aligned}$$

When substituting $\beta = \frac{a+1}{a+2} - z\frac{(a+1)^2}{a(a+2)}$ in the expression under the root sign, one gets at first sight a rather unpleasant result. But fortunately it is a nice square, namely

$$\left(a + \frac{2}{a+2} - z\frac{2(a+1)}{a(a+2)}\right)^2,$$

and after some calculations one then finds that

$$\begin{aligned} m_a(Q_{a,z}) &= \frac{1}{\log 2} \iint_{Q_{a,z}} \frac{d\alpha d\beta}{a\sqrt{a^2 + 4\alpha\beta}} \\ &= \frac{1}{\log 2} \left(\frac{1}{2} \log \frac{(a+1)^2}{a(a+2)} + \frac{z}{2a} \log \frac{a}{a+2} + \frac{z}{2a} \log \frac{a+z}{a-z} + \frac{1}{2} \log(1 - \frac{z^2}{a^2}) \right). \end{aligned}$$

For $z = 0$ this is Lévy's classical expression for the frequency of a specific partial quotient:

$$m_a(Q_a) = \frac{1}{\log 2} \log \frac{(a+1)^2}{a(a+2)};$$

see e.g. [1], page 85. Thus we have found the following result:

Theorem 2. *For almost all x the distribution function on the interval $[0, \frac{a}{a+1}]$ of the sequence*

$$|\theta_{n+1} - \theta_{n-1}|, \quad n = 1, 2, \dots$$

under the condition $a_{n+1} = a$ is

$$\frac{1}{\log 2} \left(\frac{z}{a} \log \frac{a+2}{a} + \frac{z}{a} \log \frac{a-z}{a+z} - \log(1 - \frac{z^2}{a^2}) \right),$$

and the corresponding density function

$$\frac{1}{\log 2} \left(\frac{1}{a} \log \frac{a+2}{a} + \frac{1}{a} \log \frac{a-z}{a+z} \right).$$

We call these functions D_a and d_a , respectively, and extend their definitions by

$$D_a(z) = 0 \quad \text{for } z < 0, \quad D_a(z) = \frac{1}{\log 2} \log \frac{(a+1)^2}{a(a+2)} \quad \text{for } z > \frac{a}{a+1}$$

and

$$d_a(z) = 0 \quad \text{outside } [0, \frac{a}{a+1}].$$

Observe that d_a has only one point of discontinuity, namely $z = 0$.

The functions D and d , defined by

$$D(z) = \sum_{a=1}^{\infty} D_a(z) \quad \text{and} \quad d(z) = \sum_{a=1}^{\infty} d_a(z)$$

¹Found with the help of Wolfram Mathematica Online Integrator and verified by differentiation.

are the distribution and the density function of the sequence $|\theta_{n+1} - \theta_{n-1}|$ on the interval $[0, 1]$. The function d is monotonically decreasing. Its value in 0 is

$$\frac{1}{\log 2} \sum_{a=1}^{\infty} \frac{1}{a} \log\left(1 + \frac{2}{a}\right) = 3.05615 \dots$$

and

$$d'(z) = -\frac{\pi^2}{3 \log 2} + O(z), \quad z \downarrow 0.$$

4. THE MEAN DISTANCE BETWEEN θ_{n-1} AND θ_{n+1}

We now determine, for almost all x , the mean distance between θ_{n-1} and θ_{n+1} . This means that we first have to calculate

$$\int_0^{\frac{a}{a+1}} z d_a(z) \, dz,$$

which is easy, using

$$\int z \log \frac{a-z}{a+z} \, dz = -az - \frac{1}{2}(a^2 - z^2) \log \frac{a-z}{a+z}.$$

It gives

$$\int_0^{\frac{a}{a+1}} z d_a(z) \, dz = \frac{1}{\log 2} \left(-\frac{a}{a+1} + \frac{1}{2} a \log \frac{a+2}{a} \right),$$

and so

$$\int_0^1 z d(z) \, dz = \frac{1}{\log 2} \sum_{a=1}^{\infty} \left(-\frac{a}{a+1} + \frac{1}{2} a \log \frac{a+2}{a} \right).$$

Now

$$\sum_{a=1}^n \frac{a}{a+1} = n - \log n + 1 - \gamma + O\left(\frac{1}{n}\right), \quad n \rightarrow \infty,$$

and

$$\sum_{a=1}^n a \log \frac{a+2}{a} = (n-1) \log(n+1) + n \log(n+2) - 2 \sum_{a=1}^n \log a.$$

Trivially,

$$(n-1) \log(n+1) + n \log(n+2) = 2n \log n - \log n + 3 + O\left(\frac{1}{n}\right), \quad n \rightarrow \infty,$$

and by Stirling's formula,

$$\sum_{a=1}^n \log a = n \log n - n + \frac{1}{2} \log n + \frac{1}{2} \log 2\pi + O\left(\frac{1}{n}\right), \quad n \rightarrow \infty.$$

The final outcome is that

$$\int_0^1 z d(z) \, dz = \frac{1}{\log 2} \left(\gamma + \frac{1}{2} - \frac{1}{2} \log 2\pi \right).$$

This finishes the proof of the following theorem:

Theorem 3. *For almost all x ,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N |\theta_{n+1} - \theta_{n-1}| = \frac{2\gamma + 1 - \log 2\pi}{2 \log 2} = 0.22834 \dots,$$

where γ is Euler's constant.

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OUDE LARENSEWEG 26, 7214 PC EPSE, THE NETHERLANDS

E-mail address: `epserbos@xs4all.nl`